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Flow-invariant structures on unit tangent bundles

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Abstract. We study unit tangent bundles T_1M for which the structural operator $h = \frac{1}{2}\mathfrak{L}_\xi\phi$, its characteristic derivative $h' = \nabla_\xi h$ or the characteristic Jacobi operator $\ell = R(\cdot,\xi)\xi$ is invariant under the geodesic flow generated by the characteristic vector field ξ. Also, we prove that the operator ℓ on T_1M is η-parallel if and only if the base manifold is of constant curvature.

1. Introduction

One way to study the geometry of a Riemannian manifold (M,G) is to investigate the interaction of the manifold with its unit tangent sphere bundle T_1M endowed with its standard contact Riemannian structure (η, g, ϕ, ξ). Special properties for the geometry of (M,G) will be reflected in special properties for the contact structure on T_1M and vice versa. In particular, the characteristic vector field ξ on T_1M contains crucial information about M. In fact, all the geodesics in M are controlled by the geodesic flow on T_1M which is precisely given by ξ.

Apart from the defining structure tensors η, g, ϕ and ξ, two other operators play a fundamental role in contact Riemannian geometry, namely the structural operator $h = \frac{1}{2}\mathfrak{L}_\xi\phi$ and the characteristic Jacobi operator $\ell = R(\cdot,\xi)\xi$, where $\mathfrak{L}_\xi$ denotes Lie differentiation in the characteristic direction ξ. An important topic in the study of the contact metric structure on unit tangent bundles has been

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to determine those Riemannian manifolds (M, G) for which the corresponding contact structure on T_1M enjoys symmetry properties related to the geodesic flow.

A first symmetry type for the contact metric structure occurs when the geodesic flow, generated by ξ, leaves some structure tensors invariant. This is always the case for ξ and η since $\mathfrak{L}_\xi\xi = 0$ and $\mathfrak{L}_\xi\eta = 0$. The metric g is left invariant by the flow of ξ (or equivalently, the flow consists of local isometries or ξ is a Killing vector field) if and only if the structural operator h vanishes. By definition, this corresponds precisely to $\mathfrak{L}_\xi\phi = 0$, i.e., also ϕ is preserved under the geodesic flow. Y. Tashiro proved in [14] that this happens for a unit tangent bundle $(T_1M; \eta, g, \phi, \xi)$ if and only if (M, G) has constant curvature $c = 1$. In this paper, we investigate when the operators h, ℓ and $h' = \nabla_\xi h$ on T_1M are preserved by the geodesic flow. Namely, we prove in Section 4:

Theorem 1. *Let T_1M be the unit tangent sphere bundle with the standard contact Riemannian structure (η, g, ϕ, ξ). Then T_1M satisfies $\mathfrak{L}_\xi h = 0$ if and only if (M, G) is of constant curvature $c = 1$.*

Theorem 2. *Let T_1M be the unit tangent sphere bundle with the standard contact Riemannian structure (η, g, ϕ, ξ). Then T_1M satisfies $\mathfrak{L}_\xi\ell = 0$ if and only if (M, G) is of constant curvature $c = 0$ or $c = 1$.*

Theorem 3. *Let T_1M be the unit tangent sphere bundle with the standard contact Riemannian structure (η, g, ϕ, ξ). Then T_1M satisfies $\mathfrak{L}_\xi h' = 0$ if and only if (M, G) is of constant curvature $c = -1$, $c = 0$ or $c = 1$.*

A second type of symmetry occurs when some structure tensors are covariantly parallel along the integral curves of ξ. On a contact metric space, it always holds $\nabla_\xi\xi = \nabla_\xi\eta = \nabla_\xi g = \nabla_\xi\phi = 0$, but the other structure tensors need not be parallel in the ξ-direction. Recently, it was proved that T_1M satisfies the condition $\nabla_\xi h = 0$ or, equivalently, $\nabla_\xi\ell = 0$, if and only if (M, G) is of constant curvature $c = 0$ or $c = 1$ ([12], [13]). This result can easily be verified from the formulas in this paper.

