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Lattice-valued positive vector measures with given marginals

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Abstract. Suppose E is a Dedekind complete vector lattice, X_1 and X_2 are Hausdorff completely regular spaces, and $M^+_{(o,t)}(X_1, E)$, $M^+_{(o,t)}(X_2, E)$, $M^+_{(o,t)}(X_1 \times X_2, E)$ are E-valued tight measues on X_1, X_2, and $X_1 \times X_2$ respectively, in the context of order convergence. Some Strassen type theorems are proved about these measures. Similar results are proved about τ-smooth and Baire measures.

1. Introduction and notation

The celebrated Strassen theorem ([8]) has been the subject of investigations by many authors ([3], [2], [4], [1]). In [1] the authors have considered the measures spaces of positive lattice-valued measures when the countable additivity is defined in terms of order convergence. They have established conditions for the existence of positive lattice-valued measures having given marginals. In this paper we deal with the similar matter when the measures are considered on completely regular Hausdorff spaces.

All vector spaces are taken over reals. E, in this paper, is always assumed to be Dedekind complete vector lattice. For a completely regular Hausdorff space X, $\mathcal{B}(X)$ and $\mathcal{B}_1(X)$ are the classes of Borel and Baire subsets of X, $C(X)$ (resp. $C_b(X)$) is the spaces of all real-valued, (resp. real-valued and bounded) continuous functions on X and $X^\sim$ is the Stone–Cech compactification of X. For an $f \in C_b(X)$, $f^\sim$ is its unique continuous extension to $X^\sim$.

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Let G be a Dedekind complete vector lattice. G is said to be weakly σ-distributive ([13]) if whenever $\{v_{i,j} : i = 1, 2, \dots ,\ j = 1, 2, \dots\}$ is an order bounded subset of G with $v_{i,j+1} \le v_{i,j}$ for each i and for each j then

$$\bigvee_{i=1}^{\infty} \bigwedge_{j=1}^{\infty} v_{i,j} = \bigwedge_{\phi \in N^N} \bigvee_{i=1}^{\infty} v_{i,\phi(i)}.$$

G is said to be weakly (σ, ∞)-distributive if for any infinite set L and the set N of natural numbers, we have, for any order bounded subset $\{v_{n,\lambda} : n \in N,\ \lambda \in L\}$,

$$\bigvee_{n \in N} \bigwedge_{\lambda \in L} v_{n,\lambda} = \bigwedge \Big\{ \bigvee_{n \in N} v_{n,\varphi(n)} : \varphi \in L^N \Big\}.$$

(The definition of ordinary weakly (σ, ∞)-distributivity is given in ([1]); the two are the same in case G is Dedekind complete.)

For a compact Hausdorff space X, let $\mu : \mathcal{B}(X) \to G^+$ be a countably additive (conutable additivity in the order convergence of G) Borel measure; then μ is said to be quasi-regular if for any open $V \subset X$, $\mu(V) = \sup\{\mu(C) : C \text{ compact},\ C \subset V\}$. Integration with respect to these measures is taken in the sense of ([9], [12]). There is 1-1 correspondence between these quasi-regular, positive, G-valued, Borel measures on X and positive linear mappings $\mu : C(X) \to G$ ([9], [11], [5]); $M_o^+(X, G)$ will denote the set of all these measures.

Now suppose that X is a completely regular Hausdorff space. A positive countably additive Borel measure $\mu : \mathcal{B}(X) \to G^+$ is said to be tight if for any open $V \subset X$, $\mu(V) = \sup\{\mu(C) : C \text{ compact},\ C \subset V\}$ ([6], p. 207). This measure gives a positive linear mapping $\mu^\sim : C(X^\sim) \to G$, $\mu^\sim(f) = \mu(f_{|X})$; $M^+_{(o,t)}(X, G)$ will denote the set of all these tight measures. If $\mu : \mathcal{B}(X) \to G^+$ is a countably additive Borel measure, then μ is said to be τ-smooth if for any increasing net $\{U_\alpha\}$ of open subsets of X $\mu(\cup U_\alpha) = \sup \mu(U_\alpha)$ (some properties of these measures are given in [6], p. 207). Any such measure gives a positive linear mapping $\mu^\sim : C(X^\sim) \to G$, $\mu^\sim(f) = \mu(f_{|X})$; $M^+_{(o,\tau)}(X, G)$ will denote the set of all these τ-smooth measures. If $\mu : \mathcal{B}_1(X) \to G^+$ is a countably additive Baire measure then, as in the case of τ-smooth measure, we get $\mu^\sim : C(X^\sim) \to G$, $\mu^\sim(f) = \mu(f_{|X})$; $M^+_{(o,\sigma)}(X, G)$ will denote the set of all these Baire measures.

