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Strong quasi-metric spaces and countable paracompactness of bispaces

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Abstract. T. G. Raghavan and I. L. Reilly proved that if (X, d) is a quasi-metric space such that $\tau(d)$ is countably paracompact with respect to $\tau(d^{-1})$, then (X, d) is strong (i.e. $\tau(d) \subseteq \tau(d^{-1})$). Here, we show that the converse of this result is also true. We obtain, in this way, a characterization of strong quasi-metric spaces in terms of bitopological countable paracompactness.

1. Introduction and preliminaries

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the sets of real numbers, the set of nonnegative real numbers and the set of positive integer numbers, respectively.

Our basic references for quasi-metric spaces are [3] and [7] and for general topology it is [2].

Let us recall that a *quasi-metric* on a (nonempty) set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = 0 \Leftrightarrow x = y$; and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A *quasi-metric space* is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X.

If d is a quasi-metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a quasi-metric on X called *the conjugate*

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of d. The function d^s defined on $X \times X$ by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$, is a metric on X.

Each quasi-metric d on X generates a T_1-topology $\tau(d)$ on X which has as a base the family of open d-balls $\{B_d(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A quasi-metric d on X is said to be *strong (point symmetric* in [3]) if $\tau(d) \subseteq \tau(d^{-1})$. By a *strong quasi-metric space* we mean a quasi-metric space (X,d) such that the quasi-metric d is strong. A topological space (X,τ) is called *(strongly) quasi-metrizable* if there is a (strong) quasi-metric d on X such that $\tau(d) = \tau$ on X. In this case, we say that d is *compatible* with τ.

Strong quasi-metric spaces were introduced by R. A. Stoltenberg [11], who proved, among other results, that every strongly quasi-metrizable space is developable and, hence, countably metacompact. Conversely, each quasi-metrizable developable space is strongly quasi-metrizable [3], [6]. Furthermore, a quasi-metrizable space (X,τ) is compact if and only if every quasi-metric d on X compatible with τ is strong [6]. The Niemytzki plane, the Pixley–Roy space and the Dieudonné example of a Tychonoff locally compact non-normal space are paradigmatic examples of non-metrizable strongly quasi-metrizable spaces [3], [11]. Recently, separable completely metrizable spaces have been characterized [1] in terms of quasi-metrics d such that $\tau(d^{-1})$ is compact, and hence, d^{-1} is strong (see [8] for a generalization of such a characterization to completely metrizable spaces).

The notion of a bitopological space appears in a natural way when one considers the topologies $\tau(d)$ and $\tau(d^{-1})$ generated by a quasi-metric d and its conjugate. Recall that a *bitopological space* is an ordered triple (X,P,Q) such that X is a (nonempty) set and P and Q are topologies on X [5]. In the sequel we shall use the term *bispace* instead of bitopological space.

It is well known that paracompactness is one of the most intractable notions in the setting of bispaces (see [10], and pages 910–912 and the bibliography of [7]). In particular, T. G. Raghavan and I. L. Reilly introduced in [9] the following concept in their study on metrizability of quasi-metric spaces: In the bispace (X,P,Q) P is *(countably) paracompact with respect to* Q provided that each (countable) P-open cover of X has a P-open refinement which is Q-locally finite.

Then, they proved, among other results, that if (X,d) is a quasi-metric space such that $\tau(d)$ is countably paracompact with respect to $\tau(d^{-1})$, then d is strong, and thus, $(X,\tau(d^{-1}))$ is metrizable (Proposition 7 and Theorem 2 of [9]).

The main purpose of this note is to prove that the converse of this result is also true. Thus, we obtain a somewhat unexpected characterization of strong quasi-metric spaces in terms of Raghavan–Reilly's notion of bitopological countable paracompactness.

2. The results

In order to obtain our main result we first characterize those bispaces (X,P,Q) such that P is countably paracompact with respect to Q in the style of the following classical characterization of countable metacompactness due to F. Ishikawa [4]: A topological space X is countably metacompact if and only if for any decreasing sequence $(F_n)_{n\in\mathbb{N}}$ of nonempty closed sets such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$ there is a decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of open sets such that $F_n \subseteq G_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} G_n = \emptyset$.

Prposition 1. *Let (X,P,Q) be a bispace. Then P is countably paracompact with respect to Q if and only if for any decreasing sequence $(F_n)_{n\in\mathbb{N}}$ of nonempty P-closed sets such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$ there is a decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of P-open sets such that $F_n \subseteq G_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} \mathrm{cl}_Q G_n = \emptyset$.*

Proof. If P is countably paracompact with respect to Q and $(F_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty P-closed sets satisfying $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then $\{X\backslash F_n : n \in \mathbb{N}\}$ is a countable P-open cover of X and, hence, it has a P-open refinement $\mathcal{W}$ which is Q-locally finite. For each $W \in \mathcal{W}$ let $f(W)$ be the first $n \in \mathbb{N}$ such that $W \subseteq X\backslash F_n$. Put, then,

$$V_n = \bigcup\{W \in \mathcal{W} : f(W) = n\}$$

for all $n \in \mathbb{N}$. Therefore $\{V_n : n \in \mathbb{N}\}$ is a P-open cover of X which is Q-locally finite and verifies $V_n \subseteq X\backslash F_n$ for all $n \in \mathbb{N}$. Now let

$$G_n = \bigcup\{V_k : k \geq n+1\},$$

for all $n \in \mathbb{N}$. Clearly $(G_n)_{n\in\mathbb{N}}$ is a decreasing sequence of P-open sets. Since $(F_n)_{n\in\mathbb{N}}$ is decreasing and $F_n \subseteq X\backslash V_n$, it follows that $F_n \subseteq G_n$ for all $n \in \mathbb{N}$. Furthermore, given $x \in X$ there exists a Q-neighborhood H of x which meets only a finite number of V_n's. Consequently there is $m \in \mathbb{N}$ with $H \cap G_m = \emptyset$, and, hence, $\bigcap_{n=1}^{\infty} \mathrm{cl}_Q G_n = \emptyset$.

