Publ. Math. Debrecen
70/1-2 (2007), 149–166

On the diophantine equation $x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n$

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Abstract. Let $S = \{p_1, \dots, p_s\}$ be a set of distinct primes and denote by $\mathbf{S}$ the set of non-zero integers composed only of primes from S. Further, denote by Q the product of the primes from S. Let $f \in \mathbb{Z}[X]$ be a monic quadratic polynomial with negative discriminant D_f contained in $\mathbf{S}$. Consider equation $f(x) = y^n$ (2) in integer unknowns x, y, n with $n \geq 3$ prime and $y > 1$. It follows from a general result of [13] that in (2) n can be bounded from above by an effectively computable constant depending only on Q. This bound is, however, large and is not given explicitly. Using some results of Bugeaud and Shorey [8] we derive, apart from certain exceptions, a good and completely explicit upper bound for n in (2) (see Theorems 1 and 2). Further, combining our Theorem 2 with some deep results of Cohn [12] and de Weger [25] we give all non-exceptional (see Section 1) solutions of equation $x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n$ (6), where x, y, n, α, β, γ, δ are unknown non-negative integers with $x \geq 1$, $\gcd(x, y) = 1$ and $n \geq 3$ (cf. Theorem 3). When, in (6), $\alpha \geq 1$ is also assumed then our Theorem 3 is a generalization of a result of Luca [19]. In this case all the solutions of equation (6) are listed.

1. Introduction

There are many results concerning the generalized Ramanujan–Nagell equation

$$x^2 + D = \mu y^n, \tag{1}$$

where $D > 0$ is a given integer, $\mu \in \{1, 4\}$ and x, y, n are positive integer unknowns with $n \geq 3$ and $\gcd(x, y) = 1$. First consider the case $\mu = 1$. Then the first result was due to V. A. Lebesque [15] who proved that there are no

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Mathematics Subject Classification: 11D41, 11D61.
Key words and phrases: exponential diophantine equations, primitive divisors.

solutions for $D = 1$. Ljunggren [16] solved (1) for $D = 2$, and Nagell [22], [23] solved it for $D = 3, 4$ and 5. In his elegant paper [11], Cohn gave a fine summary of work on equation (1). Further, he developed a method by which he found all solutions of the above equation for 77 positive values of $D \leq 100$. For $D = 74$ and $D = 86$, equation (1) was solved by Mignotte and de Weger [20]. By using the theory of Galois representations and modular forms Bennett and Skinner [5] solved (1) for $D = 55$ and $D = 95$. On combining the theory of linear forms in logarithms with Bennett and Skinner's method and with several additional ideas, Bugeaud, Mignotte and Siksek [7] gave all the solutions of (1) for the remaining 19 values of $D \leq 100$. Bugeaud and Shorey [8] used a beautiful result of Bilu, Hanrot and Voutier [6] to solve completely several equations of type (1) both for $\mu = 1$ and for $\mu = 4$ when D is an odd positive square-free integer, $n \geq 3$ is an odd prime not dividing the class number of the field $\mathbb{Q}(\sqrt{-D})$ and $D \not\equiv 7 \pmod 8$ if $\mu = 1$ (see Corollaries 3, 5 and 7 of [8]).

Let $S = \{p_1, \ldots, p_s\}$ denote a set of distinct primes and $\mathbf{S}$ the set of non-zero integers composed only of primes from S. Denote by P and Q the greatest and the product of the primes of S, respectively. In recent years, equation (1) has been considered also in the more general case when D is no longer fixed but $D \in \mathbf{S}$ with $D > 0$. It follows from Theorem 2 of [24] that in (1) n can be bounded from above by an effectively computable constant depending only on f, P and s. In [13] an effective upper bound was derived for n which depends only on Q. By using the powerful method of Bilu, Hanrot and Voutier [6] equation (1) can be completely solved for $\mu = 1$ and some special sets of primes S. Namely, if in (1) $D \in \mathbf{S}$ with $S = \{2\}$ then all solutions of (1) were given by Cohn [10] and Arif and Muriefah [1] and [3]. For $S = \{3\}$, equation (1) was solved completely by Arif and Muriefah [2] and Luca [18]. When $S = \{q\}$, where $q \geq 5$ is an odd prime with $q \not\equiv 7 \pmod 8$, Arif and Muriefah [4] determined all solutions of the equation $x^2 + q^{2k+1} = y^n$, where $\gcd(n, 3h_0) = 1$ and $n \geq 3$. Here h_0 denotes the class number of the field $\mathbb{Q}(\sqrt{-q})$. For $S = \{2, 3\}$, Luca [19] gave the complete solution of (1).

To formulate our results we introduce some notation. Let $f(x) = x^2 + Ax + B$ where $A, B \in \mathbb{Z}$ and denote by D_f the discriminant of f. Set

$$\Delta = \begin{cases} -\dfrac{D_f}{4} & \text{if } D_f \text{ is even,} \\ -D_f & \text{if } D_f \text{ is odd.} \end{cases}$$

Suppose that $\Delta \in \mathbf{S}$ and $\Delta > 0$. Let c and d be non-zero integers such that $\Delta = dc^2$ and $d > 0$ denotes the square-free part of Δ. Further, for any $k \in \mathbb{Z}$ and rational prime p denote by $\mathrm{ord}_p(k)$ the greatest power of p to which p divides k.

Consider the equation

$$f(x)=y^n \tag{2}$$

in integer unknowns x, y, n with $n\geq 3$ prime and $y>1$. We say that a solution (x,y,n) of (2) is *exceptional* if

$$\mathrm{ord}_2(D_f)=2,\ y \text{ is even and } d\equiv 7 \pmod 8.$$

Write h for the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d}\,)$. Further, denote by $h(-4\Delta)$ the number of classes of positive binary quadratic forms with discriminant -4Δ (for the definition see Section 2).

Theorem 1. *If* (x,y,n) *is a non-exceptional solution of* (2) *with* $x\neq -\frac{A}{2}$ *and* $\gcd(y,\Delta)=1$ *then, except for the infinite families of equations*

$$x^2+Ax+B=y^n$$

where $(A,B,x,y,\Delta,n)\in\{(A,(A^2+7)/4,(11-A)/2,2,7,5),(A,(A^2+7)/4,(181-A)/2,2,7,13),(A,(A^2+11)/4,\ (31-A)/2,3,11,5),(A,(A^2+19)/4,(559-A)/2,5,19,7)\}$, *where* A *is odd and* $(A,B,x,y,\Delta,n)\in\{(A,(A^2{+}76)/4,(44868-A)/2,55,19,5),(A,(A^2{+}1364)/4,(5519292-A)/2,377,341,5\}$, *where* A *is even, we have*

$$n=3 \quad \textit{or} \quad n\mid h(-4\Delta).$$

Further, in the latter case

$$n\leq\max\{3,P\} \quad \textit{if} \quad n\nmid h$$

and

$$n<\frac{4}{\pi}\sqrt{Q}\log(2e\sqrt{Q}\,) \quad \textit{if} \quad n\mid h.$$

We note that the assumption $x\neq -\frac{A}{2}$ is necessary. Otherwise using (2) and supposing that D_f is even we get $y^n=\Delta$, whence by $\Delta\in\mathbf{S}$ we see that n cannot be bounded.

