Publ. Math. Debrecen
43 / 3–4 (1993), 245–253

Which F loops are associative

By LEONG FOOK (Penang) and TEH PANG ENG (Penang)

1. Introduction

It is known that Moufang loops of order p, p^2, pq, p^3 [3], p^2q (odd) with $p < q$, pqr (odd), $2p^2$ [12], $p^4 (p \geq 5)$ [4] and $2pq$ $(p < q,\ p, \nmid q-1)$ [6] are all groups. On the other hand, there exist nonassociative Moufang loops of order 2^4 [3], 3^4 [2], p^5 $(p \geq 5)$ [15], 2^2q and $2pq$ $(p < q,\ p \mid q-1)$ [13].

Now we shall confine our study in a similar direction to a special class of Moufang loops called F loops whose orders are $2^\alpha {p_1}^{\alpha_1} \dots p_r^{\alpha_r}$ with $0 \leq \alpha \leq 3$; p_i are distinct odd primes such that $\alpha_i \leq 3$ if $p_i = 3$ and $\alpha_i \leq 4$ if otherwise. We shall prove that these F loops are groups if

(i) $0 \leq \alpha \leq 2$ or
(ii) $r \leq 2$

2. Definition

1. A loop $(L, \cdot)$ is a Moufang loop if $xy \cdot zx = (x \cdot yz)x$ for all $x, y, z \in L$.
2. L_a, the associator subloop of L, is the subloop generated by all the associators (x, y, z) where $xy \cdot z = (x \cdot yz)(x, y, z)$.
3. $N = N(L)$, nucleus of L, is the set of all $n \in L$ such that $(n, x, y) = (x, n, y) = (x, y, n) = 1$ for all $x, y \in L$.
4. $Z = Z(L)$, the centre of L, is the set of all $z \in N$ such that $(z, x) = 1$ where $zx = xz(z, x)$ for all $x \in L$.
5. An F loop L is a Moufang loop such that if H is a subloop generated by any three elements x, y, z of L, then $\langle (x, y, z) \rangle \subset Z(H)$, the centre of H.

Remark. It can be shown easily that $H_a = \langle (x, y, z) \rangle$ for any F loop H generated by x, y and z. [6, p. 80, Lemma]

3. Results

From now on, L is assumed to be a finite Moufang loop.

R_1 L is diassociative, i.e. $\langle x, y\rangle$ is associative for all $x, y \in L$.
[1, p. 115, Lemma 3.1]

R_2 If $(x, y, z) = 1$, then $\langle x, y, z\rangle$ is a group for any $x, y, z \in L$.
[1, p. 117, Moufang's Theorem]

R_3 N and Z are normal subloops of L. Clearly N and Z are associative.
[1, p. 114, Theorem 2.1]

R_4 There exist simple nonassociative Moufang loops $M(p^n)$ with $|M(p^n)| = p^{3n}(p^{4n} - 1)/d(p)$ where $d(2) = 1$ and $d(p) = 2$ if p is an odd prime.
[11, p. 475, Theorem 4.5]

R_5 L is simple if and only if L is a simple group or L is isomorphic with $M(p^n)$ for some prime p.
[10, p. 33, Theorem]

R_6 120 is a divisor of $|M(p^n)|$. [14]

R_7 If H is a subloop of L, $x \in L$, and d is the smallest positive integer such that $x^d \in H$, then $|\langle H, x\rangle| \geq |H|d$.
[3, p. 31, Lemma 1]

R_8 $L_a \triangleleft L$ and $L_a \subset C_L(N) = \{x \mid x \in L,\ xn = nx \text{ for all } n \in N.\}$
[5, p. 34, Corollary]

R_9 If L is an F loop, and $x, y, z \in L$,
(a) $(x, y, z) = (y, z, x) = (y, x, z^{-1})$
(b) $(x^n, y, z) = (x, y, z)^n$
[1, p. 125, Lemma 5.5]

R_{10} If L is an F loop of order $2^{\alpha_1}3^{\alpha_2}p_1^{\beta_1}\cdots p_n^{\beta_n}$ where $0 \leq \alpha_1 \leq 1$, $0 \leq \alpha_2 \leq 3$, $0 \leq \beta_i \leq 4$ and p_i are distinct primes ≥ 5, then L is a group [6, p. 81, Corollaries 2 and 3].

4. F loops of order $2^2p_1^{\alpha_1}\cdots {p_r}^{\alpha_r}$

Lemma 1. *Let L be an F loop of order $2^2p_1^{\alpha_1}\cdots p_r^{\alpha_r}$; p_i are distinct odd primes; $\alpha_i \leq 3$ if $p_i = 3$ and $\alpha_i \leq 4$ if $p_i \geq 5$. Suppose $\exists$ a maximal normal subloop K of L (In symbol, $K \triangleleft\cdot L$; it being understood that K is neither trivial nor the entire loop) such that K is associative. Then L is a group.*

Proof. Since K is a maximal normal subloop of L, L/K is simple. However, since $120 \nmid |L|$, $120 \nmid |L/K|$ and so by R_5 and R_6, L/K is a group and $L_a \subset K$.
Let $|L/K| = 2^k m$, $(2, m) = 1$.

Case 1: $k = 2$.

Then $|K|$ is odd. Let x be an element of L and $|x| = 2^\alpha$. By R_9, $(x,y,z)^{2^\alpha} = (x^{2^\alpha}, y, z) = 1$ for all $y, z \in L$. But the order of (x,y,z) is odd also since $(x,y,z) \in L_a \subset K$. Thus $(x,y,z) = 1$ and so all 2-elements lie in N. Then $K \triangleleft KN \triangleleft L$. So $L = KN$.
$L_a = (KN, KN, KN) = (K, K, K) = 1$, as K is associative.

