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A few remarks related to the four exponentials conjecture

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Abstract. We consider classical results on alternants from a combinatorial point of view. This provides an alternative approach to the analytic one. We illustrate it by studying the following question: does there exist a real number t, which is not an integer, such that both 2^t and 3^t are integers? We explain why the usual approach related to this problem does not lead to a proof.

1. Introduction

The four exponentials conjecture may be stated as follows (see [11] for a complete description).

Conjecture 1. *Let x_1, x_2 (resp. y_1, y_2) be two $\mathbb{Q}$-linearly independant complex numbers. Then at least one of the four numbers $\exp(x_i y_j)$, ($i = 1, 2$, $j = 1, 2$) is transcendental.*

A weaker result is known: the six exponentials theorem. It may be deduced from SCHNEIDER's work [10] and has also been proved by LANG [1], [2] and RAMACHANDRA [7], [8].

Theorem 1. *Let x_1, x_2, x_3 (resp. y_1, y_2) be three (resp. two) $\mathbb{Q}$-linearly independant complex numbers. Then at least one of the six numbers $\exp(x_i y_j)$, ($i = 1, 2, 3$, $j = 1, 2$) is transcendental.*

Specializing these two statements leads to the following ones.

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Conjecture 2. *Let t be a real number such that both 2^t and 3^t are integers. Then t is also an integer.*

Theorem 2. *Let t be a real number such that both 2^t, 3^t and 5^t are integers. Then t is also an integer.*

We shall investigate these questions in this article. More precisely, we shall study an important tool for this matter, the alternants. An alternant is a determinant of the form

$$D(\mathbf{f},\mathbf{x}) = \det\left(f_i(x_j)\right)_{\substack{1\le i\le n\\ 1\le j\le n}},$$

where the f_i's are entire functions and where the x_j's are complex numbers. Most of the classical estimates for alternants rely on analytical lemmas and we shall get other estimates using combinatorial lemmas. We shall explain why the usual approach does not provide a proof of Conjecture 2.

In the next section, we shall give a classical analytic approach to Theorem 2. We shall also introduce the notations and tools that we shall use in this paper. The third section will be devoted to alternants: we shall expand them to get numerous properties, and we shall apply these results to a special alternant, the Pólya alternant. In the last section, we shall focus on examples, namely Theorem 2 and Conjecture 2.

2. Preliminaries

2.1. The classical analytic approach. Let $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{y}=(y_1,\dots,y_n)$ be two n-uples of complex numbers. Define the Pólya alternant:

$$\Delta(\mathbf{x},\mathbf{y}) = \det\left(e^{x_j y_i}\right)_{\substack{1\le i\le n\\ 1\le j\le n}}.$$

With the notations of the introduction, we thus have $f_i(z)=\exp(y_i z)$.

Pólya [6] proved the following lemma by using real analysis.

Lemma 1. *Assume that all the x_j's and all the y_i's are real numbers, with $x_1 < x_2 < \dots < x_n$ and $y_1 < y_2 < \dots < y_n$. Then*

$$\Delta(\mathbf{x},\mathbf{y}) > 0.$$

One also deduces from a Schwarz's lemma an upper bound for an alternant (see Lemma 2.5 p. 37–38 in [11] for details).

Lemma 2. *Let r and R be two real numbers with $0 < r \leq R$, such that the x_j's belong to the disk $|z| \leq r$. Then*

$$\left|\det\left(f_i(x_j)\right)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}\right| \leq \left(\frac{R}{r}\right)^{-\binom{n}{2}} n! \prod_{i=1}^{n} |f_i|_R,$$

where $|f|_R = \sup_{|z|=R} |f(z)|$.

We can deduce from these two lemmas a proof of Theorem 2. Assume that t is an irrational real number such that both 2^t, 3^t and 5^t are integers. Let us take $n = L^6$, where L is a large integer. We choose

$$\mathbf{x} = (s_1 + ts_2)_{0\leq s_1, s_2 \leq L^3-1}$$

and

$$\mathbf{y} = (t_1 \log 2 + t_2 \log 3 + t_3 \log 5)_{0\leq t_1,t_2,t_3\leq L^2-1}.$$

One easily checks that the $e^{x_i y_j}$'s are integers, of the form

$$2^{s_1t_1} 3^{s_1t_2} 5^{s_1t_3} \left(2^t\right)^{s_2t_1} \left(3^t\right)^{s_2t_2} \left(5^t\right)^{s_2t_3}.$$

We want to get lower and upper bounds for Pólya's alternant. By Lemma 1, the alternant $\Delta(\mathbf{x},\mathbf{y})$ is therefore a nonzero integer and we obtain $|\Delta(\mathbf{x},\mathbf{y})| \geq 1$.