A final symmetry notion on contact Riemannian manifolds is the notion of η-parallelity. We call a $(1, 1)$-tensor T *η-parallel* if $g((\nabla_X T)Y, Z) = 0$ for all vector fields X, Y, Z orthogonal to ξ. In particular, the tensor ϕ is η-parallel if and only if the contact structure is CR-integrable, $(\nabla_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$. On a unit tangent bundle T_1M, this occurs if and only if (M, G) is of constant curvature, as can easily be verified from the formulas further on. Contact metric spaces with η-parallel structural operator h were completely classified by the first two authors in [6]. As a corollary of that result, we obtain also that the standard contact Riemannian structure (η, g, ϕ, ξ) of a unit tangent sphere bundle T_1M has η-parallel h if and only if (M, G) is of constant curvature. Now, we consider the case when the characteristic Jacobi operator ℓ on T_1M is η-parallel. We show in Section 5:

Theorem 4. *Let T_1M be the unit tangent sphere bundle with the standard contact Riemannian structure (η, g, ϕ, ξ). Then the characteristic Jacobi operator ℓ of T_1M is η-parallel if and only if (M, G) is of constant curvature.*

2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. We start by collecting some fundamental material about contact metric geometry. We refer to [2] for further details. A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, the *characteristic vector field,* satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there exists a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi \tag{1}$$

where X and Y are vector fields on M. From (1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

A Riemannian manifold M equipped with structure tensors (η, g, ϕ, ξ) satisfying (1) is said to be a contact Riemannian manifold and is denoted by $M = (M; \eta, g, \phi, \xi)$. Given a contact Riemannian manifold M, we define the *structural operator* h by $h = \frac{1}{2}\mathfrak{L}_\xi \phi$, where $\mathfrak{L}$ denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0 \quad \text{and} \quad h\phi = -\phi h, \tag{3}$$

$$\nabla_X \xi = -\phi X - \phi h X \tag{4}$$

where ∇ is the Levi–Civita connection. From (3) and (4) we see that each trajectory of ξ is a geodesic. We denote by R the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z. Along a trajectory of ξ, the Jacobi operator $\ell = R(\cdot,\xi)\xi$ is a symmetric $(1,1)$-tensor field. We call it the *characteristic Jacobi operator*. We have

$$\ell = \phi\ell\phi - 2(h^2 + \phi^2), \tag{5}$$

$$\nabla_\xi h = \phi - \phi\ell - \phi h^2. \tag{6}$$

A contact Riemannian manifold for which ξ is Killing is called a *K-contact manifold.* It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$ or, equivalently, $\ell = I - \eta \otimes \xi$.

3. The contact metric structure of the unit tangent bundle

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [9], [11], [16]). We only briefly review some notations and definitions. Let $M = (M, G)$ be an n-dimensional Riemannian manifold and let TM denote its tangent bundle with the projection $\pi : TM \to M$, $\pi(x,u) = x$. For a vector $X \in T_xM$, we denote by X^H and X^V, the *horizontal lift* and the *vertical lift*, respectively. Then we can define a Riemannian metric $\tilde{g}$, the *Sasaki metric* on TM, in a natural way by

$$\tilde{g}(X^H, Y^H) = \tilde{g}(X^V, Y^V) = G(X,Y) \circ \pi, \quad \tilde{g}(X^H, Y^V) = 0$$

for all vector fields X and Y on M. Also, a natural almost complex structure tensor J of TM is defined by $JX^H = X^V$ and $JX^V = -X^H$. Then we easily see that $(TM; \tilde{g}, J)$ is an almost Hermitian manifold. We note that J is integrable if and only if (M, G) is locally flat ([9]). Now we consider the unit tangent sphere bundle $(T_1M, \bar{g})$, which is an isometrically embedded hypersurface in $(TM, \tilde{g})$ with unit normal vector field $N = u^V$. For $X \in T_xM$, we define the *tangential lift* of X to $(x,u) \in T_1M$ by

