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A continuous analogue of the invariance principle and its almost sure version

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Abstract. We deal with random processes obtained from a homogeneous random process with independent increments by replacement of the time scale and by multiplication by a norming constant. We prove the convergence in distribution of these processes to Wiener process in the Skorohod space endowed by the topology of uniform convergence. An integral type almost sure version of this limit theorem is obtained.

1. Introduction

The usual invariance principle asserts the convergence of a sequence of the random processes $Z_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nx]} \xi_i$, $x \in [0,1]$, as $n \to \infty$, to the Wiener process W, where ξ_i are independent identically distributed centered random variables with variance 1. In this paper we study approximations of the Wiener process W by the random processes

$$X_t(x) = \frac{1}{\sqrt{t}} V(tx), \quad x \in [0,1], \tag{1}$$

where $t > 0$ is a parameter and V is a centered homogeneous random process with independent increments such that $V(0) = 0$ and $\mathbf{E}(V(1))^2 = \sigma^2$. Then almost all sample paths of X_t belong to the Skorohod space $D[0,1]$ and in Section 2 we prove that X_t converges to σW, as $t \to \infty$, in distribution in $D[0,1]$.

Section 3 deals with almost sure versions of limit theorems from Section 2. Recall the notion of an almost sure limit theorem. Let ζ_n, $n \in \mathbf{N}$ be a sequence

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of random variables defined on the probability space $(\Omega, \mathfrak{A}, \mathbf{P})$. We will denote by $\xrightarrow{w}$ the weak convergence of measures, by μ_ζ the distribution of the random element ζ and by $\mathfrak{B}(\mathbf{B})$ the σ-algebra of the Borel subsets of the metric space $\mathbf{B}$.

Usual limit theorem deals with the convergence of ζ_n. Consider the sequence of measures

$$Q_n^*[\zeta_n](\omega) = Q_n^*(\omega) = \frac{1}{D_n} \sum_{k=1}^{n} d_k \delta_{\zeta_k(\omega)}, \quad \omega \in \Omega,\ n \in \mathbf{N},$$

where $d_k,\ D_n \in \mathbb{R}$. Here and in the following δ_x denotes the measure of unit mass, concentrated at the point x. In several cases, under the conditions of a usual limit theorem, we have

$$Q_n^*[\zeta_n](\omega) \xrightarrow{w} \mu_\zeta, \quad \text{as } n \to \infty,$$

for almost all $\omega \in \Omega$. Such limit theorem is called almost sure version of the limit theorem.

Almost sure versions of functional limit theorems were studied in several papers. Here we mention only Lacey and Philipp [1], Chuprunov and Fazekas [2], Chuprunov and Fazekas [3].

We describe sequences (s_n), such that

$$Q_n(\omega) = Q_n[X_{s_n}(t)](\omega) = \frac{1}{D_n} \sum_{k=1}^{n} d_k \delta_{X_{s_k}(t,\omega)} \tag{2}$$

converges weakly to the distribution of σW in $D[0,1]$ for almost all $\omega \in \Omega$.

In Chuprunov and Fazekas [4] it described the function f such that the measures

$$Q_{T,\omega}(A) = \frac{1}{\ln(T)} \int_1^T \delta_{X_{f(t)}(1)(\omega)} \frac{1}{t} dt, \quad A \in \mathcal{B}(\mathbb{R})$$

converges weakly for almost all $\omega \in \Omega$, as $T \to \infty$, to centered Gaussian distribution with the variance σ^2.

We prove a functional version of this assertion (Theorem 1). We show that under Chuprunov and Fazekas assumption the measures

$$Q_{T,\omega}^I = \frac{1}{D(T)} \int_1^T \delta_{X_{f(t)}(\omega)} d(t) dt$$

converges weakly, as $T \to \infty$, to the distribution of σW in $D[0,1]$ for almost all $\omega \in \Omega$. The proof of this result is based on the criterion for integral type almost sure version of a limit theorem which was obtained in CHUPRUNOV and FAZEKAS [4].

In MAJOR [5] it proved that if $X(t)$ is a generated Ornstain–Uhlenbeck process then the measures

$$Q_{T,\omega}(A) = \frac{1}{\ln(T)} \int_1^T \delta_{X_t(\omega)} \frac{1}{t} dt, \quad A \in \mathcal{B}(D[0,1])$$

converges weakly, as $T \to \infty$, to the distribution of X in $D[0,1]$ for almost all $\omega \in \Omega$. So Theorem 1 is a some analogue of Major Theorem. Observe, that the proves of aur almost sure versions of limit theorems are based on CHUPRUNOV and FAZEKAS Lemma [3]. The proof of Major Theorem is based on ergogic theorem.