It is obvious that $|f_i|_R = e^{y_i R}$. We find that, for $R \geq r = L^3(1+t)$,

$$\left(\frac{R}{r}\right)^{-\binom{n}{2}} n! \prod_{i=1}^{n} |f_i|_R = \exp\left(\binom{L^6}{2}\log(L^3(1+t)/R) + \log((L^6)!) + \sum_{i=1}^{L^6} y_i R\right).$$

Note that

$$\sum_{i=1}^{L^6} |y_i| = \sum_{0\leq t_1,t_2,t_3\leq L^2-1} (t_1\log 2 + t_2 \log 3 + t_3 \log 5) = L^4\binom{L^2}{2}\log 30.$$

We then deduce from Lemma 2 that

$$\log|\Delta(\mathbf{x},\mathbf{y})| \leq \binom{L^6}{2}(3\log L - \log R) + L^4\binom{L^2}{2} R\log 30 + O(L^{12}).$$

We want to minimize the function

$$R \mapsto \binom{L^6}{2}(3\log L - \log R) + L^4\binom{L^2}{2} R\log 30$$

on the interval $[L^3(1+t),+\infty[$. The optimal choice of the parameter R occurs for $R_0=(L^4+L^2+1)/\log 30$, which is allowed for L large enough. In this case we get

$$\log|\Delta(\mathbf{x},\mathbf{y})|\le -\frac{L^{12}}{2}\log L+O(L^{12}),$$

which contradicts the property $|\Delta(\mathbf{x},\mathbf{y})|\ge 1$. Therefore the only real numbers t such that $(2^t,3^t,5^t)\in\mathbb{N}^3$ are rationals. In this case, if $t=p/q$, then $(2^t)^q$ is a power of 2, which implies that 2^t is also a power of 2 and finally t is an integer.

We can try to follow the same approach to attack Conjecture 2. Assume that t is an irrational real number such that both 2^t and 3^t are integers. Take $n=L^2$, and choose

$$\mathbf{x}=(s_1+ts_2)_{0\le s_1,s_2\le L-1}$$

and

$$\mathbf{y}=(t_1\log 2+t_2\log 3)_{0\le t_1,t_2\le L-1}.$$

The alternant $\Delta(\mathbf{x},\mathbf{y})$ is still a nonzero integer. However Schwarz's. Lemma 2 only provides a bound of the size $O(L^4)$, which is too weak to conclude. It even seems impossible to get the precise asymptotic behaviour of $\Delta(\mathbf{x},\mathbf{y})$ by this method. In the last section, we shall give the right order of magnitude of $\Delta(\mathbf{x},\mathbf{y})$: $\log|\Delta(\mathbf{x},\mathbf{y})|\asymp L^4$.

2.2. A few combinatorial objects. Let us consider special alternants. When $f_i(z)=z^{i-1}$, we get the Vandermonde determinant

$$V(\mathbf{x})=\det\left(x_j^{i-1}\right)_{\substack{1\le i\le n\\1\le j\le n}}=\prod_{1\le i<j\le n}(x_j-x_i).$$

More generally, the alternant of polynomials is an antisymmetrical polynomial in the x_i's and is therefore divisible by the Vandermonde determinant.

Let $\mathcal{P}_n$ denote the set of partitions of length at most n:

$$\mathcal{P}_n=\{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{N}^n:0\le\lambda_1\le\cdots\le\lambda_n\}.$$

When $f_i(t)=t^{\lambda_i+i-1}$, with $\lambda\in\mathcal{P}_n$, the Schur function associated to λ is a symmetrical polynomial in the x_j's. It may be defined as the quotient of two alternants:

$$S_\lambda(\mathbf{x})=\frac{1}{V(\mathbf{x})}\det\left(x_j^{\lambda_i+i-1}\right)_{\substack{1\le i\le n\\1\le j\le n}}.$$

There exist other definitions for the Schur functions. The interested reader is referred to Macdonald's book [5] for more details. A fundamental property of the Schur functions is that they are a $\mathbb{Z}$-basis of the ring of symmetric polynomials in n variables with integer coefficients. Since the quotient $D(\mathbf{f},\mathbf{x})/V(\mathbf{x})$ is symmetric in the x_i's, it is a natural problem to find its expansion in the basis of Schur functions. This will be given in the next section.

