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On a projective class of Finsler metrics

By B. NAJAFI (Tehran), ZHONGMIN SHEN (Indianapolis)
and AKBAR TAYEBI (Tehran)

Abstract. In this paper, we study a class of Finsler metrics whose Douglas curvature satisfies $D_j{}^i{}_{kl;m}y^m = T_{jkl}y^i$. It is known that this class is closed under projective change and all metrics with vanishing Douglas curvature or vanishing Weyl curvature belong to it. Thus Finsler metrics in this class are called generalized Douglas–Weyl (GDW) metrics. For a Randers metric $F = \alpha + \beta$, we find a sufficient and necessary condition for F to be a GDW metric.

1. Introduction

The Douglas (projective) curvature $D_j{}^i{}_{kl}$ and the Weyl (projective) curvature $W^i{}_k$ are two most important quantities in projective Finsler geometry. Finsler metrics with $D_j{}^i{}_{kl}=0$ are called *Douglas metrics* and Finsler metrics with $W^i{}_k=0$ are called *Weyl metrics*. It is well-known that a Finsler metric is a Weyl metric if and only if it is of scalar flag curvature, namely, the flag curvature $\mathbf{K}(P, y) = K(x, y)$ is independent of the section P containing y. Thus Weyl metrics are also called *metrics of scalar (flag) curvature*, and being of scalar flag curvature is a projective property.

Equations $D_j{}^i{}_{kl} = 0$ and $W^i{}_k = 0$ are projectively invariant, namely, if a Finsler metric F satisfies one of the equations, then any Finsler metric projectively equivalent to F must satisfy the same equation. There is another projective

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invariant equation in Finsler geometry, that is, for some tensor T_{jkl},

$$D_j{}^i{}_{kl;m}y^m = T_{jkl}y^i, \tag{1}$$

where $D_j{}^i{}_{kl;m}$ denotes the horizontal covariant derivatives of $D_j{}^i{}_{kl}$ with respect to the Berwald connection of F. Equation (1) is equivalent to that for any linearly parallel vector fields $U = U(t), V = V(t)$ and $W = W(t)$ along a geodesic $c(t)$, there is a function $T = T(t)$ such that

$$\frac{d}{dt}[D_{\dot{c}}(U, V, W)] = T\dot{c}.$$

The geometric meaning of the above identity is that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic.

For a manifold M, let $\mathcal{GDW}(M)$ denote the class of all Finsler metrics satisfying (1) for some tensor T_{jkl} (T_{jkl} not fixed). In [2], Bácsó–Papp show that $\mathcal{GDW}(M)$ is closed under projective changes. More precisely, if F is projectively equivalent to a Finsler metric in $\mathcal{GDW}(M)$, then $F \in \mathcal{GDW}(M)$.

A natural question is: how large is $\mathcal{GDW}(M)$ and what kind of interesting metrics does it have? It is obvious that all Douglas metrics belong to this class. On the other hand, all Weyl metrics (metrics of scalar flag curvature) also belong to this class. The later is really a surprising result, due to Sakaguchi [4]. In this sense, we shall call Finsler metrics in $\mathcal{GDW}(M)$ *GDW-metrics* (generalized Douglas–Weyl metrics).

In this paper, we are going to study and characterize GDW-metrics of Randers type on a manifold M. A Randers metric on a manifold M is a Finsler metric in the following form

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. One of the reasons why we would like to study Randers metrics for the above problem is because that Randers metrics are "computable".