$$X^T_{(x,u)} = X^V_{(x,u)} - G(X,u)N_{(x,u)}.$$

Clearly, the tangent space $T_{(x,u)}T_1M$ is spanned by vectors of the form X^H and X^T where $X \in T_xM$. We put

$$\bar{\xi} = -JN, \quad \bar{\phi} = J - \bar{\eta} \otimes N.$$

Then we find $\bar{g}(X, \bar{\phi}Y) = 2d\bar{\eta}(X,Y)$. By taking $\xi = 2\bar{\xi}$, $\eta = \frac{1}{2}\bar{\eta}$, $\phi = \bar{\phi}$, and $g = \frac{1}{4}\bar{g}$, we get the standard contact Riemannian structure (ϕ, ξ, η, g). Indeed, we easily check that these tensors satisfy (1). Here we notice that ξ determines the geodesic flow. The tensors ξ and ϕ are explicitly given by

$$\xi = 2u^H, \quad \phi X^T = -X^H + \frac{1}{2}\,G(X,u)\xi, \quad \phi X^H = X^T \tag{7}$$

where X and Y are vector fields on M. From now on, we consider $T_1M = (T_1M;\eta,g)$ with the standard contact Riemannian structure. We list the fundamental formulae which we need for the proof of our theorems. They are derived in, e.g., [3], [4], [5], [8], [12], [14]. The Levi–Civita connection ∇ of (T_1M,g) is given by

$$\begin{aligned} \nabla_{X^T}Y^T = -G(Y,u)X^T, \qquad \nabla_{X^T}Y^H &= \frac{1}{2}\,(K(u,X)Y)^H, \\ \nabla_{X^H}Y^T &= (D_XY)^T + \frac{1}{2}\,(K(u,Y)X)^H, \\ \nabla_{X^H}Y^H &= (D_XY)^H - \frac{1}{2}\,(K(X,Y)u)^T. \end{aligned} \tag{8}$$

For the Riemann curvature tensor R, we give only the two expressions we need for the characteristic Jacobi operator ℓ:

$$\begin{aligned} R(X^T,Y^H)Z^H &= -\frac{1}{2}\,\{K(Y,Z)(X-G(X,u)u)\}^T \\ &\quad + \frac{1}{4}\,\{K(Y,K(u,X)Z)u\}^T - \frac{1}{2}\,\{(D_YK)(u,X)Z\}^H, \\ R(X^H,Y^H)Z^H &= (K(X,Y)Z)^H + \frac{1}{2}\,\{K(u,K(X,Y)u)Z\}^H \\ &\quad - \frac{1}{4}\,\{K(u,K(Y,Z)u)X - K(u,K(X,Z)u)Y\}^H \\ &\quad + \frac{1}{2}\,\{(D_ZK)(X,Y)u\}^T \end{aligned} \tag{9}$$

for all vector fields X, Y and Z on M. In the above, we denote by D the Levi–Civita connection and by K the Riemannian curvature tensor associated with G. From (7) and (8), it follows

$$\nabla_{X^T}\xi = -2\phi X^T - (K_uX)^H, \quad \nabla_{X^H}\xi = -(K_uX)^T \tag{10}$$

where $K_u = K(\cdot,u)u$ is the Jacobi operator associated with the unit vector u. From (4) and (10), it follows that

$$hX^T = X^T - (K_uX)^T, \quad hX^H = -X^H + \frac{1}{2}\,G(X,u)\xi + (K_uX)^H. \tag{11}$$

Using the formulae (9), we get

$$\ell X^T = (K_u^2 X)^T + 2(K_u' X)^H, \quad \ell X^H = 4(K_u X)^H - 3(K_u^2 X)^H + 2(K_u' X)^T \tag{12}$$

where $K_u' = (D_u K)(\cdot, u)u$ and $K_u^2 = K(K(\cdot, u)u, u)u$. By using (6), (7) and (9) we obtain

$$\begin{aligned} h' X^T &= -2(K_u X)^H + 2(K_u^2 X)^H - 2(K_u' X)^T, \\ h' X^H &= -2(K_u X)^T + 2(K_u^2 X)^T + 2(K_u' X)^H \end{aligned} \tag{13}$$

where we put $h' = \nabla_\xi h$.