Let us present an alternative combinatorial definition of special interest. The Ferrers diagram of a partition λ is

$$\{(i,j)\in\mathbb{Z}^2 : 1\le j\le n \text{ et } 1\le i\le \lambda_j\}\ .$$

A semistandard Young tableau T is a Ferrers diagram in which each point has been replaced by an integer in such a way that one obtains nondecreasing sequences in the rows and increasing sequences in the columns, as the coordinates increase. The partition λ is called the shape of T. We put $\mathbf{x}^T=\prod_{1\le i\le n} x_i^{m_i}$, where m_i denotes the number of i's in T. We can now state a deep result.

Theorem 3. *We have*

$$S_\lambda(\mathbf{x})=\sum_T \mathbf{x}^T,$$

where the sum runs over all semistandard Young tableaux of shape λ.

The interested reader may find an approach of Schur functions based on this other definition of Schur functions in Sagan's book [9].

3. Alternants

3.1. Expansions of alternants. The results given in this subsection may also be deduced from results of Laurent [4] on interpolation determinants. For the sake of simplicity, we omit the indices for determinants: the integers i and j run from 1 to n throughout this section. The following lemma describes the expansion of an alternant in the basis of symmetric functions.

Lemma 3. *We have*

$$D(\mathbf{f},\mathbf{x})=V(\mathbf{x})\sum_{\lambda\in\mathcal{P}_n}\left(\prod_{i=1}^{n}\frac{1}{(\lambda_i+i-1)!}\right)\det\left(f_i^{(\lambda_j+j-1)}(0)\right)S_\lambda(\mathbf{x}).$$

Proof. Let us expand each f_i in a Taylor series:

$$D(\mathbf{f},\mathbf{x})=\det\left(\sum_{k=0}^{+\infty}\frac{f_i^{(k)}(0)}{k!}x_j^k\right)=\sum_{k_1,\dots,k_n\in\mathbb{N}^n}\frac{f_1^{(k_1)}(0)\dots f_n^{(k_n)}(0)}{k_1!\dots k_n!}\det\left(x_j^{k_i}\right).$$

If the k_i's are not pairwise distinct, the determinant $\det(x_j^{k_i})$ vanishes. If they are pairwise distinct, there exists an unique partition $\lambda \in \mathcal{P}_n$ and an unique permutation $\sigma \in \mathcal{S}_n$ such that $k_i = \sigma.(\lambda_i + i - 1) = \lambda_{\sigma(i)} + \sigma(i) - 1$. We thus find

$$\begin{aligned} D(\mathbf{f}, \mathbf{x}) &= \sum_{\lambda \in \mathcal{P}_n} \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{i=1}^n \frac{f_i^{(\sigma.(\lambda_i+i-1))}(0)}{(\sigma.(\lambda_i+i-1))!} \right) \det\left(x_j^{\sigma.(\lambda_i+i-1)}\right) \\ &= \sum_{\lambda \in \mathcal{P}_n} \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{i=1}^n \frac{f_i^{(\sigma.(\lambda_i+i-1))}(0)}{(\lambda_i+i-1)!} \right) \epsilon(\sigma) \det\left(x_j^{\lambda_i+i-1}\right) \\ &= \sum_{\lambda \in \mathcal{P}_n} \left(\prod_{i=1}^n \frac{1}{(\lambda_i+i-1)!} \right) \det\left(f_i^{(\lambda_j+j-1)}(0)\right) \det\left(x_j^{\lambda_i+i-1}\right). \end{aligned}$$

The lemma then follows from the first definition of Schur functions. □

We can also consider the case where $x_1 = x_2 = \cdots = x_n = x$.

Lemma 4. *We have*

$$\det\left(f_i^{(j-1)}(x)\right) = \sum_{\lambda \in \mathcal{P}_n} \left(\prod_{i=1}^n \frac{(i-1)!}{(\lambda_i+i-1)!} \right) \det\left(f_i^{(\lambda_j+j-1)}(0)\right) S_\lambda(x, \ldots, x).$$

Proof. Let us take $x_i = x + (i-1)h$, where h tends to zero. We obtain

$$V(\mathbf{x}) = h^{\binom{n}{2}} \prod_{1 \le i < j \le n} (j - i) = h^{\binom{n}{2}} \prod_{i=0}^{n-1} i!.$$

By Taylor's formula and by iterating $j-1$ the difference operator $P(x) \to P\cdot(x+h) - P(x)$, we find the relation

$$\sum_{k=1}^{j} (-1)^{j-k} \binom{j-1}{k-1} f(x_k) = f^{(j-1)}(x) h^{j-1} + O(h^j).$$

This uppertriangular system of relations (with ones on the main diagonal) enables us to write

$$D(\mathbf{f}, \mathbf{x}) = \det\left(\sum_{k=1}^{j} (-1)^{j-k} \binom{j-1}{k-1} f_i(x_k) \right) \sim \det\left(f_i^{(j-1)}(x)\right) h^{\binom{n}{2}},$$

and the lemma follows. □

3.2. General properties of the Pólya alternant. Let us recall the definition of the Pólya alternant:

$$\Delta(\mathbf{x},\mathbf{y}) = \det(e^{x_j y_i})_{\substack{1\le i\le n\\ 1\le j\le n}}.$$

Let $\mathbf{1}$ denote the vector with all coordinates equal to 1.