Equation (2) can be reduced to an equation of the type

$$X^2+\Delta=\mu Y^n, \tag{3}$$

where $\mu\in\{1,4\}$,

$$\gcd(X,Y)=\gcd(Y,\Delta)=1 \tag{4}$$

and

$$\begin{gathered}\mu = 1 \text{ if } D_f \text{ is even}, \ \mu = 4 \text{ if } D_f \text{ is odd}, \\ \Delta \in \mathbf{S}, \ \Delta > 0, \ X \geq 1, \ Y > 1, \ n \geq 3 \text{ prime}.\end{gathered} \tag{5}$$

We shall deduce Theorem 1 from the following Theorem 2. We say that a solution (X, Y, n) of (3) is *exceptional* if

$$\mu = 1, \ \mathrm{ord}_2(D_f) = 2, \ Y \text{ is even and } d \equiv 7 \pmod 8.$$

Theorem 2. *If* (X, Y, n) *is a non-exceptional solution of equation* (3) *satisfying* (4) *and* (5) *then, except for* $(\mu, Y, \Delta, n) \in \{(4,2,7,5), (4,2,7,13), (4,3,11,5), (4,5,19,7), (1,55,19,5), (1,377,341,5)\}$, *we have*

$$n = 3 \quad \text{or} \quad n \mid h(-4\Delta).$$

Further, in the latter case

$$n \leq \max\{3, P\} \quad \text{if} \quad n \nmid h$$

and

$$n < \frac{4}{\pi}\sqrt{Q}\log(2e\sqrt{Q}) \quad \text{if} \quad n \mid h.$$

This should be compared with Corollaries 5 and 7 of Bugeaud and Shorey [8], where equations of type (3) were considered with square-free $\Delta > 0$. In Corollary 5 they showed that the equation $x^2 + 4\Delta = y^n$ has no solution with $n \geq 5$. Here Δ is square-free and n is an odd prime not dividing the class number of the field $\mathbb{Q}(\sqrt{-\Delta})$. Further, in Corollary 7 of [8] the authors considered the equation (3), where $\mu \in \{1, 4\}$, Δ is an odd positive square-free integer and $n \geq 3$ is an odd prime not dividing the class number of the field $\mathbb{Q}(\sqrt{-\Delta})$. Under these assumptions they solved completely equation (3) in the case when

$$\mu = 1, \ \Delta \equiv 1 \pmod 4, \ n \geq 3 \quad \text{or} \quad \mu = 4, \ \Delta \equiv 7 \pmod 8, \ n \geq 3 \quad \text{or}$$

$$\mu = 4, \ \Delta \equiv 3 \pmod 8, \ n \geq 5.$$

In contrast with [8], in our Theorem 2 it is not assumed that Δ is square-free. Using the approach of [8] we give completely explicit upper bounds for n in (3) depending only on P and Q. This allows us to solve completely equation (3) in the case when $S = \{2, 3, 5, 7\}$ and $\mu = 1$. Combining Theorem 2 with some results of Cohn [12] and de Weger [25], we give all non-exceptional solutions of the equation

$$x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n \tag{6}$$

where x, y, n, α, β, γ, δ are unknown non-negative integers with $x \geq 1$, $y \geq 2$, $\gcd(x, y) = 1$ and $n \geq 3$. We recall that in this special case a solution is called *exceptional* if $\alpha = 0$, y is even and $3^\beta 5^\gamma 7^\delta$ is either of the form $7c^2$ or of the form $15c^2$. We note that if our equation (6) is of the form $x^2 + 7c^2 = y^n$ or $x^2 + 15c^2 = y^n$ and (x, y, n) is an exceptional solution of (6), then we cannot use the parametrization for (x, y) provided by Lemma 2 (see e.g. [11]). Hence we consider only the non-exceptional solutions of (6). We note that using another approach Bugeaud, Mignotte and Siksek [7] solved the equations $x^2 + 7c^2 = y^n$ and $x^2 + 15c^2 = y^n$ when $1 \leq 7c^2 < 15c^2 \leq 100$.

Theorem 3. *All non-exceptional solutions of equation* (6) *are listed in the table occurring in Section 4.*

If in (6) $\alpha \geq 1$ is assumed then, by $\gcd(x, y) = 1$, y is odd. Hence the solutions (x, y, n) of (6) are always non-exceptional. Thus in this case we can list all the solutions of equation (6).

Corollary. *All solutions of* (6) *with* $\alpha \geq 1$ *are listed in the table in Section 4.*

We note that the solutions of equation (6) with $\alpha \geq 1$ are those which are not marked with an asterisk in the table. Further, in this case our Theorem 3 is a generalization of a result of Luca [19] mentioned above.

2. Auxiliary results

We keep the notations of the preceding section. For a non-zero integer m denote by $\omega(m)$ the number of distinct prime factors of m. By definition, for $a, b, c \in \mathbb{Z}$, the discriminant of the binary quadratic form $aX^2 + 2bXY + cY^2$ is $4b^2 - 4ac$, thus -4Δ is the discriminant of the form $X^2 + \Delta Y^2$. We say that a binary quadratic form is positive if $a > 0$. The set of positive binary quadratic forms of discriminant -4Δ is partitioned into a finite number of equivalence classes which we denote by $h(-4\Delta)$.

The next lemma is a special case of Lemma 1 of [8] (see also Le [14]).

Lemma 1. *Consider equation*

$$X_1^2 + \Delta Y_1^2 = \mu Y^{Z_1} \tag{7}$$

in integer unknowns X_1, Y_1, Z_1 *with* $Z_1 > 0$ *and* $\gcd(X_1, Y_1) = 1$. *Then the solutions of the above equation can be put into at most* $2^{\omega(Y)-1}$ *classes. Further, in each class there is a unique solution* (X_1, Y_1, Z_1) *such that* $X_1 > 0, Y_1 > 0$ *and*

Z_1 is minimal among the solutions of the class. This minimal solution satisfies $Z_1 \mid h(-4\Delta)$, where $h(-4\Delta)$ is the number of classes of positive binary forms of discriminant -4Δ.