For $i = 1, 2$, let X_i be compact Hausdorff spaces and $\lambda \in M_o^+(X_1 \times X_2, E)$. For $i = 1, 2$. We get $\lambda^{(i)} \in M_o^+(X_i, E)$, defined by $\lambda^{(1)}(B) = \lambda(B \times X_2)$ and $\lambda^{(2)}(B) = \lambda(X_1 \times B))$ for the respective Borel sets B. This is the same as $\lambda^{(1)}(f_1) = \lambda(f_1 \otimes 1)$ and $\lambda^{(2)}(f_2) = \lambda(1 \otimes f_2)$ for $f_i \in C(X_i)$. $\lambda^{(1)}$ and $\lambda^{(2)}$ will be called the marginals of λ. If X_i are completely regular Hausdorff spaces then, for a $\lambda \in M_o^+((X_1 \times X_2)^\sim, E)$, the marginals $\lambda^{(i)} \in M_o^+(X_i^\sim, E)$, are defined by $\lambda^{(1)}(f_1) = \lambda(f_1 \otimes 1)$ and $\lambda^{(2)}(f_2) = \lambda(1 \otimes f_2)$ for $f_i \in C(X_i^\sim)$.

For completely regular Hausdorff spaces X_1, X_2, let $H = \{fg : f \in C_b(X_1), g \in C_b(X_2)\}$; the closed subspace generated by H in $(C_b(X_1 \times X_2), \|\,.\,\|)$ is denoted by H_s. It is easily verified that $H_s = (C(X_1^\sim \times X_2^\sim))_{|(X_1 \times X_2)}$. The zero-set of H_s are the closed subsets of $(X_1 \times X_2)$ of the form $\{f^{-1}(0) : f \in H_s\}$. This result about H will be used in the paper.

2. Main results

We will need the following theorem which follows from the known results.

Theorem 1. *For a completely regular Hausdorff space X, let $\phi : C_b(X) \to E$ be a positive linear map. Let ν be the quasi-regular Borel measure on $X^\sim$ given by $\nu(f) = \phi(f_{|X})$, for each $f \in C(X^\sim)$.*

(a) *ϕ is representable by a unique, positive, Borel, E-valued, tight measure on X if and only if $\phi(1) = \vee\{\nu(C) : C$ compact and $C \subset X\}$.*

(b) *If E is weakly (σ,∞)-distributive and $\nu(C) = 0$ for any compact $C \subset X^\sim \setminus X$, then ϕ is representable by a unique, positive, Borel, E-valued, τ-smooth measure on X.*

(c) *If If E is weakly σ-distributive and $\nu(C) = 0$ for any compact G_δ set $C \subset X^\sim \setminus X$, then ϕ is representable by a unique, positive, E-valued, Baire measure on X.*

Proof. (a) This is proved in ([6], Proposition 10, p. 210). In this case we will say that $\nu \in M^+_{(o,t)}(X, E)$.

(b) This follows from ([13], Lemma 2.1) and ([6], Corollary 7, p. 208). In this case we will say that $\nu \in M^+_{(o,\tau)}(X, E)$.