Lemma 5. *We have* $\Delta(\mathbf{x},\mathbf{y}) = \Delta(\mathbf{y},\mathbf{x})$ *and*

$$\Delta(\mathbf{x}+a\mathbf{1},\mathbf{y}) = e^{a(y_1+\cdots+y_n)}\Delta(\mathbf{x},\mathbf{y}),$$

for any complex number a.

Proof. Since the determinant is invariant under transposition, the symmetry is obvious. The second property is also easy:

$$\begin{aligned}\Delta(\mathbf{x}+a\mathbf{1},\mathbf{y}) &= \det\left(e^{(x_j+a)y_i}\right) = \det\left(e^{x_jy_i+ay_i}\right)\\ &= \left(\prod_{i=1}^n e^{ay_i}\right)\det\left(e^{x_jy_i}\right) = e^{a(y_1+\cdots+y_n)}\Delta(\mathbf{x},\mathbf{y}).\end{aligned}$$ □

The Lemma 3 now becomes (see [3] for a direct proof).

Lemma 6. *We have*

$$\Delta(\mathbf{x},\mathbf{y}) = V(\mathbf{x})V(\mathbf{y})\sum_{\lambda\in\mathcal{P}_n}\left(\prod_{i=1}^n\frac{1}{(\lambda_i+i-1)!}\right)S_\lambda(\mathbf{x})S_\lambda(\mathbf{y}).$$

Proof. In this special case we have $f_i^{(k)}(0) = y_i^k$. We thus get

$$\begin{aligned}\Delta(\mathbf{x},\mathbf{y}) &= V(\mathbf{x})\sum_{\lambda\in\mathcal{P}_n}\left(\prod_{i=1}^n\frac{1}{(\lambda_i+i-1)!}\right)\det\left(y_i^{\lambda_j+j-1}\right)S_\lambda(\mathbf{x})\\ &= V(\mathbf{x})V(\mathbf{y})\sum_{\lambda\in\mathcal{P}_n}\left(\prod_{i=1}^n\frac{1}{(\lambda_i+i-1)!}\right)S_\lambda(\mathbf{x})S_\lambda(\mathbf{y}),\end{aligned}$$

using the definition of Schur functions. □

Let us also give the analog of Lemma 4.

Lemma 7. *We have*

$$e^{x(y_1+\cdots+y_n)} = \sum_{\lambda\in\mathcal{P}_n}\left(\prod_{i=1}^n\frac{(i-1)!}{(\lambda_i+i-1)!}\right)S_\lambda(\mathbf{y})S_\lambda(x,\dots,x).$$

Proof. In this special case we have $f_i^{(k)}(t) = y_i^k e^{y_i t}$. We thus find

$$\det\left(f_i^{(j-1)}(x)\right) = \det\left(y_i^{j-1} e^{y_i x}\right) = e^{x(y_1+\cdots+y_n)} V(\mathbf{y}).$$

We also get

$$\det\left(f_i^{(\lambda_j+j-1)}(0)\right) = \det\left(y_i^{\lambda_j+j-1}\right) = V(\mathbf{y}) S_\lambda(\mathbf{y}).$$

We deduce that

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}_n}\left(\prod_{i=1}^{n}\frac{(i-1)!}{(\lambda_i+i-1)!}\right)\det\left(f_i^{(\lambda_j+j-1)}(0)\right) S_\lambda(x,\ldots,x)\\ = V(\mathbf{y})\sum_{\lambda\in\mathcal{P}_n}\left(\prod_{i=1}^{n}\frac{(i-1)!}{(\lambda_i+i-1)!}\right) S_\lambda(\mathbf{y}) S_\lambda(x,\ldots,x),\end{aligned}$$

and the lemma follows. □

We can also deduce from Lemma 6 an alternative proof of Pólya's Lemma 1 (see [4] for a more general property).

Proof of Lemma 1. By Lemma 5, we may assume that all the x_j's and all the y_i's are positive. By Theorem 3, the sum

$$\sum_{\lambda\in\mathcal{P}_n}\prod_{i=1}^{n}\frac{1}{(\lambda_i+i-1)!} S_\lambda(\mathbf{x}) S_\lambda(\mathbf{y})$$

is also positive. It now follows from Lemma 6 that $\Delta(\mathbf{x},\mathbf{y})$ is nonzero, with the same sign than the product $V(\mathbf{x})V(\mathbf{y})$. □

We can also get estimates.