Conversely, let $\{W_n : n \in \mathbb{N}\}$ be a countable P-open cover of X. Assume, without loss of generality that, for each $n \in \mathbb{N}$, $W_n \neq X$. Now put

$$F_n = X\backslash \bigcup\{W_k : 1 \leq k \leq n\},$$

for all $n \in \mathbb{N}$. Then $(F_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty P-closed sets with $\bigcap_{n=1}^{\infty} F_n = \emptyset$. So, there is a decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of P-open sets such that $F_n \subseteq G_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} \mathrm{cl}_Q G_n = \emptyset$.

Define $V_1 = W_1$, and $V_n = W_n \cap G_{n-1}$ for all $n \geq 2$. Thus, it is clear that $\{V_n : n \in \mathbb{N}\}$ is a countable P-open cover of X such that $V_n \subseteq W_n$ for all $n \in \mathbb{N}$. It remains to show that $\{V_n : n \in \mathbb{N}\}$ is Q-locally finite. To this end, given $x \in X$ we choose the first $n \in \mathbb{N}$ such that $x \in X \backslash \mathrm{cl}_Q G_n$. Then, there exists a Q-neighborhood H of x satisfying $H \cap G_n = \emptyset$. Thus $H \cap G_{n+k} = \emptyset$ for all $k \in \mathbb{N} \cup \{0\}$, and, therefore, $H \cap V_{n+k} = \emptyset$ for all $k \in \mathbb{N}$. Hence, the collection $\{V_n : n \in \mathbb{N}\}$ is Q-locally finite. We conclude that P is countably paracompact with respect to Q. □

Corollary. *Let (X, P, Q) be a bispace. If P is countably paracompact with respect to Q, then (X, P) is countably metacompact.*

The following result provides a partial converse to the above corollary. Recall that a bispace (X, P, Q) is said to be pairwise normal [5] if given a P-closed set A and a disjoint Q-closed set B, there exist a P-open set G and a disjoint Q-open set H such that $A \subseteq H$ and $B \subseteq G$.

It is well known [5] that if (X, d) is a quasi-metric space, then the bispace $(X, \tau(d), \tau(d^{-1}))$ is pairwise normal.

Prposition 2. *Let (X, P, Q) be a pairwise normal bispace such that $P \subseteq Q$. If (X, P) is countably metacompact then P is countably paracompact with respect to Q.*

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be a decreasing sequence of nonempty P-closed sets with $\bigcap_{n=1}^{\infty} F_n = \emptyset$. By countable metacompactness of (X, P), there is a decreasing sequence $(W_n)_{n\in\mathbb{N}}$ of P-open sets such that $F_n \subseteq W_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} W_n = \emptyset$. Since $P \subseteq Q$, each F_n is Q-closed, so, by pairwise normality of (X, P, Q), there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of P-open sets satisfying $F_n \subseteq G_n \subseteq \mathrm{cl}_Q G_n \subseteq W_n$ for all $n \in \mathbb{N}$. Therefore $\bigcap_{n=1}^{\infty} \mathrm{cl}_Q G_n = \emptyset$. By Proposition 1, we conclude that P is countably paracompact with respect to Q. □

Theorem. *A quasi-metric space (X, d) is strong if and only if $\tau(d)$ is countably paracompact with respect to $\tau(d^{-1})$.*

Proof. Suppose that (X, d) is strong. Then $\tau(d) \subseteq \tau(d^{-1})$. Moreover, the bispace $(X, \tau(d), \tau(d^{-1}))$ is pairwise normal and $(X, \tau(d))$ is countably metacompact (see Section 1). So, by Proposition 2, $\tau(d)$ is countably paracompact with respect to $\tau(d^{-1})$. The converse was proved by Raghavan and Reilly in [9], as we indicated in Section 1. □

The above result (compare Proposition 4 of [9]) suggests the following natural question: Let (X,d) be a strong quasi-metric space. Is it $\tau(d)$ paracompact with respect to $\tau(d^{-1})$?.

We conclude the paper with an example which shows that this question has a negative answer.

Example. Let d be the quasi-metric on $\mathbb{R}$ given by:

$$d(x,x)=0 \quad \text{for all } x \in X,$$

$$d(x,y)=1/(y+1) \quad \text{if } x \text{ is irrational and } y \in \mathbb{N},$$

$$d(x,y)=1 \quad \text{otherwise.}$$

Clearly $\tau(d^{-1})$ is the discrete topology on $\mathbb{R}$, so (X,d) is strong. Consider the following $\tau(d)$-open cover of X:

$$\mathcal{W}=\{B_d(x,1) : x \text{ is irrational}\} \cup \{\{x\} : x \text{ is rational with } x \notin \mathbb{N}\}.$$

Suppose that $\mathcal{W}$ has a $\tau(d)$-open refinement $\mathcal{V}$ which is $\tau(d^{-1})$-locally finite. Then each $x \in \mathbb{R}$, and thus each $x \in \mathbb{N}$, is only in finitely many elements of $\mathcal{V}$, which, obviously, is not possible. We conclude that $\tau(d)$ is not paracompact with respect to $\tau(d^{-1})$.

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