(c) This follows from([11], Theorem N) and ([6], Corollary 3, p. 206). In this case we will say that $\nu \in M^+_{(o,\sigma)}(X, E)$. □

Theorem 2. *Let X_1 and X_2 be Hausdorff completely regular spaces. For $i = 1, 2$, let $\mu_i \in M_o^+(X_i^\sim, E)$ be such that $\mu_i(X_i^\sim) = v \in E$; also take a $\gamma \in E$, $0 < \gamma \le v$. Put $W = (X_1 \times X_2)^\sim$ or $W = X_1^\sim \times X_2^\sim$ (the theorem will work for both) and take a non-empty closed subset S of W. Then there exists a $\lambda \in M_o^+(W, E)$ such that $\lambda(S) \ge \gamma$ and $\lambda^{(i)} = \mu_i$, $i = 1, 2$, if and only if for any $f_i \in C_b(X_i^\sim)$, $f_i \ge 0$ $(i = 1, 2)$, and $f_1(x_1) + f_2(x_2) \ge 1$ on S, we have $\mu_1(f_1) + \mu_2(f_2) \ge \gamma$.*

Proof. The condition is trivially necessary.

Let $F = \{f \in C(W) : f = f_1 + f_2,\ f_i \in C(X_i^\sim),\ i = 1, 2\}$ (note that $C(X_1^\sim \times X_2^\sim)$ can be considered a subspace of $C(W)$. F is a majorizing ([7], p. 47) subspace of $C(W)$. Define $T_0 : F \to E$, $T_0(f_1 + f_2) = \mu_1(f_1) + \mu_2(f_2)$. T_0 is a well-defined positive linear operator on F. Define $\theta : C(W) \to E$, $\theta(f) = \inf\{T_0(g); g \in F,\ g \geq f\}$. It is easily verified that θ is monotone and sublinear and $\theta(f) = T_0(f)$, $\forall f \in F$ ([7], p. 47, Corollary 1.5.9). Let $K = \{f \in C(W)$, $f \geq 0,\ f_{|S} \geq 1\}$. K is convex. Define $\tau : K \to E$, $\tau(f) = \gamma$, $\forall f \in K$. It is a obvious that τ is concave and $\tau(f) \leq \theta(f)$ for every $f \in K$. As done in ([7], Lemma 1.51, p. 44), define $\rho : C(W) \to E$, $\rho(f) = \inf\{\theta(f + tk) - t\tau(k) : t \in [0, \infty),\ k \in K\}$. By ([7], Lemma 1.51, p. 44), ρ is sublinear and $\rho \leq \theta$. We claim that $T_0 \leq \rho$ on F: fix an $f \in F$ and take a $k \in K$ and a $t \in (0, \infty)$. For any $g \in F$ with $g \geq f + tk$ we have $\frac{g-f}{t} \geq k$ and so $T_0(\frac{g-f}{t}) \geq \gamma$. This means $T_0(g) - t\tau(k) \geq T_0(f)$, $\forall t \in [0, \infty)$. This proves the claim.

So the mapping $T_0 : F \to E$ satisfies the condition $T_0 \leq \rho$. By ([7], Theorem 2.5.4, p. 45), it can be extended to a linear mappling $\lambda : C(W) \to E$ such that $\lambda \leq \rho$. By ([7] Lemma 1.51., p. 44), this implies that $\lambda \leq \theta$ and, on K, $\lambda \geq \tau$. Now we will prove that λ is positive. Take an $f \leq 0$. Now $\lambda(f) \leq \theta(f) \leq \theta(0) = 0$ (note that θ is monotone). This proves that λ is positive. Thus λ is an E-valued quasi-regular Borel measure on the compact Hausdorff space W; it is easy to see that its marginals are μ_i $(i = 1, 2)$. To prove $\lambda(S) \geq \gamma$, note $\lambda \geq \tau$ on K.

Now we prove the existence of measures with given marginals. First we consider tight measures.