Lemma 8. *If all the x_j's and all the y_i's are nonnegative real numbers, we have*

$$|\Delta(\mathbf{x},\mathbf{y})| \geq |V(\mathbf{x})V(\mathbf{y})| \frac{1}{0!1!\ldots(n-1)!}.$$

Proof. Since all the x_j's and all the y_i's are nonnegative, all the terms

$$\left(\prod_{i=1}^{n}\frac{1}{(\lambda_i+i-1)!}\right) S_\lambda(\mathbf{x}) S_\lambda(\mathbf{y})$$

are nonnegative by Theorem 3, and we get

$$\begin{aligned}|\Delta(\mathbf{x},\mathbf{y})| &= |V(\mathbf{x})V(\mathbf{y})| \sum_{\lambda\in\mathcal{P}_n}\left(\prod_{i=1}^{n}\frac{1}{(\lambda_i+i-1)!}\right)S_\lambda(\mathbf{x})S_\lambda(\mathbf{y})\\ &\geq |V(\mathbf{x})V(\mathbf{y})|\,\frac{1}{0!1!\dots(n-1)!},\end{aligned}$$

since the sum is bounded below by its first term (the one indexed by the partition with all the parts equal to zero). □

Lemma 9. *If $|x_j|\leq X$ for every j, we have*

$$|\Delta(\mathbf{x},\mathbf{y})| \leq |V(\mathbf{x})V(\mathbf{y})|\,\frac{e^{X(|y_1|+\dots+|y_n|)}}{0!1!\dots(n-1)!}.$$

Proof. By Lemma 6 and Theorem 3, we get

$$|\Delta(\mathbf{x},\mathbf{y})| \leq |V(\mathbf{x})V(\mathbf{y})| \sum_{\lambda\in\mathcal{P}_n}\left(\prod_{i=1}^{n}\frac{1}{(\lambda_i+i-1)!}\right)S_\lambda(X,\dots,X)S_\lambda(|y_1|,\dots,|y_n|).$$

We then use Lemma 7 to get the announced result. □

4. Examples from diophantine approximation

4.1. Theorem 2. We shall need the following lemma in this section.

Lemma 10. *We have*

$$\sum_{i=0}^{n-1}\log(i!) = \binom{n}{2}\log n - \frac{3n^2}{4} + O(n),$$

when n goes to infinity.

Proof. Let us recall Stirling's formula: $\log(i!) = i\log i - i + o(i)$. By summation, we get

$$\begin{aligned}\sum_{i=1}^{n-1} i\log i &= \sum_{i=1}^{n-1}\left(\binom{i+1}{2}-\binom{i}{2}\right)\log i\\ &= \sum_{i=1}^{n-1}\left(\binom{i+1}{2}\log(i+1)-\binom{i}{2}\log i\right) - \sum_{i=1}^{n-1}\left(\binom{i+1}{2}\log\left(1+\frac{1}{i}\right)\right)\\ &= \binom{n}{2}\log n - \sum_{i=1}^{n-1}\frac{i+1}{2} + O(n) = \binom{n}{2}\log n - \frac{n^2}{4} + O(n).\end{aligned}$$

□

As in Subsection 2.1, we assume that t is an irrational real number such that both 2^t, 3^t and 5^t are integers. We choose $n = L^6$ with L large enough,

$$\mathbf{x} = (s_1 + ts_2)_{0 \leq s_1, s_2 \leq L^3 - 1}$$

and

$$\mathbf{y} = (t_1 \log 2 + t_2 \log 3 + t_3 \log 5)_{0 \leq t_1, t_2, t_3 \leq L^2 - 1},$$

so that $0 \leq x_j \leq (1+t)L^3 := X$ and $0 \leq y_i \leq L^2 \log(30) := Y$. We thus have $X(|y_1| + \cdots + |y_n|) = O(L^{11})$. We easily obtain

$$\log |V(\mathbf{x})| \leq \binom{n}{2} \log X = 3\binom{n}{2} \log L + O(n^2) = 3\binom{L^6}{2} \log L + O(L^{12})$$

and

$$\log |V(\mathbf{y})| \leq \binom{n}{2} \log Y = 2\binom{n}{2} \log L + O(n^2) = 2\binom{L^6}{2} \log L + O(L^{12}).$$

By Lemma 10, we get the formulas (see [3] for similar ones)

$$\sum_{i=0}^{L^6 - 1} \log i! = \binom{L^6}{2} \log(L^6) + O(L^{12})$$

and

$$\log |\Delta(\mathbf{x}, \mathbf{y})| \leq -\binom{L^6}{2} \log L + O(L^{12}),$$

by Lemma 9. Note that we find here the same upper bound than in Subsection 2.1.