The above formulae (10)–(13) are also found in [5]. Finally, from (8) and (12) we compute

$$\begin{aligned} \ell' X^T &= 4(K_u' K_u X + K_u K_u' X)^T + 4(K_u'' X + K_u^2 X - K_u^3 X)^H, \\ \ell' X^H &= 8(K_u' X - K_u' K_u X - K_u K_u' X)^H + 4(K_u'' X + K_u^2 X - K_u^3 X)^T \end{aligned} \tag{14}$$

where $\ell' = (\nabla_\xi R)(\cdot, \xi)\xi$.

4. Invariance under the geodesic flow

In this section, we prove Theorems 1, 2 and 3. We start with Theorem 1. Suppose that $T_1 M = (T_1 M; \eta, g)$ satisfies

$$\mathfrak{L}_\xi h = 0. \tag{15}$$

The definition of the Lie differential yields

$$\begin{aligned} (\mathfrak{L}_\xi h)X &= \mathfrak{L}_\xi (hX) - h(\mathfrak{L}_\xi X) \\ &= [\xi, hX] - h[\xi, X] = (\nabla_\xi h)X - \nabla_{hX}\xi + h\nabla_X \xi. \end{aligned} \tag{16}$$

Together with (4), we see that the condition (15) is equivalent to

$$h' = 2(h\phi - h^2\phi). \tag{17}$$

Since $h' = \nabla_\xi h$ is a self-adjoint operator, from (17) we see that $h^2 = 0$, which implies that $T_1 M$ is Sasakian and $c = 1$ (cf. [14]).

Next, we prove Theorem 2. From the definition of Lie differentiation, we see that the condition $\mathfrak{L}_\xi \ell = 0$ is equivalent to

$$\ell' = \ell\phi - \phi\ell + \ell\phi h - \phi h\ell. \tag{18}$$

From (18), by using (7), (11), (12) and (14) a straightforward computation yields

$$0 = (2K'_uX - 5K'_uK_uX - 3K_uK'_uX)^H + (2K''_uX + 4K^2_uX - 4K^3_uX)^T,$$

$$0 = (2K'_uX + K'_uK_uX + 3K_uK'_uX)^T + (2K''_uX + 4K_uX - 4K^2_uX)^H$$

for all vector fields X on M.

These equations are equivalent to the conditions

$$0 = 2K'_uX - 5K'_uK_uX - 3K_uK'_uX, \qquad 0 = 2K'_uX + K'_uK_uX + 3K_uK'_uX,$$

$$0 = 2K''_uX + 4K^2_uX - 4K^3_uX, \qquad 0 = 2K''_uX + 4K_uX - 4K^2_uX$$

for all tangent vectors X to M. The first two of these are equivalent to

$$0 = K'_uK_uX + K_uK'_uX, \tag{19}$$

$$0 = K'_uK_uX - K'_uX \tag{20}$$

and the last two imply

$$0 = K^3_uX - 2K^2_uX + K_uX. \tag{21}$$

Now we replace X by K'_uX in (21) and use first (19) and then (20) to compute

$$\begin{aligned} 0 &= K^3_uK'_uX - 2K^2_uK'_uX + K_uK'_uX \\ &= -K'_uK^3_uX - 2K'_uK^2_uX - K'_uK_uX = -6K'_uX. \end{aligned}$$

This implies that (M, G) is a locally symmetric space ([10], [15]). Further, we see from (21) that the eigenvalues of K_u are constant and equal to 0 or 1, i.e., (M, G) is a globally Osserman space (i.e., the eigenvalues of K_u do not depend on the point p and not on the choice of unit vector u at p). However, a locally symmetric globally Osserman space is locally flat or locally isometric to a rank one symmetric space ([1], [8]). Therefore, we conclude that M is a space of constant curvature $c = 0$ or $c = 1$.