Theorem 3. *Suppose X_1 and X_2 are Hausdorff completely regular spaces and $\lambda \in M_o^+((X_1 \times X_2)^\sim, E)$.*

(a) *If the marginals of λ are in $M^+_{(o,t)}(X_i, E)$ $(i = 1, 2)$, then $\lambda \in M^+_{(o,t)}(X_1 \times X_2, E)$.*

(b) *Suppose, for $i = 1, 2$, $\mu_i \in M^+_{(o,t)}(X_i, E)$ and $\mu_i(X_i) = e \in E$. Take a $\gamma \in E$, $0 < \gamma \leq e$ and a non-void closed subset $S \subset (X_1 \times X_2)$. There exists a $\lambda \in M^+_{(o,t)}(X_1 \times X_2, E)$ with marginals μ_1 and μ_2 and $\lambda(S) \geq \gamma$ if and only if for any $f_i \in C_b(X_i^\sim)$, $f_i \geq 0$ $(i = 1, 2)$, and $f_1(x_1) + f_2(x_2) \geq 1$ on S, we have $\mu_1(f_1) + \mu_2(f_2) \geq \gamma$.*

Proof. (a) For $i = 1, 2$ let $\mu_i \in M^+_{(o,t)}(X_i, E)$ be the marginals of λ. Let $\phi : (X_1 \times X_2)^\sim \to (X_1^\sim \times X_2^\sim)$ be the extension of the identity mapping $X_1 \times X_2 \to (X_1^\sim \times X_2^\sim)$. Because of this, $C(X_1^\sim \times X_2^\sim)$ can be considered a subspace of $C((X_1 \times X_2)^\sim)$. For $i = 1, 2$ take an increasing net $\{C^i_\alpha\}$ of compact subsets of X_i such that $\mu_i(X_i \setminus C^i_\alpha) \leq u_\alpha$ with $u_\alpha \downarrow 0$. Fix α and take $f_i \in C_b(X_i)$, $0 \leq f_i \leq 1$,

$f_i \geq \chi_{C^i_\alpha}$. From $1-f_1f_2 = 1-f_1+f_1(1-f_2)$, we have $1-f_1^\sim f_2^\sim \leq (1-f_1^\sim)+(1-f_2^\sim)$ on $(X_1\times X_2)^\sim$. This means $\lambda(1-f_1^\sim f_2^\sim) \leq \mu_1(X_1\setminus C^1_\alpha)+\mu_2(X_2\setminus C^2_\alpha) \leq u_\alpha+u_\alpha$. Because of the regularity of λ, taking limits over $f_i^\sim$ as they decrease to $\chi_{C^i_\alpha}$, we get $\lambda((X_1\times X_2)^\sim \setminus C^1_\alpha \times C^2_\alpha) \leq u_\alpha+u_\alpha$. Taking the order-limit over α and using Theorem 1(a), we prove that $\lambda \in M^+_{o,t}(X_1\times X_2, E)$.

(b) The condition is trivially necessary. Now, for $i = 1, 2$, $\mu_i^\sim \in M^+_o(X_i^\sim, E)$. Let $\bar{S}$ be the closure of S in $(X_1 \times X_2)^\sim$. Using the given hypothesis and Theorem 2, we get a $\lambda^\sim \in M^+_o((X_1\times X_2)^\sim, E)$ such that $\lambda^\sim(\bar{S}) \geq \gamma$ and its marginals are $\mu_i^\sim$ $(i = 1, 2)$. By (a) $\lambda^\sim$ arises from a $\lambda \in M^+_{(o,t)}(X_1 \times X_2, E)$. Take a net $\{f_\alpha\} \subset C_b(X_1 \times X_2)$, $f_\alpha \downarrow \chi_S$; this means $f_\alpha^\sim \downarrow$ and $\lim f_\alpha^\sim \geq \chi_{\bar{S}}$. Now $\lambda(S) = \lim \lambda(f_\alpha) = \lambda^\sim(f_\alpha^\sim) \geq \lambda^\sim(\bar{S}) \geq \gamma$. This proves the result. □

Now we consider τ-smooth measures.

Theorem 4. *Suppose X_1 and X_2 are Hausdorff completely regular spaces, E is weakly (σ,∞)-distributive and $\lambda \in M^+_o((X_1 \times X_2)^\sim, E)$.*

(a) *If $\lambda^{(1)} \in M^+_{(o,t)}(X_1, E)$ and $\lambda^{(2)} \in M^+_{(o,\tau)}(X_2, E)$ then $\lambda \in M^+_{(o,\tau)}(X_1 \times X_2, E)$.*