Let

$$s_{ij} := \frac{1}{2}\left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right).$$

Clearly, β is closed if and only if $s_{ij} = 0$. It is known that $F = \alpha + \beta$ is a Douglas metric if and only if β is closed (see [1]). On the other hand, Shen–Yildirim [6] recently find a sufficient and necessary condition for $F = \alpha + \beta$ to be of scalar flag curvature, that is,

$$\bar{R}^i{}_k = \left(\lambda - \frac{1}{n-1}t^m{}_m\right)\left\{\alpha^2\delta^i_k - a_{jk}y^iy^j\right\}$$

$$+ \alpha^2 t^i{}_k + t_{00}\delta^i_k - t_{k0}y^i - t^i{}_0 y_k - 3s^i{}_0 s_{k0}, \tag{2}$$

$$s_{ij|k} = \frac{1}{n-1}\big\{a_{ik}s^m{}_{j|m} - a_{jk}s^m{}_{i|m}\big\}, \tag{3}$$

where $t_{ij} := s_{im}s^m_j$, $\bar{R}^i{}_k$ denotes the Riemann curvature tensor of α and $\lambda = \lambda(x)$ is a scalar function on M. Here $s_{ij|k}$ denote the coefficients of the covariant derivative of s_{ij} with respect to α. We use a_{ij} and a^{ij} to lower or lift the indices of a tensor.

Theorem 1.1. *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. F is a GDW metric if and only if* (3) *holds.*

Thus any Randers metrics of scalar curvature belongs to $\mathcal{G}DW(M)$. This verifies Sakaguchi's theorem for Randers metrics. The following Randers metric actually satisfies both (2) and (3).

Example 1.1 ([5]). Let $a \in \mathrm{R}^n$ be a constant vector. Define $F = \alpha + \beta$ on an open ball $\mathrm{B}^n(1/\sqrt{|a|})$ in R^n by

$$F := \frac{\sqrt{(1 - |a|^2|x|^4)|y|^2 + (|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle)^2}}{1 - |a|^2|x|^4} - \frac{|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle}{1 - |a|^2|x|^4}.$$

Then F is of scalar flag curvature. Thus it satisfies (2) and (3). See more examples in [3].

So far, we have not found a Randers metric satisfying (3), but not (2). We conjecture that such examples exist.

2. Randers metrics

Let $F = \alpha + \beta$ be a Randers metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. The spray coefficients of F are given by

$$G^i = \bar{G}^i + \frac{r_{00} - 2s_0\alpha}{2F}y^i + \alpha s^i{}_0,$$

where $\bar{G}^i$ denote the spray coefficients of α. Let

$$\Pi^i := G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}, \quad \bar{\Pi}^i := \bar{G}^i - \frac{1}{n+1}\frac{\partial \bar{G}^m}{\partial y^m}.$$

Observe that

$$\frac{\partial(\alpha s^m{}_0)}{\partial y^m} = \frac{y_m}{\alpha} s^m{}_0 + \alpha s^m{}_m = 0.$$

Thus we have

$$\Pi^i = \bar{\Pi}^i + \alpha s^i{}_0. \tag{4}$$

By definition, the Douglas curvature is given by

$$D_j{}^i{}_{kl} := \frac{\partial^3 \Pi^i}{\partial y^j \partial y^k \partial y^l}.$$

Since $\bar{\Pi}^i$ are always quadratic in y, we get

$$D_j{}^i{}_{kl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(\alpha s^i{}_0\right) = \alpha_{jkl} s^i{}_0 + \alpha_{jk} s^i{}_l + \alpha_{jl} s^i{}_k + \alpha_{kl} s^i{}_j, \tag{5}$$

where

$$\begin{aligned}
\alpha_j &= \alpha^{-1} y_j \\
\alpha_{jk} &= \alpha^{-3} A_{jk} \\
\alpha_{jkl} &= -\alpha^{-5}\{A_{jk} y_l + A_{jl} y_k + A_{kl} y_j\},
\end{aligned}$$

where $A_{ij} := \alpha^2 a_{ij} - y_i y_j$.

It is easy to show that F is a Douglas metric if and only if $s^i{}_0 = 0$. This fact is due to Bácsó–Matsumoto [1].