2. Functional limit theorems

We will denote by $\xrightarrow{d}$ the convergence in distribution. In the paper we will denote by the same symbol the random process and the random element corresponding to this random process.

Using the KOLMOGOROV representation (see [6], Sect. 18) we can assume that the characteristic function of the centered homogeneous random process $V(t)$ with independent increments and with finite variance is

$$\phi_{V(t)}(x) = \mathbf{E}\big(e^{ixV(t)}\big) = \exp\left(t \left\{ \int_{-\infty}^{+\infty} (e^{ixy} - 1 - ixy) \frac{1}{y^2} dK(y) \right\} \right), \quad x \in \mathbf{R}. \quad (3)$$

Here $K(y)$ is a bounded increasing function such that $K(-\infty) = 0$.

We will consider the sequence of the random processes

$$Y_n(t) = X_{s_n}(t) = \frac{1}{\sqrt{s_n}} V(s_n t), \quad (4)$$

where

$$s_n \to \infty, \quad \text{as } n \to \infty. \quad (5)$$

We will use the following preliminary result.

Proposition 1. *Let Y_n be defined by* (4) *and assume that* (s_n) *satisfies* (5). *Let $\sigma^2 = K(+\infty)$. Then we have*

$$Y_n \xrightarrow{d} \sigma W, \quad \text{as } n \to \infty,$$

in $D[0,1]$ endowed by the topology of uniform convergence.

Proof. Let $0 \le t_1 < t_2 < \infty$. By the convergence criterion in [6], Sect. 19, we have

$$Y_n(t_2) - Y_n(t_1) \xrightarrow{d} \sigma(W(t_2) - W(t_1)), \quad \text{as } n \to \infty. \tag{6}$$

Let $0 \le t_0 < t_1 < \dots < t_k < \infty$. Introduce the notation $\Delta Y_{ni} = Y_n(t_i) - Y_n(t_{i-1})$ and $\Delta W_i = W(t_i) - W(t_{i-1})$. Since ΔY_{ni}, $1 \le i \le k$, are independent random variables, from (6) we obtain

$$(\Delta Y_{n1}, \dots, \Delta Y_{nk}) \xrightarrow{d} (\sigma \Delta W_1, \dots, \sigma \Delta W_k).$$

Consequently, the finite dimensional distributions of Y_n converge to the finite dimensional distributions of σW. Also we have

$$\limsup_{n\to\infty} \mathbf{E}|Y_n(t_2) - Y_n(t_1)|^2 = \sigma^2 |t_2 - t_1|. \tag{7}$$

But (7) together with the convergence of the finite dimensional distributions gives the weak convergence of Y_n to σW (see Billingsley [7], Theorem 15.6) in $D[0,1]$ with Skorohod's J_1-topology. However, in our case the limit process is a continuous one. Hence, (see Pollard [8], p. 137, and the discussion in Billingsley [7], Sect. 18), the weak convergence in Skorohod's J_1-topology actually implies the weak convergence in the uniform topology of $D[0,1]$. The proof is complete. □

Using Proposition 1, we can prove the following proposition.

Proposition 2. *Let X_t be defined by* (1). *Then it holds that*

$$X_t \xrightarrow{d} \sigma W, \quad \text{as } t \to \infty,$$

in $D[0,1]$ endowed by the topology of uniform convergence.

Proof. Consider $D[0,1]$ in the topology of uniform convergence and the space M of distributions on $D[0,1]$ with the topology of convergence in distribution. Then M is a metric space and denote by ρ_M a metric which defines this topology on M. Then, by Proposition 1, $\rho_M(\mu_{X_{s_n}}, \mu_{\sigma W}) \to 0$, as $s_n \to \infty$. Therefore $\rho_M(\mu_{X_t}, \mu_{\sigma W}) \to 0$, as $t \to \infty$. The proof is complete. □

Let ξ_i, $i \in \mathbf{N}$, be independent identically distributed random variables with the expectation a and the variance b^2 and let $\pi(t)$, $t \in \mathbf{R}^+$, be a Poisson process with the intensity 1, such that the families ξ_i, $i \in \mathbf{N}$ and $\pi(t)$, $t \in \mathbf{R}^+$ are independent. Then

$$V'(x) = \sum_{i=1}^{\pi(x)} \xi_i - ax, \quad x \in [0,1],$$

is a centered homogeneous random process with independent increments such that $V'(0) = 0$ and $\mathbf{E}(V'(1))^2 = a^2 + b^2$

So from Proposition 2 we obtain the following corollary.