(b) *Suppose, $\mu_1 \in M^+_{(o,t)}(X_i, E)$, $\mu_2 \in M^+_{(o,\tau)}(X_i, E)$ and, for $i = 1, 2$, $\mu_i(X_i) = e \in E$. Take a $\gamma \in E$, $0 < \gamma \leq e$ and a non-void closed subset $S \subset (X_1 \times X_2)$. There exists a $\lambda \in M^+_{(o,\tau)}(X_1 \times X_2, E)$ with marginals μ_1 and μ_2 and $\lambda(S) \geq \gamma$ if and only if for any $f_i \in C_b(X_i)$, $f_i \geq 0$ $(i = 1, 2)$, and $f_1(x_1) + f_2(x_2) \geq 1$ on S, we have $\mu_1(f_1) + \mu_2(f_2) \geq \gamma$.*

Proof. Proof. (a) $Q = C(X_1^\sim \times X_2^\sim)$ can be considered a subspace of $C((X_1 \times X_2)^\sim)$. Taking $\nu = \lambda_{|Q}$, we get $\nu \in M^+_o(X_1^\sim \times \tilde{X}_2, E)$. Now, for $i = 1, 2$, $\lambda^{(i)} \in M^+_o(X_i^\sim)$ and we have $\lambda^{(1)}(f) = \nu(f \otimes 1)$, $f \in C(X_1^\sim)$, and $\lambda^{(2)}(f) = \nu(1 \otimes f)$, $f \in C(X_2^\sim)$. This means for any compact B_i in $X_i^\sim$, $i = 1, 2$, $\nu(B_1 \times X_2^\sim) = \lambda^{(1)}(B_1 \cap X_1)$ and $\nu(X_1^\sim \times B_2) = \lambda^{(2)}(B_2 \cap X_2)$. Take an increasing net $\{C_\alpha\}$ of compact subsets of X_1 such that $\nu((X_1^\sim \setminus C_\alpha) \times X_2^\sim) \downarrow 0$ (here we are very much using that $\lambda^{(1)} \in M^+_{(o,t)}(X_1, E)$). First we prove that for any compact K of $(X_1^\sim \times X_2^\sim)$, $K \subset (X_1^\sim \times (\tilde{X}_2 \setminus X_2))$, $\nu(K) = 0$. Let $\psi_1 : X_1^\sim \times X_2^\sim \to X_1^\sim$ and $\psi_2 : X_1^\sim \times X_2^\sim \to X_2^\sim$ be the canonical mappings; they are continuous. $K_i = \psi_i(K)$ are compact subsets of $X_i^\sim$, $i = 1, 2$, $K \subset K_1 \times K_2$ and $K_2 \subset (\tilde{X}_2 \setminus X_2)$. Since μ_2 is τ-smooth, $\nu(X_1^\sim \times K_2) = 0$. This means $\nu(K_1 \times K_2) = 0$ and so $\nu(K) = 0$, proving the result. Now take any compact $K \subset (X_1^\sim \times \tilde{X}_2) \setminus (X_1 \times X_2)$. This means $K \cap (C_\alpha \times X_2^\sim) \subset (X_1^\sim \times (\tilde{X}_2 \setminus X_2))$, for all α and so $\nu(K \cap (C_\alpha \times X_2^\sim)) = 0$, for all α. Now $\nu(K) = \nu(K \cap (C_\alpha \times$

$\tilde{X_2})) + \nu(K \cap ((\tilde{X_1} \setminus C_\alpha) \times \tilde{X_2})) \leq \nu((\tilde{X_1} \setminus C_\alpha) \times \tilde{X_2}) = \lambda^{(1)}(X_1^\sim \setminus C_\alpha)$. Taking limit over α and using the tightness property of μ_1, we get $\nu(K) = 0$.