Let $\tilde{G} = y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i \frac{\partial}{\partial y^i}$ where

$$\tilde{G}^i := \bar{G}^i + \alpha s^i{}_0. \tag{6}$$

Let "$\|$" and "$|$" denote the covariant differentiations with respect to $\tilde{G}$ and $\bar{G}$ respectively. Then

$$\begin{aligned}
D_j{}^i{}_{kl\|m} y^m = {} & D_j{}^i{}_{kl|m} y^m - 2\alpha \frac{\partial}{\partial y^p}(D_j{}^i{}_{kl}) s^p{}_0 \\
& + \alpha^{-1}[\alpha^2 s^i{}_p + y_p s^i{}_0] D_j{}^p{}_{kl} - \alpha^{-1}[\alpha^2 s^p{}_j + s^p{}_0 y_j] D_p{}^i{}_{kl} \\
& - \alpha^{-1}[\alpha^2 s^p{}_k + s^p{}_0 y_k] D_j{}^i{}_{pl} - \alpha^{-1}[\alpha^2 s^p{}_l + s^p{}_0 y_l] D_j{}^i{}_{kp}.
\end{aligned} \tag{7}$$

Since "|" is a differentiation with respect to α, $a_{ij|m} = 0$. Thus

$$\alpha_{|m} = 0, \quad \alpha_{j|m} = 0, \quad \alpha_{jk|m} = 0, \quad \alpha_{jkl|m} = 0.$$

We obtain

$$\begin{aligned} D_j{}^i{}_{kl|m} y^m &= \alpha_{jkl} s^i{}_{0|0} + \alpha_{jk} s^i{}_{l|0} + \alpha_{jl} s^i{}_{k|0} + \alpha_{kl} s^i{}_{j|0} \\ &= -\alpha^{-5}\big\{A_{jk} y_l + A_{jl} y_k + A_{kl} y_j\big\} s^i{}_{0|0} + \alpha^{-3}\big\{A_{jk} s^i{}_{l|0} + A_{jl} s^i{}_{k|0} + A_{kl} s^i{}_{j|0}\big\}. \end{aligned}$$

Differentiating (5) yields

$$\frac{\partial}{\partial y^p}(D_j{}^i{}_{kl}) = \alpha_{jklp} s^i{}_0 + \alpha_{jkl} s^i{}_p + \alpha_{jkp} s^i{}_l + \alpha_{jlp} s^i{}_k + \alpha_{klp} s^i{}_j,$$

where

$$\begin{aligned} \alpha_{jklp} = 3\alpha^{-5} y_p \{a_{jk} y_l + a_{jl} y_k + a_{kl} y_j\} - \alpha^{-3}\big\{a_{jk} a_{lp} + a_{jl} a_{kp} + a_{kl} a_{jp}\big\} \\ + 3\alpha^{-5}\big\{y_k y_l a_{jp} + y_j y_l a_{kp} + y_j y_k a_{lp}\big\} - 15\alpha^{-7} y_j y_k y_l y_p. \end{aligned}$$

We get

$$\begin{aligned} \alpha_{jklp} s^p{}_0 &= -\alpha^{-5}\big\{A_{jk} s_{l0} + A_{jl} s_{k0} + A_{kl} s_{j0}\big\} \\ &\qquad + 2\alpha^{-5}\big\{s_{j0} y_k y_l + s_{k0} y_i y_l + s_{l0} y_j y_k\big\} \\ \alpha_{jkp} s^p{}_0 &= -\alpha^{-3}\big\{y_k s_{j0} + y_j s_{k0}\big\} \\ \alpha_{jlp} s^p{}_0 &= -\alpha^{-3}\big\{y_l s_{j0} + y_j s_{l0}\big\}, \quad \alpha_{klp} s^p{}_0 = -\alpha^{-3}\big\{y_k s_{l0} + y_l s_{k0}\big\}. \end{aligned}$$