Proof. See [8]. □

Lemma 2. *Suppose that equation (3) has a solution under the assumptions (4) and (5) with $\mu = 1$. Denote by $d > 0$ the square-free part of $\Delta = dc^2$. If $d \not\equiv 7 \pmod 8$ or $d \equiv 7 \pmod 8$ and Y is odd then one of the following cases holds:*

(a) *there exist $a_1, b_1 \in \mathbb{Z}$ with $b_1 \mid c$, $b_1 \neq \pm c$ such that $Y = a_1^2 + b_1^2 d$ and $\pm X + c\sqrt{-d} = (a_1 + b_1\sqrt{-d}\,)^n$;*

(b) *$n \mid h$, where h denotes the class number of the field $\mathbb{Q}(\sqrt{-d}\,)$;*

(c) *$d \equiv 3 \pmod 8$, $n = 3$ and there exist odd integers A_1, B_1 with $B_1 \mid c$ such that $Y = \frac{1}{4}(A_1^2 + B_1^2 d)$, $\pm X + c\sqrt{-d} = \frac{1}{8}(A_1 + B_1\sqrt{-d}\,)^3$;*

(d) *$(n, \Delta, X) = (3, 3u^2 \pm 8, u^3 \pm 3u)$ or $(n, \Delta, X) = (3, 3u^2 \pm 1, 8u^3 \pm 3u)$, where $u \in \mathbb{Z}$;*

(e) *$(n, \Delta, X) = (5, 19, 22434)$ or $(n, \Delta, X) = (5, 341, 2759646)$.*

Proof. If $d \not\equiv 7 \pmod 8$ then the lemma is a reformulation of a theorem of Cohn [12]. So, it remains the case when in (3) $d \equiv 7 \pmod 8$ and Y is odd. In this case we may apply a result of Ljunggren [17] (pp. 593–594) to conclude that if in equation (3) $n \nmid h$ then there exist $a_1, b_1 \in \mathbb{Z}$ such that

$$\pm X + c\sqrt{-d} = \left(\frac{a_1 + b_1\sqrt{-d}}{2}\right)^n, \quad a_1 \equiv b_1 \pmod 2. \tag{8}$$

If in (8) a_1 and b_1 are both odd then since $d \equiv 7 \pmod 8$, we get

$$a_1^2 + db_1^2 \equiv 0 \pmod 8,$$

whence, by

$$Y = \frac{a_1^2 + db_1^2}{4},$$

it follows that Y is even, a contradiction. So a_1 and b_1 are both even and the lemma is proved. □

The next lemma provides an upper bound for the class number of an imaginary quadratic field.

Lemma 3. *Let $\mathcal{D} > 0$ be a square-free integer, and denote by h the class number of the field $\mathbb{K} = \mathbb{Q}(\sqrt{-\mathcal{D}}\,)$. Then*

$$h < \frac{4}{\pi}\sqrt{\mathcal{D}}(\log 2e\sqrt{\mathcal{D}}\,).$$

Proof. Denote by $h(-4\mathcal{D})$ the class number of the unique quadratic order in $\mathbb{K}$ with discriminant $-4\mathcal{D}$. Then $h(-4\mathcal{D})$ is the number of classes of positive quadratic forms of discriminant $-4\mathcal{D}$ (see e.g. Cohen [9], Definition 5.2.7). Further, we have

$$h(-4\mathcal{D}) < \frac{4}{\pi}\sqrt{\mathcal{D}}(\log 2e\sqrt{\mathcal{D}}\,)$$

(cf. e.g. Proposition 1 of [8]). Since $h \mid h(-4\mathcal{D})$ (see e.g. [21]), the assertion follows. □

Lemma 4. *Denote by $h(-4\Delta)$ the number of classes of positive binary forms of discriminant -4Δ. Then, for $d \equiv 3 \pmod 4$,*

$$h(-4\Delta) = h(-4c^2d) = h2c\prod_{p|2c}\left(1 - \frac{(-d/p)}{p}\right)\frac{1}{u},$$

where $u = 3$, if $d = 3$ and $u = 1$ otherwise; for $d \equiv 1, 2 \pmod 4$,

$$h(-4\Delta) = h(-4c^2d) = hc\prod_{p|c}\left(1 - \frac{(-4d/p)}{p}\right)\frac{1}{u},$$

where $u = 2$, if $d = 1$ and $u = 1$ otherwise. Here $(\frac{\cdot}{p})$ denotes the Kronecker symbol.

Proof. See Mollin [21]. □

The next lemma is a deep result of de Weger [25]. It will be utilized in the proof of Theorem 3.

Lemma 5. *Let $S = \{2, 3, 5, 7\}$. Consider the equation $U + V = W^2$ in unknowns U, V, W, where U, V or $-V \in \mathbf{S} \cap \mathbb{Z}_{>0}$, $W \in \mathbb{Z}_{>0}$. Suppose that $U \geq V$ and that $\gcd(U, V)$ is square-free. Then the above equation has exactly 388 solutions which are given explicitly in* [25].

Proof. This is Theorem 7.2 of [25]. □

3. Proofs of theorems

Proof of Theorem 2. Consider equation (3) satisfying (4) and (5). We follow the approach of [8] and we introduce two infinite sets. Denote by F_k the Fibonacci sequence defined by $F_0 = 0, F_1 = 1$ and satisfying $F_k = F_{k-1} + F_{k-2}$

for all $k \geq 2$ and by L_k the Lucas sequence defined by by $L_0 = 2$, $L_1 = 1$ and satisfying $L_k = L_{k-1} + L_{k-2}$ for all $k \geq 2$. Then

$$\mathrm{F} := \{(F_{k+\varepsilon}, L_{k-\varepsilon}, F_k) \mid k \geq 2,\ \varepsilon \in \{\pm 1\}\},$$

and

$$\mathrm{H} := \{(1, \Delta, Y) \mid \text{ there exist } r, s \in \mathbb{Z}_{>0} \text{ such that } s^2 + \Delta = \mu Y^r \text{ and } 3s^2 - \Delta = \mp\mu\}.$$

If (X, Y, n) is a non-exceptional solution of (3) then it corresponds to a solution $(X_1, Y_1, Z_1) = (X, 1, n)$ of (7). Since by Lemma 1 the solutions of (7) can be put into at most $2^{\omega(Y)-1}$ classes we have to distinguish two cases. Firstly, if $(X, 1, n)$ is the minimal solution in the class then by Lemma 1 we have $n \mid h(-4\Delta)$. Secondly, if $(X, 1, n)$ is not the minimal solution then there exist at least two solutions of (7) in the class. By using Theorem 2 of [8] and noting that n is an odd prime we see that in this case either $(\mu, Y, \Delta, n) \in \{(4, 2, 7, 3), (4, 7, 3, 3), (4, 2, 7, 5), (4, 2, 7, 13), (4, 3, 11, 5), (4, 5, 19, 7), (1, 55, 19, 5), (1, 377, 341, 5)\}$ or we have

$$n \in \{1, 5\} \quad \text{and} \quad (1, \Delta, Y) \in \mathrm{F}$$

or

$$n \in \{r, 3r\} \quad \text{and} \quad (1, \Delta, Y) \in \mathrm{H}, \quad \text{with } r \in \mathbb{Z}_{>0}.$$

Since n is an odd prime we obtain that

$$n = 5 \quad \text{and} \quad (1, \Delta, Y) \in \mathrm{F} \qquad \text{or} \quad n = 3 \quad \text{and} \quad (1, \Delta, Y) \in \mathrm{H}.$$