Therefore $\Delta(\mathbf{x}, \mathbf{y})$ tends to zero when L goes to infinity. Since this alternant is an integer (see Subsection 2.1), this implies it vanishes for L large enough, which contradicts Lemma 8 (or Lemma 1).

4.2. Conjecture 2. In Subsection 2.1, we assumed that t is an irrational real number such that both 2^t and 3^t are integers. Here we can withdraw the hypothesis 3^t integer to prove the following result.

Theorem 4. *Assume that t is an irrational real number such that 2^t is an integer. Take $n = L^2$ and choose*

$$\mathbf{x} = (s_1 + ts_2)_{0 \leq s_1, s_2 \leq L - 1}$$

and

$$\mathbf{y} = (t_1 \log 2 + t_2 \log 3)_{0 \leq t_1, t_2 \leq L - 1}.$$

There exist two constants $C_1(t)$ and $C_2(t)$ such that

$$C_1(t) \le \liminf_{L\to\infty} \frac{\log|\Delta(\mathbf{x},\mathbf{y})|}{L^4} \le \limsup_{L\to\infty} \frac{\log|\Delta(\mathbf{x},\mathbf{y})|}{L^4} \le C_2(t).$$

More precisely, we can choose

$$\begin{aligned} C_1(t) = \frac{3}{4} + \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} &(1-|v_1|)(1-|v_2|) \\ &\times \log\left(|v_1+tv_2|\,|v_1\log 2 + v_2\log 3|\right) \mathrm{d}\,v_1\,\mathrm{d}\,v_2 \end{aligned}$$

and $C_2(t) = C_1(t) + (1+t)\log 6/2$.

Proof. Here we have $0 \le x_j \le (1+t)L = X$ and $0 \le y_i$. We get

$$\sum_{i=1}^{L^2} |y_i| = \binom{L}{2} L\log 6,$$

so that $X(|y_1| + \cdots + |y_n|) \sim L^4(1+t)\log 6/2$.

In order to use Lemmas 8–10, we need to get the precise asymptotic behaviour of $V(\mathbf{x})$ and $V(\mathbf{y})$. We find

$$\begin{aligned} \log|V(\mathbf{x})| - \binom{n}{2}\log L &= \sum_{\substack{0\le s_1,s_2,s_1',s_2'\le L-1\\ s_1+ts_2<s_1'+ts_2'}} \log|(s_1+ts_2)-(s_1'+ts_2')| - \binom{n}{2}\log L \\ &= \frac{1}{2}\sum_{\substack{0\le s_1,s_2,s_1',s_2'\le L-1\\ (s_1,s_2)\ne(s_1',s_2')}} \log\left|\frac{s_1-s_1'}{L} + t\frac{s_2-s_2'}{L}\right| \\ &= \frac{1}{2}\sum_{\substack{-(L-1)\le i,j\le L-1\\ (i,j)\ne(0,0)}} (L-|i|)(L-|j|)\log\left|\frac{i}{L}+t\frac{j}{L}\right| \\ &= \frac{L^4}{2}\int_{-1}^{1}\int_{-1}^{1}(1-|v_1|)(1-|v_2|)\log|v_1+tv_2|\,\mathrm{d}\,v_1\,\mathrm{d}\,v_2 + o(L^4). \end{aligned}$$

We used here a convergence result for Riemann sums. The validity of this relation is not quite obvious and we shall show it in the next subsection. Similarly we also have

$$\begin{aligned} &\log|V(\mathbf{y})| - \binom{n}{2}\log L \\ &\quad = \frac{L^4}{2}\int_{-1}^{1}\int_{-1}^{1}(1-|v_1|)(1-|v_2|)\log|v_1\log 2 + v_2\log 3|\,\mathrm{d}\,v_1\,\mathrm{d}\,v_2 + o(L^4). \end{aligned}$$

We deduce from these last estimates and from Lemma 10 that

$$\log |V(\mathbf{x})V(\mathbf{y})| - \sum_{i=0}^{n-1} \log(i!) \sim C_1(t)L^4.$$