Corollary 1. *Let ξ_i, $i \in \mathbf{N}$, be independent identically distributed random variables with the expectation a and the variance b^2. Let*

$$X'_t(x) = \frac{\sum_{i=1}^{\pi(tx)} \xi_i - atx}{\sqrt{t}}, \quad x \in [0,1]. \tag{8}$$

Then one has

$$X'_t \xrightarrow{d} \sqrt{a^2 + b^2}\, W, \quad \text{as } t \to \infty,$$

in $D[0,1]$ with the topology of uniform convergence.

For $\xi_i = 1$ from Corollary 1 we obtain the following.

Corollary 2. *Let*

$$X^*_t(x) = \frac{\pi(tx) - tx}{\sqrt{t}}, \quad x \in [0,1]. \tag{9}$$

Then one has

$$X^*_t \xrightarrow{d} W, \quad \text{as } t \to \infty,$$

in $D[0,1]$ with the topology of uniform convergence.

3. Almost sure versions of functional limit theorems

We will consider the sequence of measures defined by (2) and connected with the random processes $X_t(x)$.

For the sequence s_n we will assume the following property:

(A) for some $\beta > 0$, $\dfrac{s_n}{n^\beta}$ is an increasing sequence.

We will consider $d_k \in \mathbb{R}$ with the properties:

(B) $$0 \le d_k \le \ln\left(\frac{k+1}{k}\right), \quad \sum_{k=1}^{\infty} d_k = \infty.$$

Denote $D_n = \sum_{k=1}^{n} d_k$.

Proposition 3. *Let* (A) *and* (B) *be valid. Then it holds that*

$$Q_n(\omega) \xrightarrow{w} \mu_{\sigma W} \quad \text{if } n \to \infty \tag{10}$$

for almost all $\omega \in \Omega$.

Proof. By (A) (s_n) is an increasing sequence.

Let $l < k$. Let

$$Y_{kl}(x) = \begin{cases} 0, & 0 \le x < \dfrac{s_l}{s_k}, \\ Y_k(x) - \dfrac{V(s_l)}{\sqrt{s_k}}, & \dfrac{s_l}{s_k} \le x \le 1. \end{cases}$$

Then $Y_{kl}(x)$, $0 \le x \le 1$, and $Y_l(x)$, $0 \le x \le 1$, are independent random processes. Let ρ be the metric of $D[0,1]$. Using the moment inequality from [8], Sect. 5, p. 231 we obtain

$$\begin{aligned} \mathbf{E}\rho(Y_k, Y_{kl}) &\le \mathbf{E} \sup_{0\le x\le 1} |Y_k(x) - Y_{kl}(x)| \le \mathbf{E} \sup_{0\le x\le \frac{s_l}{s_k}} |Y_k(x)| \\ &\le 4\mathbf{E}\left|Y_k\left(\frac{s_l}{s_k}\right)\right| \le 4\frac{\sqrt{\mathbf{E}(V(s_l))^2}}{\sqrt{s_k}} \le 4\sigma\sqrt{\frac{s_l}{s_k}} \le 4\sigma\left(\frac{l}{k}\right)^{\beta/2}. \end{aligned} \tag{11}$$

Lemma 1 in [3], Proposition 1 and (11) imply (10). The proof is complete. □

For the function f we will consider the following property

(C) for some $\beta > 0$, $\dfrac{f(x)}{x^\beta}$ is an increasing function.

Now we will prove the integral type almost sure version of Proposition 2. We will consider the random processes

$$Y_t(x) = \frac{V(f(t)x)}{\sqrt{f(t)}}, \quad x \in [0,1].$$

Let the function $d(s)$ is a decreasing such that the condition

(D) $\displaystyle\int_k^{k+1} d(s)ds \le \ln\left(\sqrt{\frac{k+1}{k}}\right)$ for all $k \in \mathbf{N}$ and $\displaystyle\int_1^\infty d(s)ds = +\infty$

are valid.