If $n = 5$ and $(1, \Delta, Y) \in \mathrm{F}$ then by the definition of the set F we get

$$F_{k-2} = 1,\ L_{k+1} = \Delta,\ F_k = Y$$

or

$$F_{k+2} = 1,\ L_{k-1} = \Delta,\ F_k = Y.$$

We see that $F_{k+2} = 1$ cannot hold since in this case it follows that $k + 2 \in \{1, 2\}$ and hence $k = 0$. Thus $F_0 = Y = 0$ follows which contradicts the assumption $Y > 1$. If $F_{k-2} = 1$ we get $k - 2 \in \{1, 2\}$, whence $k \in \{3, 4\}$ which implies by $F_k = Y$ that

$$(Y, \Delta) \in \{(2, 7), (3, 11)\}.$$

Hence using (3) we get

$$X^2 + 7 = \mu \cdot 2^5 \quad \text{and} \quad X^2 + 11 = \mu \cdot 3^5.$$

We see that if $\mu = 4$ the above equations have solutions which are already listed (i.e. $(\mu, Y, \Delta, n) \in \{(4,2,7,5), (4,3,11,5)\}$). If $\mu = 1$ then $X^2 + 11 = 3^5$ is impossible, while the equation $X^2 + 7 = 2^5$ leads to an exceptional solution of (3), which contradicts the assumption that (X, Y, n) is non-exceptional. Hence we obtain that

$$n \mid h(-4\Delta) \quad \text{or} \quad n = 3,$$

according as $(X, 1, n)$ is the minimal solution in the class or not. We recall that n is an odd prime and $\Delta = dc^2 \in \mathbf{S}$. Thus if $n \mid h(-4\Delta)$ but $n \nmid h$ then by Lemma 4 we obtain that n cannot exceed the greatest prime lying in $S = \{p_1, \dots, p_s\}$. Hence

$$n \le \max\{3, P\}.$$

If $n \mid h$ then since d is the square-free part of Δ we have

$$d \le Q = p_1 \cdots p_s.$$

Hence using Lemma 3 the assertion follows. □

Proof of Theorem 1. Put $f(x) = x^2 + Ax + B$, where $A, B \in \mathbb{Z}$. One can easily see that equation (2) leads to the equation of type (3)

$$X^2 + \Delta = \mu Y^n, \tag{9}$$

where

$$(X, \Delta, \mu, Y) = \begin{cases} \left(x + \dfrac{A}{2}, -\dfrac{D_f}{4}, 1, y\right) & \text{if } D_f \text{ is even,} \\ (2x + A, -D_f, 4, y) & \text{if } D_f \text{ is odd.} \end{cases}$$

According to the definition of Δ and the assumption $x \ne -\frac{A}{2}$, we may suppose that in equation (9)

$$\Delta \in \mathbf{S},\ \Delta > 0,\ X \ge 1,\ n \ge 3 \text{ prime}. \tag{10}$$

Since, by assumption, $\gcd(Y, \Delta) = 1$ we can apply Theorem 2 to equation (9) and we get Theorem 1. □

Proof of Theorem 3. There is no loss of generality by supposing that in (6) $n = 4$ or n is an odd prime. Keeping the notations of the preceding sections we have $dc^2 = \Delta = 2^\alpha 3^\beta 5^\gamma 7^\delta$, where $d \in \mathcal{H}$ with $\mathcal{H} = \{1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\}$. Assume that n is an odd prime. Since (x, y, n) is a non-exceptional solution of (6) and for every $d \in \mathcal{H}$ the class number h of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d}\,)$ is 1 or a power of 2 by Theorem 2 we get $n \le 7$. Hence (6) can have a solution only if $n \in \{3,4,5,7\}$.

The case $n \in \{5,7\}$. We recall that $dc^2 = \Delta = 2^\alpha 3^\beta 5^\gamma 7^\delta$, where $d \in \mathcal{H}$ with $\mathcal{H} = \{1,2,3,5,6,7,10,14,15,21,30,35,42,70,105,210\}$.

Consider equation (6) with $n = 5$. Assume first that $\alpha \ge 0$, $\beta \ge 0$, $\gamma \ge 6$, $\delta \ge 0$. By Lemma 2 we get

$$\pm x + c\sqrt{-d} = (a + b\sqrt{-d}\,)^5 \tag{11}$$

where $a, b \in \mathbb{Z}$, $b \mid c$, $y = a^2 + db^2$. Hence, by comparing the imaginary parts of (11) we obtain

$$5a^4 b - 10a^2 b^3 d + b^5 d^2 = c. \tag{12}$$

Since $\gamma \ge 6$ we have $\mathrm{ord}_5(c) \ge 3$.

Case 1. $1 \le \mathrm{ord}_5(b) \le \mathrm{ord}_5(c) - 2$

Since $b \mid c$ and $\mathrm{ord}_5(b) \le \mathrm{ord}_5(c) - 2$, we see that $\frac{c}{5b} \in \mathbb{Z}$ and $\mathrm{ord}_5\left(\frac{c}{5b}\right) \ge 1$. Using (12) we get

$$a^4 - 2a^2 b^2 d + \frac{b^4}{5} d^2 = \frac{c}{5b}. \tag{13}$$

Since $\mathrm{ord}_5\left(\frac{c}{5b}\right) \ge 1$ and $\mathrm{ord}_5(b) \ge 1$, we obtain by (13) that $5 \mid a^4$, whence by $a \mid x$ we get $5 \mid x$. Thus using equation (6) and the assumption $\gamma \ge 6$, we obtain that $5 \mid y$ which is impossible since x and y are relatively prime.

Case 2. $\mathrm{ord}_5(b) = \mathrm{ord}_5(c)$.

In this case we have $\mathrm{ord}_5(b) \ge 3$ since $\mathrm{ord}_5(c) \ge 3$. By (12) it follows that

$$5a^4 - 10a^2 b^2 d + b^4 d^2 = \frac{c}{b}. \tag{14}$$

Hence the left-hand side of (14) is divisible by 5 but the right-hand side is not, a contradiction.

Case 3. $\mathrm{ord}_5(b) = 0$

By $\mathrm{ord}_5(b) = 0$ we have $\mathrm{ord}_5(c) = \mathrm{ord}_5\left(\frac{c}{b}\right) \ge 3$. Thus using (14) we see that $5 \mid b^4 d^2$ follows, whence by $\mathrm{ord}_5(b) = 0$ we get $5 \mid d$. Hence from (14) we infer that

$$a^4 - 2a^2 b^2 d + \frac{b^4 d^2}{5} = \frac{c}{5b}. \tag{15}$$

Clearly $\frac{b^4 d^2}{5}$ and $\frac{c}{5b}$ are integers and $\mathrm{ord}_5\left(\frac{c}{5b}\right) \ge 2$. Thus by (15) it follows that $5 \mid a^4$, whence by $a \mid x$ we get $5 \mid x$. Thus using equation (6) and the assumption $\gamma \ge 6$, we obtain that $5 \mid y$ which contradicts $\gcd(x, y) = 1$.