Conversely, when (M, G) is of constant curvature c, we find the following explicit expressions for h, ℓ, h' and ℓ' from (11)–(14):

$$\begin{aligned} hX^T &= (1-c)X^T, \quad & hX^H &= (c-1)\Big(X^H - \frac{1}{2}\,G(X,u)\xi\Big), \\ \ell X^T &= c^2X^T, & \ell X^H &= (4c-3c^2)\Big(X^H - \frac{1}{2}\,G(X,u)\xi\Big), \\ h'X^T &= 2(c^2-c)\Big(X^H - \frac{1}{2}\,G(X,u)\xi\Big), \quad & h'X^H &= 2(c^2-c)X^T, \end{aligned}$$

$$\ell' X^T = 4(c^2 - c^3)\Big(X^H - \frac{1}{2}G(X,u)\xi\Big), \quad \ell' X^H = 4(c^2 - c^3)X^T \tag{22}$$

for vector fields X on M. From these, we easily check that T_1M satisfies (18) when $c = 0$ or $c = 1$.

Finally, we prove Theorem 3. Suppose that T_1M satisfies $\mathfrak{L}_\xi h' = 0$. Then we see that the condition is equivalent to

$$\nabla_\xi h' = 2h'\phi - \phi(hh' + h'h). \tag{23}$$

(We use the commutation relation $\phi h' + h'\phi = 0$ here. It follows from the second equation of (3) and by using $\nabla_\xi \phi = 0$.) If we take the skew-symmetric part of (23), we obtain $\phi(hh' + h'h) = 0$, which implies

$$hh' + h'h = 0. \tag{24}$$

So, (23) reduces to

$$\nabla_\xi h' = 2h'\phi.$$

Next, we start from (6), written in the equivalent form $\ell = \phi h' - \phi^2 - h^2$ to compute

$$\ell' = \phi\,\nabla_\xi h' - (hh' + h'h) = 2\phi h'\phi = 2h'. \tag{25}$$

Using the expressions (13) and (14) for h' and ℓ', this condition is equivalent to the system

$$0 = K_u' X + K_u' K_u X + K_u K_u' X, \qquad 0 = K_u' X - 2K_u' K_u X - 2K_u K_u' X,$$

$$0 = K_u'' X + K_u X - K_u^3 X$$

for all tangent vectors X to M. In a similar way as before, we can conclude that the manifold (M, G) must be locally symmetric ($K_u' = 0$) and of constant curvature c equal to -1, 0 or 1 ($K_u^3 - K_u X = 0$).

Conversely, when (M, G) has constant curvature c equal to -1, 0 or 1, we use again the expressions (22) to show that (23) holds. This proves Theorem 3.

5. η-parallel characteristic Jacobi operator on T_1M

If the characteristic Jacobi operator ℓ of a given contact Riemannian manifold satisfies $g((\nabla_{\bar X}\ell)\bar Y, \bar Z) = 0$ for all vector fields $\bar X$, $\bar Y$ and $\bar Z$ orthogonal to ξ, then we say that ℓ is *η-parallel*. In this section, we determine the unit tangent sphere bundles with η-parallel characteristic Jacobi operator by proving Theorem 4.

From (8) and (12), we compute the covariant derivatives of the characteristic Jacobi operator ℓ:

$$\begin{aligned}(\nabla_{X^H}\ell)Y^H = &\Big(4(D_XK)(Y,u)u - 3(D_XK)(K_uY,u)u\\ &- 3K_u((D_XK)(Y,u)u) + K(u,K'_uY)X + K'_u(K(X,Y)u)\Big)^H\\ &+\Big(-2K(X,K_uY)u + \frac{3}{2}K(X,K_u^2Y)u\\ &+2(D^2_{Xu}K)(Y,u)u + \frac{1}{2}K_u^2(K(X,Y)u)\Big)^T,\end{aligned} \tag{26}$$