Let $X = (X_1 \times X_2)^\sim$. This means $\lambda \in M_o^+(X, E)$. Let $\phi : X \to (X_1^\sim \times \tilde{X_2})$ be the unique continuous extension of the identity mapping $(X_1 \times X_2) \to (X_1^\sim \times X_2^\sim)$; ϕ maps $X \setminus (X_1 \times X_2)$ onto $(X_1^\sim \times X_2^\sim) \setminus (X_1 \times X_2)$. It is easily verified that for any $f \in C(X_1^\sim \times X_2^\sim)$, $\nu(f) = \lambda(f \circ \phi)$. By regularity, we get $\nu(K) = \lambda(\phi^{-1}(K))$, for any compact $K \subset (X_1^\sim \times X_2^\sim)$. Take a compact $C \subset X \setminus (X_1 \times X_2)$. Then $C_1 = \phi^{-1}(\phi(C))$ is compact and contains C, and $\phi(C)$ is disjoint from $(X_1 \times X_2)$. Now $\lambda(C) \leq \lambda(C_1) = \nu(\phi(C)) = 0$. By Theorem 1(b), $\lambda \in M^+_{(o,\tau)}(X_1 \times X_2)$. This proves the result.

(b) The condition is trivially necessary. Now, for $i = 1, 2$, define $\mu_i^\sim(f) = \mu_i(f_{|X_i})$, $\forall f \in C(X_i^\sim)$; $\mu_i^\sim \in M_o^+(X_i^\sim, E)$. Let $\bar{S}$ be the closure of S in $(X_1 \times X_2)^\sim$. Using the given hypothesis and Theorem 2, we get a $\lambda^\sim \in M_o^+((X_1 \times X_2)^\sim, E)$ such that $\lambda^\sim(\bar{S}) \geq \gamma$ and its marginals are $\mu_i^\sim$ $(i = 1, 2)$. By (a) $\lambda^\sim$ arises from a $\lambda \in M^+_{(o,\tau)}(X_1 \times X_2, E)$. Take a net $\{f_\alpha\} \subset C_b(X_1 \times X_2)$, $f_\alpha \downarrow \chi_S$; this means $f\alpha^\sim \downarrow$. Proceeding as in part (b) Theorem 3, we get $\lambda(S) \geq \gamma$. This proves the result. $\square$

Before the next theorem we introduce some new notations. Let Baire(H) be the smallest σ-algebra in $X_1 \times X_2$ relative to which all functions in H are measurable. It is a simple verification that Baire(H) = Baire(H_s). Also
Baire(H) = (σ-algebra of Baire subsets of $\tilde{X_1} \times \tilde{X_2}) \cap (X_1 \times X_2)$.
Baire$(H) \supset \{B_1 \times B_2 : B_1$ a Baire set in X_1, B_2 a Baire set in $X_2\}$.

Theorem 5. *Suppose X_1 and X_2 are Hausdorff completely regular spaces, E is weakly σ-distributive and $\lambda \in M_o^+(X_1^\sim \times X_2^\sim, E)$.*

(a) *If $\lambda^{(1)} \in M^+_{(o,t)}(X_1, E)$ and $\lambda^{(2)} \in M^+_{(o,\sigma)}(X_2, E)$ then λ can be considered as $\lambda :$ Baire$(H) \to E$ and is countably additive.*

(b) *Suppose, $\mu_1 \in M^+_{(o,t)}(X_i, E)$, $\mu_2 \in M^+_{(o,\sigma)}(X_i, E)$ and, for $i = 1, 2$, $\mu_i(X_i) = e \in E$. Take a $\gamma \in E$, $0 < \gamma \leq e$ and a non-void H_s-zero-set $S \subset (X_1 \times X_2)$. There exists a a countably additive $\lambda :$ Baire$(H) \to E$ with marginals μ_1 and μ_2 and $\lambda(S) \geq \gamma$ if and only if for any $f_i \in C_b(X_i)$, $f_i \geq 0$ $(i = 1, 2)$, and $f_1(x_1) + f_2(x_2) \geq 1$ on S, we have $\mu_1(f_1) + \mu_2(f_2) \geq \gamma$.*