Case 4. $\mathrm{ord}_5(b) = \mathrm{ord}_5(c) - 1$

By assumption we see that in (15) $\frac{b^4d^2}{5}$ and $\frac{c}{5b}$ are integers and $\mathrm{ord}_5(\frac{c}{5b})$=0. If now $\mathrm{ord}_2\left(\frac{c}{5b}\right)\geq 1$ then clearly $\alpha\geq 1$ and by (15) we get

$$a^4\equiv \frac{b^4d^2}{5} \pmod 2. \tag{16}$$

We may suppose that a is odd since otherwise we obtain by (6), $a\mid x$ and $\alpha\geq 1$ that $2\leq \gcd(x,y)$ contradicting the assumption $\gcd(x,y)=1$. Thus by (16) we see that b and d are odd integers, whence it follows that $2\mid y=a^2+db^2$. Using equation (6) and $\alpha\geq 1$ we get $2\mid x$ which cannot hold since x and y are relatively prime integers.

Suppose now that $\mathrm{ord}_2\left(\frac{c}{5b}\right)=0$ and $\mathrm{ord}_3\left(\frac{c}{5b}\right)\geq 1$. Then obviously $\beta\geq 1$. We may assume that $3\nmid d$ and $3\nmid b$ since otherwise we get by (15) that $3\mid a^4$ whence $3\mid x$. Thus by (6) and $\beta\geq 1$ we see that $3\mid y$ which leads to a contradiction.

By $\Delta=dc^2, y=a^2+db^2$ and (6) we have

$$x^2+dc^2=(a^2+db^2)^5 \tag{17}$$

Clearly $3\nmid x$ since otherwise we obtain a contradiction by (6), $\beta\geq 1$ and $\gcd(x,y)=1$. Thus by $3\nmid x$ and $\beta\geq 1$ we have

$$x^2+dc^2\equiv 1 \pmod 3. \tag{18}$$

If $3\mid a$, then by $a\mid x,\beta\geq 1$ and (6) we obtain a contradiction. Hence

$$(a^2+db^2)^5\equiv (1+d)^5 \pmod 3. \tag{19}$$

Since for every $d\in\mathcal{H}$ with $3\nmid d$ we have $1+d\equiv -1,0 \pmod 3$ we see by (19) that

$$(a^2+db^2)^5\equiv -1,0 \pmod 3. \tag{20}$$

Combining (17),(18) and (20) we get a contradiction.

If $\mathrm{ord}_2\left(\frac{c}{5b}\right)=0$ and $\mathrm{ord}_3\left(\frac{c}{5b}\right)=0$ then we have by (15) that

$$a^4-2a^2b^2d+\frac{b^4d^2}{5}=\pm 7^{\delta'}, \tag{21}$$

for some non-negative integer δ'. If $\delta'\geq 1$ and $7\mid d$ then by (21) we infer that $7\mid a^4$, whence by $a\mid x$ we have $7\mid x$ and $7\mid y=a^2+db^2$. This cannot hold by $\gcd(x,y)=1$.

If $\delta' \geq 1$ and $7 \nmid d$ then (21) is impossible mod 7 for every $d \in \mathcal{H}$.
If $\delta' = 0$ then (21) is a Thue equation. By solving (21) for every $d \in \mathcal{H}$ we obtain the solution $(\Delta, y, n) = (2 \cdot 5^3, 11, 5)$. It remains the case when in (6) $n = 5$ and $\gamma \in \{0, 1, 2, 3, 4, 5\}$. If $\gamma \in \{0, 3, 4, 5\}$ then we may apply the argument used in the *Case 4* to the equation

$$5a^4 - 10a^2b^2d + b^4d^2 = \frac{c}{b} \quad \text{and} \quad a^4 - 2a^2b^2d + \frac{b^4d^2}{5} = \frac{c}{5b},$$

respectively. Thus in this case there are no other solutions with $n = 5$.
If $\gamma \in \{1, 2\}$ we see that $\mathrm{ord}_5(c) = 0$ and $\mathrm{ord}_5(d) = 1$, or $\mathrm{ord}_5(c) = 1$ and $\mathrm{ord}_5(d) = 0$. This leads to a contradiction in view of (12).

In the case when in (6) $n = 7$ we can work as above and we conclude that there are no solutions to (6) with $n = 7$.

The case $n \in \{3, 4\}$.
First consider equation (6) with $n = 4$. Then factorizing in (6) we get

$$(y^2 + x)(y^2 - x) = \Delta. \tag{22}$$

Hence $y^2 + x \in \mathbf{S}$ and $y^2 - x \in \mathbf{S}$, where $S = \{2, 3, 5, 7\}$. Thus we have the following equations

$$\begin{cases} \dfrac{y^2 + x}{2} + \frac{y^2 - x}{2} = y^2, & \text{if } x \equiv y \pmod 2, \\ 2(y^2 + x) + 2(y^2 - x) = (2y)^2, & \text{if } x \not\equiv y \pmod 2. \end{cases} \tag{23}$$

Since x and y are relatively prime we have $\gcd\left(\frac{y^2+x}{2}, \frac{y^2-x}{2}\right) = 1$ and $\gcd(2(y^2 + x), 2(y^2 - x)) = 2$. Further, since $x > 0$ we get that $\frac{y^2+x}{2} \geq \frac{y^2-x}{2}$ and $2(y^2 + x) \geq 2(y^2 - x)$ always holds. Thus we see that equations (23) satisfies the conditions of Lemma 5 with

$$(U, V, W) = \left(\frac{y^2 + x}{2}, \frac{y^2 - x}{2}, y\right) \qquad \text{and}$$

$$(U, V, W) = (2(y^2 + x), 2(y^2 - x), 2y).$$

By applying Lemma 5 to (23) we obtain all non-exceptional solutions of (6) with $n = 4$ (see the table).

Next suppose that in (6) $n = 3$. By Lemma 2 we see that (6) can have a solution with $n = 3$ only in the following cases:

(I) $3a_1^2b_1 - b_1^3d = c$, where $a_1, b_1 \in \mathbb{Z}$ and $b_1 \mid c$,

(II) $3A_1^2B_1 - B_1^3d = 8c$, where A_1, B_1 are odd integers and $B_1 \mid c$,

(III) $\Delta = 3u^2 \pm 8$ and $x = u^3 \pm 3u$, where $u \in \mathbb{Z}$,

(IV) $\Delta = 3u^2 \pm 1$ and $x = 8u^2 \pm 3u$, where $u \in \mathbb{Z}$.

Each of the above cases leads to an equation of the form

$$U + V = W^2, \quad U, V \in \mathbf{S}$$

with the following choices of the triple (U, V, W).

Case (I) In this case we have the equation

$$3a_1^2b_1 - b_1^3d = c,$$

where $a_1, b_1 \in \mathbb{Z}$ and $b_1 \mid c$. We distinguish three subcases according to $3 \mid d$ or $3 \nmid db_1$ or $3 \nmid d$ and $3 \mid b_1$.