By Lemmas 8–9, we thus get

$$C_1(t) \le \liminf_{L\to\infty} \frac{\log |\Delta(\mathbf{x},\mathbf{y})|}{L^4} \le \limsup_{L\to\infty} \frac{\log |\Delta(\mathbf{x},\mathbf{y})|}{L^4} \le C_2(t),$$

with $C_1(t)$ and $C_2(t)$ as defined above. □

As a matter of fact, one can compute $C_1(t)$, by the formula

$$\int_0^1 \int_0^1 (1-x)(1-y) \log |x+yt| \,\mathrm{d}\,x \,\mathrm{d}\,y$$
$$= \frac{(2t^4+8t^3)\log |1+1/t| + (12t^2+8t+2)\log |t+1| - 2t^3 - 25t^2 - 2t}{48t^2}.$$

Tedious calculations show that $C_1(t)$ is an increasing function of t for $t \ge 1$. The only zero of $C_1(t)$ in this range occurs at $t = 3.278662088\ldots$ and therefore $C_1(t)$ is positive for $2^t \ge 10$.

This explains why the usual approach for the alternant $\Delta(\mathbf{x},\mathbf{y})$ does not allow to show Conjecture 2: the constant $C_1(t)$ may be positive. The property $\Delta(\mathbf{x},\mathbf{y}) \in \mathbb{N}^*$, which follows from the hypothesis $3^t \in \mathbb{N}$, leads to the inequality $\Delta(\mathbf{x},\mathbf{y}) \ge 1$, which is weaker than the general lower bound given in Theorem 4, for $2^t \ge 10$.

When comparing the attempt of proof of Conjecture 2 with the proof of Theorem 2, one notices that the Vandermonde determinants are the main factors (in the right-hand side of the formula in Lemma 6). Here the points defining $\mathbf{x}$ and $\mathbf{y}$ are not dense enough to get very small Vandermonde determinants. It seems impossible to me to get sufficiently small Vandermonde determinants by another choice of points. Another possibility would be to show that $\Delta(\mathbf{x},\mathbf{y})$ is quite large, by exhibiting a large divisor. It is rather easy to find prime factors dividing $\Delta(\mathbf{x},\mathbf{y})$, but it seems very difficult to get a divisor with the correct order of magnitude.

4.3. Convergence of the Riemann sums. The aim of this subsection is to prove the following result.

Lemma 11. *For any real irrational number t such that 2^t is an integer, we have*

$$\sum_{\substack{-(L-1)\le i,j\le L-1\\(i,j)\neq(0,0)}}\left(1-\left|\frac{i}{L}\right|\right)\left(1-\left|\frac{j}{L}\right|\right)\log\left|\frac{i}{L}+t\frac{j}{L}\right|$$
$$=L^2\int_{-1}^{1}\int_{-1}^{1}(1-|v_1|)(1-|v_2|)\log|v_1+tv_2|\,\mathrm{d}\,v_1\,\mathrm{d}\,v_2+O\left(L(\log L)^2\right).$$

Since t is the quotient of the logarithms of two integers ($t=\log a/\log 2$ for some integer a), we have the bounds

$$-(\log L)^2\ll\log|i+jt|\ll\log L, \tag{1}$$

for $1\le|j|\le L$ and $i\in\{-\lceil Lt\rceil-1,\dots,-1,0\}$, by the theory of linear forms of logarithms (see [11], pp. 187–188). Also note that t is greater than 1.

Let us introduce further notations. Put $\varphi(x,y)=(1-|x|)(1-|y|)\log|x+yt|$ and

$$\delta(i,j)=\varphi\left(\frac{i}{L},\frac{j}{L}\right)-L^2\int_{i/L}^{(i+1)/L}\int_{j/L}^{(j+1)/L}\varphi(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y,$$

for $-(L-1)\le i,\,j\le L-1$, when $(0,0)\notin[i/L,(i+1)/L]\times[j/L,(j+1)/L]$. We also define the sets

$$I_j=\{-\lceil(j+1)t\rceil-1,\dots,-\lfloor jt\rfloor-1,-\lfloor jt\rfloor\},$$

for $j\in\mathbb{Z}$. We then have the following properties:

$$\forall i\notin I_j,\ \forall(x,y)\in[i/L,(i+1)/L]\times[j/L,(j+1)/L],$$
$$\begin{cases}L(x+yt)\le i+1+\lceil(j+1)t\rceil\le-1 & \text{if } i<-\lceil(j+1)t\rceil-1,\\ L(x+yt)\ge i+\lfloor jt\rfloor\ge 1 & \text{if } i>-\lfloor jt\rfloor.\end{cases} \tag{2}$$

We also have, for $xy\neq 0$:

$$\frac{\partial\varphi}{\partial x}(x,y)=\pm(1-|y|)\log|x+yt|+\frac{(1-|x|)(1-|y|)}{x+yt}$$

and

$$\frac{\partial\varphi}{\partial y}(x,y)=\pm(1-|x|)\log|x+yt|+\frac{t(1-|x|)(1-|y|)}{x+yt}.$$