Then we obtain

$$\begin{aligned} -2\alpha \frac{\partial}{\partial y^p}(D_j{}^i{}_{kl}) s^p{}_0 &= 2\alpha^{-4}\big\{A_{jk} s_{l0} + A_{jl} s_{k0} + A_{kl} s_{j0}\big\} s^i{}_0 \\ &+ 2\alpha^{-4}\{A_{jk} y_l + A_{jl} y_k + A_{kl} y_j\} t^i{}_0 - 4\alpha^{-4}\big\{y_k y_l s_{j0} + y_j y_l s_{k0} + y_j y_k s_{l0}\big\} s^i{}_0 \\ &+ 2\alpha^{-2}\big\{y_k s_{j0} + y_j s_{k0}\big\} s^i{}_l + 2\alpha^{-2}\big\{y_l s_{j0} + y_j s_{l0}\big\} s^i{}_k \\ &+ 2\alpha^{-2}\big\{y_k s_{l0} + y_l s_{k0}\big\} s^i{}_j. \end{aligned}$$

By (5), we can also easily get

$$\begin{aligned} \alpha^{-1}[\alpha^2 s^i{}_p + y_p s^i{}_0] D_j{}^p{}_{kl} &= -\alpha^{-2}\{A_{jk} y_l + A_{jl} y_k + A_{kl} y_j\} t^i{}_0 \\ &+ \alpha^{-2}\big\{A_{jk} t^i{}_l + A_{jl} t^i{}_k + A_{kl} t^i{}_j\big\} - \alpha^{-4}\big\{A_{jk} s_{l0} + A_{jl} s_{k0} s^i{}_0 + A_{kl} s_{j0}\big\} s^i{}_0. \\ -\alpha^{-1}[\alpha^2 s^p{}_j + s^p{}_0 y_j] D^i_{pkl} &= \alpha^{-2}\big\{y_k s_{lj} + y_l s_{kj}\big\} + 2\alpha^{-4} y_k y_l s_{j0} s^i{}_0 \end{aligned}$$

$$
\begin{aligned}
&- \alpha^{-2}(\alpha^2 s_{kj} - s_{0j}y_k){s^i}_l - \alpha^{-2}(\alpha^2 s_{lj} - s_{0j}y_l){s^i}_k \\
&+ \alpha^{-4}\big\{y_j y_k s_{l0} + y_j y_l s_{k0}\big\}{s^i}_0 - \alpha^{-2}y_j s_{k0}{s^i}_l - \alpha^{-2}y_j s_{l0}{s^i}_k \\
&- \alpha^{-4}y_j A_{kl}{t^i}_0 - \alpha^{-4}A_{kl}s_{j0}{s^i}_0 - \alpha^{-2}A_{kl}{t^i}_j.
\end{aligned}
$$

$$
\begin{aligned}
- \alpha^{-1}[\alpha^2 {s^p}_k + {s^p}_0 y_k]D^i_{plj} &= \alpha^{-2}\big\{y_l s_{jk} + y_j s_{lk}\big\} + 2\alpha^{-4}y_l y_j s_{k0}{s^i}_0 \\
&- \alpha^{-2}(\alpha^2 s_{lk} - s_{0k}y_l){s^i}_j - \alpha^{-2}(\alpha^2 s_{jk} - s_{0k}y_j){s^i}_l \\
&+ \alpha^{-4}\big\{y_k y_l s_{j0} + y_k y_j s_{l0}\big\}{s^i}_0 - \alpha^{-2}y_k s_{l0}{s^i}_j - \alpha^{-2}y_k s_{j0}{s^i}_l \\
&- \alpha^{-4}y_k A_{jl}{t^i}_0 - \alpha^{-4}A_{lj}s_{k0}{s^i}_0 - \alpha^{-2}A_{jl}{t^i}_k.
\end{aligned}
$$