If $3 \mid d$ then obviously $3 \mid c$ and hence we get from the above equation

$$a_1^2 = \frac{c}{3b_1} + \frac{d}{3}b_1^2.$$

We see that $\gcd\left(\frac{c}{3b_1}, \frac{d}{3}b_1^2\right)$ is square-free since $\frac{c}{3b_1}$ and b_1^2 are relatively prime and $\frac{d}{3}$ is square-free. The last two subcases can be reduced in a similar way. Thus we have to solve for every $d \in \mathcal{H}$ the following equations:

$$(U,V,W) = \begin{cases} \left(\frac{d}{3}b_1^2, \frac{c}{3b_1}, a_1\right), & \text{if } 3 \mid d \text{ and } \frac{d}{3}b_1^2 \geq \frac{c}{3b_1}, \\ \left(\frac{c}{3b_1}, \frac{d}{3}b_1^2, a_1\right), & \text{if } 3 \mid d \text{ and } \frac{c}{3b_1} > \frac{d}{3}b_1^2, \\ \left(3db_1^2, \frac{3c}{b_1}, 3a_1\right), & \text{if } 3 \nmid d,\ 3 \nmid b_1 \text{ and } 3db_1^2 \geq \frac{3c}{b_1}, \\ \left(\frac{3c}{b_1}, 3db_1^2, 3a_1\right), & \text{if } 3 \nmid d,\ 3 \nmid b_1 \text{ and } \frac{3c}{b_1} > 3db_1^2, \\ \left(3db_1'^2, \frac{c}{9b_1'}, a_1\right), & \\ \quad b_1 = 3b_1', & \text{if } 3 \nmid d,\ 3 \mid b_1, \text{ and } 3db_1'^2 \geq \frac{c}{9b_1'}, \\ \left(\frac{c}{9b_1'}, 3db_1'^2, a_1\right), & \\ \quad b_1 = 3b_1', & \text{if } 3 \nmid d,\ 3 \mid b_1, \text{ and} \frac{c}{9b_1'} > 3db_1'^2. \end{cases} \tag{24}$$

Case (II) In this case we deal only with those values of $d \in \mathcal{H}$ for which $d \equiv 3 \pmod 8$. Thus $d \in \{3, 35\}$ and we get the following equations:

$$(U,V,W) = \begin{cases} \left(B_1^2, \frac{8c}{3B_1}, A_1\right), & \text{if } d = 3 \text{ and } B_1^2 \geq \frac{8c}{3B_1}, \\ \left(\frac{8c}{3B_1}, B_1^2, A_1\right), & \text{if } d = 3 \text{ and } \frac{8c}{3B_1} > B_1^2, \\ \left(105B_1^2, \frac{24c}{B_1}, 3A_1\right), & \text{if } d = 35,\ 3 \nmid B_1 \text{ and } 105B_1^2 \geq \frac{24c}{B_1}, \\ \left(\frac{24c}{B_1}, 105B_1^2, 3A_1\right), & \text{if } d = 35,\ 3 \nmid B_1 \text{ and } \frac{3c}{B_1} > 105B_1^2, \\ \left(105B_1'^2, \frac{8c}{9B_1'}, A_1\right), & \\ \quad B_1 = 3B_1', & \text{if } d = 35,\ 3 \mid b_1, \text{ and } 105B_1'^2 \geq \frac{8c}{9B_1'}, \\ \left(\frac{8c}{9B_1'}, 105B_1'^2, A_1\right), & \\ \quad B_1 = 3B_1', & \text{if } d = 35,\ 3 \mid b_1, \text{ and } \frac{8c}{9B_1'} > 105B_1'^2. \end{cases} \tag{25}$$

Case (III)

$$(U,V,W) = \begin{cases} (3\Delta, \pm 24, 3u), & \text{if } \alpha \in \{0,1\}, \\ \left(\frac{3\Delta}{4}, \pm 6, \frac{3u}{2}\right), & \text{if } \alpha \geq 2. \end{cases} \tag{26}$$

Case (IV)

$$(U,V,W) = (3\Delta, \pm 3, 3u). \tag{27}$$

One can easily see that each of the above equations satisfies the conditions of Lemma 5. By applying Lemma 5 to equations (24)–(27) we get all non-exceptional solutions to (6) with $n = 3$ (see the table). □

4. Non-exceptional solutions of equation (6)

The solutions of (6) with $\alpha = 0$ are marked with an asterisk.

Table

Δ	y	n	Δ	y	n	Δ	y	n
2^2	5	3	$3^7 7^2$ *	67	3	$2^8 3^7 5^4 7^2$	50401	3
3^4 *	13	3	$3^5 5^2 7^2$ *	79	3	$3^{13} 5^2 7^4$ *	59539	3
7^2 *	65	3	$2^6 3^7 7^2$	193	3	$3^{13} 5^6 7^2$ *	60799	3
$2^4 3^4$	193	3	$2^4 3^7$	73	3	$2^2 3^{13} 5^4 7^6$	93349	3
$3^6 7^2$ *	585	3	$3^7 5^2$ *	91	3	$2^8 3^5 5^6 7^6$	129649	3