$$\begin{aligned}(\nabla_{X^H}\ell)Y^T = &\Big(\frac{1}{2}K(u,K_u^2Y)X + 2(D^2_{Xu}K)(Y,u)u\\ &- 2K_u(K(u,Y)X) + \frac{3}{2}K_u^2(K(u,Y)X)\Big)^H\\ &+\Big((D_XK)(K_uY,u)u + K_u((D_XK)(Y,u)u)\\ &- K(X,K'_uY)u - K'_u(K(u,Y)X)\Big)^T,\end{aligned} \tag{27}$$

$$\begin{aligned}(\nabla_{X^T}\ell)Y^H = &\Big(4K(Y,X)u + 4K(Y,u)X + 2K(u,X)K_uY - 3K_u(K(Y,X)u)\\ &- 3K_u(K(Y,u)X) - 3K(K_uY,X)u\\ &- 3K(K_uY,u)X - \frac{3}{2}K(u,X)K_u^2Y\\ &- 2K_u(K(u,X)Y) + \frac{3}{2}K_u^2(K(u,X)Y)\Big)^H\\ &+\Big(2(D_XK)(Y,u)u + 2(D_uK)(Y,X)u\\ &+ 2(D_uK)(Y,u)X - K'_u(K(u,X)Y)\Big)^T\\ &+G(X,u)\Big(12(K_u^2Y)^H - 8(K_uY)^H - 6(K'_uY)^T\Big),\end{aligned} \tag{28}$$

$$\begin{aligned}(\nabla_{X^T}\ell)Y^T = &\Big(2(D_XK)(Y,u)u + 2(D_uK)(Y,X)u\\ &+ 2(D_uK)(Y,u)X + K(u,X)K'_uY\Big)^H\\ &+\Big(K_u(K(Y,X)u) + K_u(K(Y,u)X)\\ &+ K(K_uY,X)u + K(K_uY,u)X\Big)^T\\ &-G(X,u)\Big(4(K_u^2Y)^T + 6(K'_uY)^H\Big)\\ &+G(Y,u)\Big((K_u^2X)^T + 2(K'_uX)^H\Big).\end{aligned} \tag{29}$$

Now, we suppose that the Jacobi operator ℓ of T_1M is η-parallel. Then from (26)–(29), we obtain for tangent vectors X, Y, Z orthogonal to u:

$$\begin{aligned} 0 &= 4G((D_XK)(Y,u)u,Z) - 3G((D_XK)(K_uY,u)u,Z) \\ &\quad - 3G(K_u((D_XK)(Y,u)u),Z) + G(K(u,K'_uY)X,Z) \\ &\quad + G(K'_u(K(X,Y)u),Z), \end{aligned} \tag{30}$$

$$\begin{aligned} 0 &= -4G(K(X,K_uY)u,Z) + 3G(K(X,K_u^2Y)u,Z) \\ &\quad + 4G((D^2_{Xu}K)(Y,u)u,Z) + G(K_u^2(K(X,Y)u),Z), \end{aligned} \tag{31}$$

$$\begin{aligned} 0 &= G(K(u,K_u^2Y)X,Z) + 4G((D^2_{Xu}K)(Y,u)u,Z) \\ &\quad - 4G(K_u(K(u,Y)X),Z) + 3G(K_u^2(K(u,Y)X),Z), \end{aligned} \tag{32}$$

$$\begin{aligned} 0 &= G((D_XK)(K_uY,u)u,Z) + G(K_u((D_XK)(Y,u)u),Z) \\ &\quad - G(K(X,K'_uY)u,Z) - G(K'_u(K(u,Y)X),Z), \end{aligned} \tag{33}$$

$$\begin{aligned} 0 &= 8G(K(Y,X)u,Z) + 8G(K(Y,u)X,Z) \\ &\quad + 4G(K(u,X)K_uY,Z) - 6G(K_u(K(Y,X)u),Z) \\ &\quad - 6G(K_u(K(Y,u)X),Z) - 6G(K(K_uY,X)u,Z) \\ &\quad - 6G(K(K_uY,u)X,Z) - 3G(K(u,X)K_u^2Y,Z) \\ &\quad - 4G(K_u(K(u,X)Y),Z) + 3G(K_u^2(K(u,X)Y),Z), \end{aligned} \tag{34}$$