$$
\begin{aligned}
- \alpha^{-1}[\alpha^2 {s^p}_l + {s^p}_0 y_l]D^i_{pkj} &= \alpha^{-2}\big\{y_k s_{jl} + y_j s_{kl}\big\} + 2\alpha^{-4}y_k y_j s_{l0}{s^i}_0 \\
&- \alpha^{-2}(\alpha^2 s_{kl} - s_{0l}y_k){s^i}_j - \alpha^{-2}(\alpha^2 s_{jl} - s_{0l}y_j){s^i}_k \\
&+ \alpha^{-4}\big\{y_l y_k s_{j0} + y_l y_j s_{k0}\big\}{s^i}_0 - \alpha^{-2}y_l s_{k0}{s^i}_j - \alpha^{-2}y_l s_{j0}{s^i}_k \\
&- \alpha^{-4}y_l A_{jk}{t^i}_0 - \alpha^{-4}A_{kj}s_{l0}{s^i}_0 - \alpha^{-2}A_{jk}{t^i}_l.
\end{aligned}
$$

Plugging the above identities into (7), we get

$$
{D_j}^i{}_{kl\|m}y^m = \alpha^{-5}\big\{A_{jk}{H^i}_l + A_{jl}{H^i}_k + A_{kl}{H^i}_j\big\}, \tag{8}
$$

where

$$
{H^i}_j := \alpha^2 {s^i}_{j|0} - y_j {s^i}_{0|0}.
$$

3. Proof of Theorem 1.1

First we are going to prove the following

Theorem 3.1. *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. F is a GDW-metric if and only if*

$$
\alpha^2 s_{ij|0} = s_{i0|0}y_j - s_{j0|0}y_i. \tag{9}
$$

Proof. Let $F = \alpha + \beta$ be a Randers metric on a manifold. Let G denote the spray of F and $\tilde{G}$ the spray defined in (6). Since $\tilde{G}$ and G are projectively equivalent, the following conditions are equivalent

(i) there is a tensor T_{jkl} such that ${D_j}^i{}_{kl;m}y^m = T_{jkl}y^i$,

(ii) there is a tensor D_{jkl} such that

$$
{D_j}^i{}_{kl\|m}y^m = D_{jkl}y^i, \tag{10}
$$

where $D_j{}^i{}_{kl;m}$ and $D_j{}^i{}_{kl\|m}$ denote the covariant derivatives of $D_j{}^i{}_{kl}$ with respect to the Berwald connections of G and $\tilde{G}$, respectively. This equivalence is essentially proved in [2]. Thus the argument is omitted here.

Assume that F is a GDW-metric. Then (10) holds for some tensor D_{jkl}. By (8), we have

$$D_{jkl}y^i = \alpha^{-5}\big\{A_{jk}H^i{}_l + A_{jl}H^i{}_k + A_{kl}H^i{}_j\big\}. \tag{11}$$

Contracting (11) with y_i yields

$$D_{jkl} = -\alpha^{-5}\big\{A_{jk}s_{l0|0} + A_{jl}s_{k0|0} + A_{kl}s_{j0|0}\big\}. \tag{12}$$

Plugging (12) into (11), we get

$$A_{jk}\big\{H^i{}_l + s_{l0|0}y^i\big\} + A_{jl}\big\{H^i{}_k + s_{k0|0}y^i\big\} + A_{kl}\big\{H^i{}_j + s_{j0|0}y^i\big\} = 0. \tag{13}$$

Contracting (13) with a^{kl} we obtain

$$H^i{}_j + s_{j0|0}y^i = 0. \tag{14}$$

This is obviously equivalent to (9).

Conversely, if (9) holds, or equivalently, (14) holds, it follows from (8) that

$$D^i_{jkl\|m}y^m = D_{jkl}y^i,$$

where D_{jkl} are given by (12). Thus F is a GDW-metric. □

To prove Theorem 1.1, one just needs to prove the equivalence between (3) and (9).