From (2), we deduce that, for $i < -\lceil (j+1)t\rceil - 1$:

$$\begin{aligned}|\delta(i,j)| &= L^2 \left| \int_{i/L}^{(i+1)/L} \int_{j/L}^{(j+1)/L} \left(\varphi\left(\frac{i}{L}, \frac{j}{L}\right) - \varphi(x,y)\right) \mathrm{d}\,x \,\mathrm{d}\,y \right| \\ &\le L^2 \int_{i/L}^{(i+1)/L} \int_{j/L}^{(j+1)/L} \left| \varphi\left(\frac{i}{L}, \frac{j}{L}\right) - \varphi(x,y)\right| \mathrm{d}\,x \,\mathrm{d}\,y \\ &\ll L \int_{i/L}^{(i+1)/L} \int_{j/L}^{(j+1)/L} \max_{\substack{x\in[i/L,(i+1)/L] \\ y\in[j/L,(j+1)/L]}} \left(\left|\frac{\partial\varphi}{\partial x}(x,y)\right| + \left|\frac{\partial\varphi}{\partial y}(x,y)\right| \right) \mathrm{d}\,x \,\mathrm{d}\,y \\ &\ll \frac{\log L}{L} - \frac{1}{i+1+\lceil (j+1)t\rceil}.\end{aligned}$$

The same kind of bound is also valid for $i > -\lfloor jt\rfloor$. We thus get by summation:

$$\sum_{\substack{-(L-1)\le i\le L-1 \\ i\notin I_j}} |\delta(i,j)| \ll \log L,$$

for any j, since $\sum_{1\le k\le n} 1/k \ll \log n$ when n goes to infinity. We find a first estimate:

$$\left| \sum_{\substack{-(L-1)\le i,j\le L-1 \\ i\notin I_j}} \delta(i,j) \right| \ll L\log L. \tag{3}$$

For any $j \in \{-(L-1),\dots,L-1\}$, we have $|I_j| \ll 1$. It follows from (1) that

$$\sum_{i\in I_j} \left| \varphi\left(\frac{i}{L}, \frac{j}{L}\right) \right| \ll (\log L)^2.$$

By summation, this gives

$$\sum_{\substack{-(L-1)\le i,j\le L-1 \\ (i,j)\ne(0,0),\ i\in I_j}} \left| \varphi\left(\frac{i}{L}, \frac{j}{L}\right) \right| \ll L(\log L)^2. \tag{4}$$

For any $j \in \{-(L-1),\dots,L-1\}$, we also have

$$\begin{aligned}\left| \sum_{i\in I_j} \int_{i/L}^{(i+1)/L} \int_{j/L}^{(j+1)/L} \varphi(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \right| &\le \int_{(-\lceil (j+1)t\rceil -1)/L}^{(-\lfloor jt\rfloor +1)/L} \int_{j/L}^{(j+1)/L} |\varphi(x,y)|\,\mathrm{d}\,x\,\mathrm{d}\,y \\ &\le \int_{j/L}^{(j+1)/L} \left(\int_{-yt-(t+2)/L}^{-yt+(t+2)/L} \big|\log|x+yt|\big|\,\mathrm{d}\,x \right) \mathrm{d}\,y\end{aligned}$$

$$\ll \int_{j/L}^{(j+1)/L} \frac{\log L}{L} \, \mathrm{d}\, y \ll \frac{\log L}{L^2},$$

from which we deduce

$$\sum_{\substack{-(L-1)\le i,j\le L-1\\ (i,j)\neq(0,0),\, i\in I_j}} \left| L^2 \int_{i/L}^{(i+1)/L} \int_{j/L}^{(j+1)/L} \varphi(x,y) \, \mathrm{d}\, x \, \mathrm{d}\, y \right| \ll L \log L. \tag{5}$$

Since

$$\sum_{\substack{-(L-1)\le i,j\le L-1\\ (i,j)\neq(0,0)}} \delta(i,j) = \sum_{\substack{-(L-1)\le i,j\le L-1\\ (i,j)\neq(0,0)}} \left(1 - \left|\frac{i}{L}\right|\right) \left(1 - \left|\frac{j}{L}\right|\right) \log \left|\frac{i}{L} + t\frac{j}{L}\right|$$
$$- L^2 \int_{-1}^{1} \int_{-1}^{1} (1 - |v_1|)(1 - |v_2|) \log |v_1 + t v_2| \, \mathrm{d}\, v_1 \, \mathrm{d}\, v_2,$$

Lemma 11 then follows from (3–5).

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