$$\begin{aligned} 0 &= 2G((D_XK)(Y,u)u,Z) + 2G((D_uK)(Y,X)u,Z) \\ &\quad + 2G((D_uK)(Y,u)X,Z) - G(K'_u(K(u,X)Y),Z), \end{aligned} \tag{35}$$

$$\begin{aligned} 0 &= 2G((D_XK)(Y,u)u,Z) + 2G((D_uK)(Y,X)u,Z) \\ &\quad + 2G((D_uK)(Y,u)X,Z) + G(K(u,X)K'_uY,Z), \end{aligned} \tag{36}$$

$$\begin{aligned} 0 &= G(K_u(K(Y,X)u),Z) + G(K_u(K(Y,u)X),Z) \\ &\quad + G(K(K_uY,X)u,Z) + G(K(K_uY,u)X,Z). \end{aligned} \tag{37}$$

If we multiply (33) by 3 and sum with (30), we get

$$\begin{aligned} 0 &= 4G((D_XK)(Y,u)u,Z) + G(K(u,K'_uY)X,Z) \\ &\quad + G(K'_u(K(X,Y)u),Z) - 3G(K(X,K'_uY)u,Z) \\ &\quad - 3G(K'_u(K(u,Y)X),Z). \end{aligned} \tag{38}$$

By using the first Bianchi indentity, (38) is rewritten as follows.

$$0 = 4G((D_XK)(Y,u)u,Z) + 3G(K(u,X)K'_uY,Z)$$

$$- G(K'_u(K(u,X)Y),Z) + 2G(K(K'_uY,u)X,Z) - 2G(K'_u(K(u,Y)X),Z). \tag{39}$$

If we apply (35) and (36) in (39), we obtain

$$\begin{aligned} 0 = {} & 2G((D_XK)(Y,u)u,Z) + 2G((D_uK)(Y,X)u,Z) \\ & + 4G((D_uK)(Y,u)X,Z) - G(K(K'_uY,u)X,Z) \\ & + 2G((D_YK)(X,u)u,Z) + 2G((D_uK)(X,u)Y,Z). \end{aligned} \tag{40}$$

We suppose that $X = Y = Z$ are orthogonal to u. Then from (40), we find that $(D_XK)(\cdot,X)X = 0$ for all tangent vectors X. From this, we conclude that the base manifold is locally symmetric. In the case when dim $M = 2$, we at once see that M is of constant curvature.

Next, multiplying (37) by 6 and summing with (34), we get

$$\begin{aligned} 0 = {} & 8G(K(Y,X)u,Z) + 8G(K(Y,u)X,Z) \\ & + 4G(K(u,X)K_uY,Z) - 3G(K(u,X)K_u^2Y,Z) \\ & - 4G(K_u(K(u,X)Y),Z) + 3G(K_u^2(K(u,X)Y),Z). \end{aligned} \tag{41}$$

From (31) and (32), and using the fact that the base manifold M is locally symmetric, we obtain

$$\begin{aligned} 0 = {} & 8G(K(Y,X)u,Z) + 8G(K(Y,u)X,Z) \\ & - G(K_u^2(K(Z,Y)u),X) - G(K(u,K_u^2X)Y,Z). \end{aligned} \tag{42}$$

In (42), we put $Y = Z$. Then $G(K(Y,X)Y,u) = 0$ for any orthogonal triple u, X, Y. By Cartan's theorem ([7]), the base manifold (M,G) must have constant curvature if $\dim M \geq 3$. We conclude that M is of constant curvature for all dimensions.

Conversely, we can use the expressions (22) to show that T_1M has η-parallel characteristic Jacobi-operator ℓ when the manifold M is of constant curvature c.

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