Δ	y	n	Δ	y	n	Δ	y	n
$3^6 5^2\ *$	2701	3	$3^7 5^2 7^2\ *$	151	3	$2^{10} 3^7 5^8 7^4$	362401	3
$2^6 3^4 7^2$	37633	3	$2^4 3^5 5^4 7^4$	1801	3	$2^{18} 3^5 5^2 7^6$	1053721	3
2	3	3	$3^5 5^2 7^2\ *$	211	3	$2^8 3^5 5^4 7^8$	5762401	3
$2 \cdot 5^4$	11	3	$2^6 3^5 5^2$	241	3	$3^{17} 5^8 7^2\ *$	19136251	3
$2^3 5^2$	9	3	$2^6 3^7 5^4$	481	3	$5 \cdot 7^2\ *$	9	3
$2 \cdot 3^4 5^2$	19	3	$2^6 3^5 5^2 7^2$	361	3	$2^2 5 \cdot 7^2$	29	3
$2^5 5^2$	41	3	$3^7 5^4 7^2\ *$	499	3	$3^4 \cdot 5\ *$	61	3
$2^7 5^6$	129	3	$2^2 3^5 5^2 7^2$	421	3	$3^4 5 \cdot 7^4\ *$	109	3
$2^3 3^4$	97	3	$2^2 3^9 5^2 7^2$	589	3	$2^2 5 \cdot 7^6$	141	3
$2^3 3^4 5^4$	121	3	$2^2 3^5 5^4 7^2$	541	3	$2^2 5^3 7^2$	669	3
$2 \cdot 3^6 5^2$	211	3	$2^4 3^5 5^4$	601	3	$2^4 3^4 5 \cdot 7^4$	1009	3
$2^5 3^4 5^4$	409	3	$3^9 5^4\ *$	679	3	$2^6 3^4 \cdot 5$	3841	3
$2^9 5^2$	681	3	$2^2 3^5 5^4 7^2$	709	3	$5^5 7^6\ *$	4281	3
$2^3 3^6 5^8$	1489	3	$2^8 3^3 5^4 7^2$	849	3	$2^4 3^6 5 \cdot 7^4$	8689	3
$2 \cdot 3^4 5^6 7^2$	1051	3	$2^2 3^9 7^2$	757	3	$3^6 5^3 7^8\ *$	15901	3
$2 \cdot 3^4 5^2 7^2$	1171	3	$2^8 3^9 5^2$	889	3	$2^8 3^4 5 \cdot 7^8$	17761	3
$2 \cdot 3^8 5^6$	1819	3	$2^4 3^9 5^2 7^2$	1009	3	$3^8 5 \cdot 7^2\ *$	238141	3
$2^9 3^4 5^{12}$	21769	3	$2^8 3^5 5^2 7^2$	1129	3	$2 \cdot 3^3$	7	3
$2^9 3^6 5^2$	6129	3	$2^2 3^7 5^2 7^2$	1261	3	$2^3 3^3 7^2$	25	3
$2^5 3^4 5^2$	9601	3	$2^4 3^7 5^2 7^4$	2041	3	$2 \cdot 3^3 5^2 7^4$	151	3
$2^7 3^6 5^4$	13849	3	$2^2 3^7 5^2 7^4$	2221	3	$2 \cdot 3^5 7^2$	79	3
$2^5 3^4 5^8 7^2$	19441	3	$2^6 3^5 7^4$	2353	3	$2^5 3^3 7^2$	121	3
$2^3 3^6 5^4 7^2$	42361	3	$3^3 5^4 7^4\ *$	2451	3	$2 \cdot 3^3 5^2$	199	3
$2 \cdot 3^6 5^{14} 7^4$	440491	3	$2^8 3^1 17^4$	4993	3	$2 \cdot 3^5 7^6$	415	3
$2^7 3^4 5^2 7^4$	92198401	3	$3^9 5^2 7^4\ *$	2671	3	$2^3 3^5 7^4$	337	3
$3^5\ *$	7	3	$2^{10} 3^5 5^2 7^4$	3361	3	$2^7 3^3 7^2$	505	3
$3^5 5^2\ *$	19	3	$2^4 3^{11} 5^4 7^2$	5161	3	$2 \cdot 3^7 7^2$	655	3
$2^2 3^5$	13	3	$2^{10} 3^7 7^2$	4033	3	$2^5 3^5$	1153	3
$2^2 3^5 7^2$	37	3	$2^6 3^{11} 5^2$	6481	3	$2^9 3^3 7^6$	1705	3
$2^2 3^7 5^2$	61	3	$3^9 5^6 7^4\ *$	12979	3	$2^3 3^5 5^2 7^4$	7249	3
$2^4 3^5 5^2$	49	3	$3^7 5^6 7^2\ *$	15751	3	$2^3 3^3 5^2 7^2$	39201	3
$3^5 5^2\ *$	31	3	$2^{12} 3^5 5^6$	16009	3	$2^7 3^7 7^8$	43873	3
$2^4 3^5 5^2 7^2$	169	3	$2^{12} 3^9 5^2 7^2$	17329	3	$2 \cdot 3^{11} 7^{10}$	69295	3
$3^5 7^2\ *$	43	3	$2^{10} 3^9 5^6 7^2$	27721	3	$2^{15} 3^3 7^2$	131065	3
$2^2 3^5 5^2 7^2$	109	3	$2^{14} 3^{11} 5^4 7^2$	51361	3	$5^2 7\ *$	11	3
$2^2 5^4 7$	29	3	$2^2 3^9 5^4 7^3$	70189	3	$2^{10} 3^4 7^2$	65	4
$2^2 7$	37	3	$2^4 3^{23} 5^2 7^3$	607849	3	$2^{10} 3^2 5^2 7^2$	113	4
$3^4 5^2 7\ *$	79	3	$2^{10} 3^{19} 5^2 7$	723361	3	$2^{12} 3^4 5^4 7^2$	337	4
$5^{10} 7^3\ *$	1499	3	$3^{31} 5^4 7$	4800469	3	$2^{10} 3^2 5^4 7^4$	1201	4
$3^6 5^6 7\ *$	631	3	$2 \cdot 3^7 5$	31	3	2^5	3	4
$2^4 3^4 5^4 7$	1369	3	$2^3 3^7 5$	169	3	$2^7 7^2$	9	4
$2^4 3^4 5^8 7$	1969	3	$2^3 3^{11} 5$	241	3	$2^5 3^2 7^2$	11	4

Δ	y	n
$3^{4}5^{2}7^{3}\ \ast$	4111	3
$2^{6}3^{4}5^{4}7$	5401	3
$2^{10}5^{2}7^{3}$	9569	3
$2^{4}3^{6}7$	12097	3
$2^{2}3^{8}5^{6}7^{3}$	26341	3
$2^{2}5^{8}7^{5}$	89429	3
$2^{4}5^{14}7^{3}$	182441	3
$2^{8}3^{6}5^{12}7$	209161	3
$2^{12}3^{4}5^{4}7^{3}$	16859161	3
$2\cdot 5\cdot 7^{2}$	11	3
$2\cdot 5^{3}7^{2}$	331	3
$2^{3}3^{4}5$	481	3
$2^{3}3^{4}5\cdot 7^{4}$	529	3
$2^{5}3^{4}5\cdot 7^{4}$	1969	3
$2^{3}3^{6}5\cdot 7^{8}$	6721	3
$2^{7}3^{4}5^{3}7^{12}$	309649	3
$2^{9}3^{6}5\cdot 7^{4}$	276529	3
$2^{3}3^{8}5^{3}7^{4}$	972049	3
$2^{3}3^{4}7$	673	3
$2\cdot 3^{10}7^{3}$	122479	3
$2^{3}3^{10}5^{4}7$	306180001	3
$3^{3}5\ \ast$	19	3
$2^{6}3^{7}5$	103681	3
$3^{5}7\ \ast$	25	3
$3^{7}7\ \ast$	37	3
$2^{2}3^{9}7$	85	3
$2^{2}3^{5}7$	109	3
$3^{5}7\ \ast$	253	3
$2^{4}3^{15}7$	1177	3
$2^{4}3^{7}7^{3}$	385	3
$2^{4}3^{7}7$	457	3
$3^{5}5^{2}7^{3}\ \ast$	721	3
$3^{11}5^{2}7\ \ast$	781	3
$2^{6}3^{11}7$	1873	3
$2^{2}3^{5}7^{3}$	5485	3
$2^{8}3^{11}7^{3}$	6601	3
$2^{4}3^{7}5^{2}7$	11209	3
$2^{8}3^{5}7$	64513	3
$2^{5}3^{2}5$	7	4