We will consider the measures

$$Q^I_{S,\omega}(\omega) = \frac{1}{D(S)} \int_1^S \delta_{Y_s(\omega)} d(s)ds,$$

where $D(S) = \int_1^S d(s)ds$.

Theorem 1. *Let* (C) *and* (D) *be valid. Then we have*

$$Q_S^I(\omega) \xrightarrow{w} \mu_{\sigma W}, \quad \text{as } S \to \infty, \tag{12}$$

for almost all $\omega \in \Omega$.

Proof. Let $0 < l < k$, $l, k \in \mathbf{N}$, $k \le t \le k+1$. Introduce the notation

$$Y_{lkt}(s) = \begin{cases} 0, & 0 \le s \le \dfrac{f(l)}{f(t)}, \\ Y_t(s) - \dfrac{V(f(l))}{\sqrt{f(t)}}, & \dfrac{f(l)}{f(t)} \le s \le 1. \end{cases}$$

Then $\{Y_{lkt}(s) : k \le t < k+1\}$ and $\{Y_t(s) : l \le t < l+1\}$ are independent families. Repeating the proof of (11), we obtain

$$\mathbf{E}\rho(Y_t, Y_{lkt}) \le \mathbf{E} \sup_{0 \le s \le 1} |Y_t(s) - Y_{lkt}(s)| \le 4 \cdot 2^{\beta/2} \sigma \left(\frac{l}{k}\right)^{\beta/2}.$$

By Corollary 2.1, from Chuprunov and Fazekas [4] this and Proposition 2 implies (12). The proof is complete. □

Theorem 1 and Corollary 1 (resp. Corollary 2) of Proposition 2 imply the following corollaries.

Corollary 1. *Let* X' *be defined by* (8) *and let* f *be a function with the property* (C). *Suppose* (D) *is valid. Then one has*

$$\frac{1}{\ln(S)} \int_1^S \delta_{X'_{f(s)}(\omega)} d(s) ds \xrightarrow{w} \mu_{bW}, \quad \text{as } S \to \infty,$$

for almost all $\omega \in \Omega$.

Corollary 2. *Let* X^* *be defined by* (9) *and let* f *be a function with the property* (C). *Suppose* (D) *is valid. Then it holds*

$$\frac{1}{\ln(S)} \int_1^S \delta_{X^*_{f(s)}(\omega)} d(s) ds \xrightarrow{w} \mu_W, \quad \text{as } S \to \infty,$$

for almost all $\omega \in \Omega$.

Remark 1. Corollary 1 of Proposition 2 is a functional limit theorem for random sums. So Corollary 1 of Theorem 1 is an integral type almost sure version of a functional limit theorem for random sums. (For limit theorems for random sums see Korolev and Kruglov [9].)

References

[1] M. T. Lacey and W. Philipp, A note on the almost sure central limit theorem, *Statistics and Probability Letters* **2** (1990), 201–205.

[2] A. Chuprunov and I. Fazekas, Almost sure versions of some analogues of the invariance principle, *Publicationes Mathematicae Debrecen* **54**(3–4) (1999), 457–471.

[3] A. Chuprunov and I. Fazekas, Almost sure limit theorems for the Pearson statistic, *Teor. Veroyatnost. i Primenen.* **48**(1) (2003), 162–169.

[4] A. Chuprunov and I. Fazekas, Integral analogues of almost sure limit theorems, *Periodica Mathematica Hungarica* **50**(1–2) (2005), 61–78.

[5] P. Major, Almost sure functional limit theorems Part I. The general case, *Studia Scientiarum Mathematicarum Hungarica* **34** (1998), 273–304.

[6] B. V. Gnedenko and A. N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables, *Addison-Wesley, Reading*, *Massachusetts*, 1954.

[7] P. Billingsley, Convergence of probability measures, *John Wiley and Sons*, *New York*, 1968.

[8] D. Pollard, Convergence of stochastic processes, *Springer-Verlag*, *New York*, 1984.

[9] N. N. Vakhania, V. I. Tarieladze and S. A. Chobanian, Probability distributions in Banach spaces, *Nauka, Moscow*, 1985 (in *Russian*).

[10] V. M. Kruglov and V. Yu. Korolev, Limit theorems for random sums, *Moscow University Press*, *Moscow*, 1990 (in *Russian*).

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