Lemma 3.2. *(3) is equivalent to (9).*

Proof. Suppose that (3) holds. Then

$$s_{ij|k} = \lambda\big\{a_{ik}s^m{}_{j|m} - a_{jk}s^m{}_{i|m}\big\}, \tag{15}$$

where $\lambda = 1/(n-1)$ (in fact, λ can be any scalar function). Contracting it with y^k yields

$$s_{ij|0} = \lambda\big\{y_i s^m{}_{j|m} - y_j s^m{}_{i|m}\big\}. \tag{16}$$

Contracting (16) with y^j yields

$$s_{i0|0} = \lambda\big\{y_i s^m{}_{0|m} - \alpha^2 s^m{}_{i|m}\big\}. \tag{17}$$

Thus

$$s_{j0|0} = \lambda\big\{y_j s^m_{\ 0|m} - \alpha^2 s^m_{\ j|m}\big\}. \tag{18}$$

By (17)–(18),

$$\begin{aligned} s_{i0|0}y_j - s_{j0|0}y_i &= \lambda\alpha^2\big\{s^m_{\ j|m}y_i - s^m_{\ i|m}y_j\big\} \\ &= \lambda\alpha^2\big\{a_{ik}s^m_{\ jm} - a_{jk}s^m_{\ m|i}\big\}y^k = \alpha^2 s_{ij|0}. \end{aligned}$$

The last identity follows from (16). Then we obtain (9).

Conversely, assume that (9) holds. First differentiating (9) with respect to y^k, y^l and y^m yields

$$\begin{aligned} &2a_{kl}s_{ij|m} + 2a_{km}s_{ij|l} + 2a_{lm}s_{ij|k} \\ &\quad = s_{ik|l}a_{jm} + s_{ik|m}a_{jl} + s_{il|k}a_{jm} + s_{im|k}a_{jl} + s_{il|m}a_{jk} + s_{im|l}a_{jk} \\ &\quad - s_{jk|l}a_{im} - s_{jk|m}a_{il} - s_{jl|k}a_{im} - s_{jm|k}a_{il} - s_{jl|m}a_{ik} - s_{jm|l}a_{ik}. \end{aligned}$$

Contracting it with a^{lm}, we get

$$ns_{ij|k} = s_{ik|j} - s_{jk|i} + a_{ik}s^m_{\ j|m} - a_{jk}s^m_{\ im}. \tag{19}$$

It follows from (19) that

$$ns_{ik|j} = s_{ij|k} + s_{jk|i} + a_{ij}s^m_{\ km} - a_{jk}s^m_{\ im} \tag{20}$$

$$ns_{jk|i} = -s_{ij|k} + s_{ik|j} + a_{ij}s^m_{\ km} - a_{ik}s^m_{\ jm}. \tag{21}$$

Subtracting (21) from (20), we get

$$s_{ik|j} - s_{jk|i} = \frac{2}{n+1}s_{ij|k} + \frac{1}{n+1}\big\{a_{ik}s^m_{\ jm} - a_{jk}s^m_{\ im}\big\}. \tag{22}$$

Plugging (22) back into (19) yields

$$s_{ij|k} = \frac{1}{n-1}\big\{a_{ik}s^m_{\ jm} - a_{jk}s^m_{\ im}\big\}.$$

We are done. $\square$

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B. NAJAFI
DEPARTMENT OF SCIENCE
SHAHED UNIVERSITY
TEHRAN
IRAN

E-mail: b_najafi@cic.aut.ac.ir

ZHONGMIN SHEN
DEPARTMENT OF MATHEMATICAL SCIENCES
INDIANA UNIVERSITY PURDUE UNIVERSITY INDIANAPOLIS (IUPUI)
402 N. BLACKFORD STREET
INDIANAPOLIS, IN 46202-3216
USA

E-mail: zshen@math.iupui.edu

AKBAR TAYEBI
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
AMIRKABIR UNIVERSITY
TEHRAN
IRAN

E-mail: akbar_tayebi@cic.aut.ac.ir

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