Proof. (a) We have $\lambda^{(1)}(f) = \lambda(f \otimes 1)$, $f \in C(X_1^\sim)$, and $\lambda^{(2)}(f) = \lambda(1 \otimes f)$, $f \in C(X_2^\sim)$. This means for any compact B_i in $X_i^\sim$, $i = 1, 2$, $\lambda(B_1 \times X_2^\sim) = \lambda^{(1)}(B_1 \cap X_1)$ and $\lambda(X_1^\sim \times B_2) = \lambda^{(2)}(B_2 \cap X_2)$. Take an increasing net $\{C_\alpha\}$ of compact subsets of X_1 such that $\lambda((X_1^\sim \setminus C_\alpha) \times X_2^\sim) \downarrow 0$ (here we are very much using that $\lambda^{(1)} \in M^+_{(o,t)}(X_1, E)$). First we prove that for any compact G_δ-set $Z \subset$

$(X_1^\sim \times (\tilde{X}_2 \setminus X_2))$, $\lambda(Z) = 0$. Let $\psi_1 : X_1^\sim \times X_2^\sim \to X_1^\sim$ and $\psi_2 : X_1^\sim \times X_2^\sim \to X_2^\sim$ be the canonical mappings; they are continuous and open. This means $\psi_2(Z)$ is a compact G_δ subset of $(\tilde{X}_2 \setminus X_2)$. Now $\lambda(Z) \le \lambda(X_1^\sim \times (\psi_2(Z)) = 0$ and so $\lambda(Z) = 0$ (note $\lambda^{(2)} \in M^+_{(o,\sigma)}(X_2, E))$. Fix any compact G_δ-set $Z \subset (X_1^\sim \times \tilde{X}_2) \setminus (X_1 \times X_2)$ and take a compact $C \subset X_1$. This means $Z \cap (C \times X_2^\sim) \subset (X_1^\sim \times (X_2^\sim \setminus X_2))$ and $\psi_2(Z \cap (C \times \tilde{X}_2))$ is compact G_δ-set in $(X_2^\sim \setminus X_2)$. Thus $\lambda(Z \cap (C \times X_2^\sim) = 0$, for every compact $C \subset X_1$. Now $\lambda(Z) = \lambda(Z \cap (C \times X_2^\sim)) + \lambda(Z \cap ((X_1^\sim \setminus C) \times X_2^\sim)) \le \lambda((X_1^\sim \setminus C) \times X_2^\sim) = \lambda^{(1)}(X_1^\sim \setminus C)$. Taking sup over C as C increases over compact subsets of X_1 and using the tightness property of $\lambda^{(1)}$, we get $\lambda(Z) = 0$. Since E is a weakly σ-distributive vector lattice, we get $\lambda(B) = 0$ for every Baire set $B \subset \tilde{X}_1 \times \tilde{X}_2 \setminus (X_1 \times X_2)$ ([11]).

Now it is easy to define λ : Baire$(H) \to E$, $\lambda(B) = \lambda(B^0)$, B^0 being any Baire set in $X_1^\sim \times X_2^\sim$ such that $(X_1 \times X_2) \cap B^0 = B$; it is easily verified that it is well-defined and countably additive. Other things are easy to verify.

(b) The condition is trivially necessary. Now, for $i = 1, 2$, define $\mu_i^\sim(f) = \mu_i(f_{|X_i})$, $\forall f \in C(X_i^\sim)$; $\mu_i^\sim \in M_o^+(X_i^\sim, E)$. Let $\bar{S}$ be the closure of S in $X_1^\sim \times X_2^\sim$. Using the given hypothesis and Theorem 2, we get a $\lambda^\sim \in M_o^+((X_1^\sim \times X_2^\sim, E)$ such that $\lambda^\sim(\bar{S}) \ge \gamma$ and its marginals are $\mu_i^\sim$ $(i = 1, 2)$. By (a) $\lambda^\sim$ arises from the countably additive λ : Baire$(H) \to E$, $\lambda(B) = \lambda(B^0)$, B^0 being any Baire set in $X_1^\sim \times \tilde{X}_2$ such that $(X_1 \times X_2) \cap B^0 = B$.

Take a sequence $\{f_n\} \subset H_s$, $f_n \downarrow \chi_S$; this means $f_n^\sim \ge \chi_{\bar{S}}$, $\forall n$ and $f_n^\sim \downarrow$. Now $\lambda(S) = \lim \lambda(f_n) = \lim \lambda^\sim(f_n^\sim) \ge \lambda^\sim(\bar{S}) \ge \gamma$. This proves the result. □

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