Δ	y	n
$2^{5}3^{7}5$	649	3
$2\cdot 3^{11}5^{3}$	919	3
$2^{3}3^{5}5$	1441	3
$2^{7}3^{15}5$	3289	3
$2\cdot 3^{7}5^{5}$	24991	3
$2^{5}3^{11}5\cdot 7^{2}$	31441	3
$2^{3}3^{23}5\cdot 7^{2}$	66889	3
$2^{3}3^{15}5^{3}7^{2}$	196729	3
$2^{9}3^{7}5^{3}$	256009	3
$2^{4}3^{4}5\cdot 7$	6721	3
$2\cdot 3^{5}5^{6}7$	379	3
$2\cdot 3^{5}5^{2}7$	499	3
$2^{5}3^{3}5^{4}7^{3}$	721	3
$2^{3}3^{5}5^{4}7$	2041	3
$2^{7}3^{3}5^{8}7$	4209	3
$2^{3}3^{5}5^{12}7$	17641	3
$2\cdot 3^{9}5^{2}7$	40819	3
$2^{11}3^{3}5^{4}7^{3}$	57169	3
$2\cdot 3^{3}5^{6}7^{5}$	134331	3
$2\cdot 3^{7}5^{10}7^{3}$	219139	3
$2^{7}3^{5}5^{2}7$	806401	3
$2^{3}3^{4}5\cdot 7$	3361	3
$2^{4}3^{5}5\cdot 7$	20161	3
$2^{3}3^{7}5\cdot 7$	1129	3
$2^{3}3^{11}5\cdot 7$	1201	3
$2^{5}3^{15}5\cdot 7$	5209	3
$2^{3}3^{23}5^{3}7$	87049	3
$2^{7}3^{7}5\cdot 7$	17929	3
$2^{3}3^{5}5^{3}7$	252001	3
$2^{6}3^{2}$	5	4
$7^{2}\ \ast$	5	4
$2^{6}3^{2}5^{2}7^{2}$	29	4
$2^{6}3^{2}5^{2}$	13	4
$2^{8}3^{2}5^{2}$	17	4
$2^{8}3^{2}7^{2}$	25	4
$2^{6}3^{2}5^{2}7^{2}$	37	4
$2^{6}3^{4}5^{2}7^{2}$	53	4
$2^{8}3^{4}5^{2}$	41	4
$3^{3}5\ \ast$	4	4

Δ	y	n
$2^{7}3^{2}$	17	4
$2^{5}3^{2}5^{2}7^{2}$	43	4
$2^{5}5^{2}7^{4}$	51	4
$2^{9}5^{2}7^{2}$	57	4
$2^{9}3^{4}7^{4}$	113	4
$2^{5}5^{2}7^{2}$	99	4
$2^{5}3$	5	4
$2^{6}3$	7	4
$2^{4}3^{5}5^{2}7^{2}$	49	4
$2^{8}37^{2}$	97	4
$2^{4}3^{3}5^{4}7^{2}$	133	4
$2^{4}5$	3	4
$2^{4}3^{2}5$	7	4
$2^{6}5$	9	4
$5^{3}7^{2}\ \ast$	21	4
$2^{4}3^{2}5\cdot 7^{2}$	47	4
$2^{8}3^{4}5$	161	4
$2^{5}3\cdot 5^{2}$	7	4
$2^{7}3\cdot 5^{2}$	11	4
$2^{5}3^{3}5^{4}$	29	4
$2^{7}3\cdot 5^{2}$	49	4
$2^{9}3^{3}5^{2}$	59	4
$2^{7}3\cdot 5^{4}7^{2}$	73	4
$2^{5}3^{3}5^{2}7^{2}$	103	4
$2^{11}3\cdot 5^{6}$	131	4
$2^{3}3^{7}7^{2}$	175	4
$2^{9}3\cdot 5^{2}7^{4}$	4801	4
$7\ \ast$	2	4
$2^{3}7$	3	4
$5^{2}7\ \ast$	4	4
$2^{6}3^{2}7$	11	4
$5^{4}7^{3}\ \ast$	22	4
$2^{8}3^{4}7$	23	4
$2^{6}3^{6}7$	29	4
$2^{6}3^{2}5^{2}7$	53	4
$2^{10}3^{2}7$	127	4
$2^{6}3^{10}5^{2}7^{3}$	443	4
$2^{12}3^{8}5^{2}7$	431	4
$2^{11}3^{3}5\cdot 7^{2}$	263	4

Δ	y	n	Δ	y	n	Δ	y	n
$2^5 3^4 5$	11	4	$3 \cdot 5^3$ *	8	4	$2^9 3 \cdot 5^3 7^6$	407	4
$2^9 3^6 5$	37	4	$2^6 3 \cdot 5 \cdot 7^2$	17	4	$2^5 3^5 5^3 7^2$	493	4
$2^7 3^2 5$	13	4	$2^8 3 \cdot 5$	31	4	$2^6 3^2 5 \cdot 7$	71	4
$2^5 3^2 5$	19	4	$2^6 3^3 5 \cdot 7^2$	47	4	$2^5 3 \cdot 7$	13	4
$2^7 3^4 5 \cdot 7^2$	89	4	$3^7 5^3$ *	34	4	$2^5 3^7 5^4 7$	8749	4
$2^5 3^6 5^3 7^2$	223	4	$2^4 3 \cdot 7$	5	4	$2^5 3^2 5 \cdot 7$	17	4
$2^5 3^{10} 5 \cdot 7^2$	247	4	$2^4 3 \cdot 5^2 7$	11	4	$2^5 3^4 5 \cdot 7$	19	4
$2^{11} 3^2 5^3$	253	4	$2^4 3^3 5^2 7$	17	4	$2^7 3^6 5 \cdot 7$	43	4
$2^3 3^2 7$	5	4	$2^6 3 \cdot 5^4 7$	31	4	$2^9 3^2 5 \cdot 7$	67	4
$2^5 5^2 7$	9	4	$2^6 3 \cdot 5^2 7$	19	4	$2^5 3^{10} 5^3 7$	257	4
$2^3 3^4 7$	13	4	$2^4 3 \cdot 5^2 7$	23	4	$2^5 3^2 5^3 7$	251	4
$2^7 7$	15	4	$2^8 3 \cdot 5^2 7$	37	4	$2^6 3 \cdot 5 \cdot 7$	11	4
$2^9 5^4 7$	39	4	$2^6 3^3 7$	55	4	$2^8 3 \cdot 5 \cdot 7$	13	4
$2^3 3^2 7^3$	19	4	$2^{10} 3^3 5^6 7$	721	4	$2^{10} 3 \cdot 5 \cdot 7$	19	4
$2^5 3^2 5^2 7$	23	4	$2^4 3^5 5^4 7^7$	293	4	$2^6 3^3 5 \cdot 7$	31	4
$2^3 3^8 7^5$	173	4	$2^{14} 3 \cdot 5^2 7^3$	2053	4	$2^6 3 \cdot 5 \cdot 7$	41	4
$2^5 3^2 5^6 7$	127	4	$2^5 3 \cdot 57^2$	13	4	$2^{14} 3^3 5 \cdot 7$	71	4
$2^7 3^4 5^4 7$	137	4	$2^5 3 \cdot 5$	11	4	$2^{12} 3^3 5^3 \cdot 7$	157	4
$2^9 3^2 5^2 7$	449	4	$2^7 3 \cdot 5 \cdot 7^2$	23	4	$2^{10} 3 \cdot 5^3 7^3$	359	4
$3 \cdot 5$ *	2	4	$2^5 3^3 5 \cdot 7^4$	59	4	$2^{20} 3 \cdot 5 \cdot 7^3$	517	4
						$2^5 3 \cdot 5 \cdot 7$	29	4
						$2 \cdot 5^3$	11	5

Acknowledgements. The author is grateful to Kálmán Győry, Ákos Pintér, Lajos Hajdu and the referee for their help and numerous valuable remarks.

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(Received May 24, 2005; revised January 9, 2006)