Case 2: $k = 1$.

Then L/K is a group of order $2m$. But a group of order $2m$ has a normal subgroup L_0/K of order m. Then $K \triangleleft L_0 \triangleleft L$, a contradiction. Hence L is a group.

Case 3: $k = 0$.

Then L/K is a simple group of odd order and hence isomorphic to C_p where $p = p_j$ for some j. Let x, y, $z \in L$. Then x^p, y^p, $z^p \in K$. Using R_9 again, as K is a group,

$$(x^{p^3}, y, z) = (x, y, z)^{p^3} = (x^p, y^p, z^p) = 1$$

Thus $x^{p^3} \in N$ and L/N is a p-loop. As $|L/N| = p^\alpha \mid p_j^{\alpha_j}$, L/N is an Abelian group because of the restriction on α_j. So $L_a \subset N$. By R_8, as $L_a \subset C_L(N)$, L_a is an Abelian group. Also since $(x,y,z)^{p^3} = 1$ for all x, y, z in L, L_a is a p-group. Let P/L_a be a Sylow p-subgroup of L/L_a. Then $|P| = p_j^{\alpha_j}$. Since $|L| = |N|p^\alpha$ and $N \cap P$ is a p-group in N, $|N \cap P| \leq p^{\alpha_j - \alpha}$. Now

$$|PN| = \frac{|P|\ |N|}{|N \cap P|} \geq p_j^{\alpha_j} \frac{|L|}{p^\alpha p^{\alpha_j - \alpha}} = |L|.$$

So $L_a = (PN, PN, PN) = (P, P, P) = P_a = 1$ since a Moufang loop of this restricted order is a group. Hence L is a group.

Theorem 1. *Let L be an F loop of order $2^2 p_1^{\alpha_1} \ldots p_r^{\alpha_r}$; p_i are distinct odd primes; $\alpha_i \leq 3$ if $p_i = 3$ and $\alpha_i \leq 4$ if $p_i \geq 5$. Then L is a group.*

Proof. Suppose L is not associative. Since 120 is not a divisor of $|L|$, L is not simple. Let $L_1 \triangleleft \cdot L$. If L_1 is a group, then L would be a group by Lemma 1, a contradiction. If L_1 is not a group, since $120 \nmid |L_1|$, L_1 is not simple. Let $L_2 \triangleleft \cdot L_1$. In this manner, we have a series of subloops $L_{j+1} \triangleleft \cdot L_j \triangleleft \cdot \ldots L_2 \triangleleft \cdot L_1 \triangleleft \cdot L$ where L_i is nonassociative for $i \leq j$ and L_{j+1} is associative. (Note that 4 is a divisor of $|L_i|$ for the nonassociative loop L_i by R_{10}). This series terminates as $|L|$ is finite. Now by Lemma 1, L_j would be a group, a contradiction. So L must be a group.

5. F loops of order $2^3p^\alpha q^\beta$

Lemma 2. *Let L be an F loop, x a p-element and $x \in L - N$. Then $\exists$ a nonassociative subloop P of order p^m in L with $m \geq 4$ if $p = 2$ or 3 and $m \geq 5$ if $p \geq 5$.*

Proof. Since $x \notin N$, $\exists y, z \in L$ such that $(x, y, z) \neq 1$. Using R_9, we can assume the order of (x, y, z) is p^r for some r. Let $H = \langle x, y, z\rangle$. Then $H_a = \langle (x, y, z)\rangle = C_{p^r} \subset Z(H) \subset N(H) \subset H$.

Let $f, g, h \in H$.

Then $(f, g, h) = (x, y, z)^j$ for some j.

$$(f, g, h)^{p^r} = (x, y, z)^{jp^r} = 1.$$

So $(f^{p^r}, g, h) = 1$ or $f^{p^r} \in N(H)$ for all $f \in H$. Therefore $H/N(H)$ is a group of exponent dividing p^r. Let $|H/N(H)| = p^\theta$ and $|N(H)| = m_0p^\gamma, (m_0, p) = 1$.

Let P/H_a be a Sylow p-subgroup of H/H_a. As $|H/H_a| = \frac{|H|}{|H_a|} = \frac{m_0p^{\gamma+\theta}}{p^r} = m_0p^{\gamma+\theta-r}$, $|P/H_a| = p^{\gamma+\theta-r}$ or $|P| = p^{\gamma+\theta}$. Since $P \cap N(H)$ is a p-subgroup of $N(H)$, $|P \cap N(H)| \leq p^\gamma$. Then $|PN(H)| = \frac{|P|\ |N(H)|}{|P\cap N(H)|} \geq \frac{p^{\theta+\gamma}m_0p^\gamma}{p^\gamma} = p^{\theta+\gamma}m_0 = |H|$. Thus $PN(H) = H$.

$$H_a = (PN(H),\ PN(H),\ PN(H)) = (P, P, P) = P_a.$$

As $H_a \neq 1$, $P_a \neq 1$ and P is not associative.

As P is a nonassociative Moufang p-loop, $|P| = p^m$ with $m \geq 4$ if $p = 2$ or 3 by [3] and with $m \geq 5$ if $p \geq 5$, by [4].

Lemma 3. *If H and K are subloops of an F loop L with order m and n such that $(m, n) = 1$, then $|HK| = mn$.*

Proof. Suppose $x_1y_1 = x_2y_2$ with $x_i \in H$, $y_i \in K$. $\therefore x_1^{-1}(x_1y_1) = x_1^{-1}(x_2y_2)$.

$$y_1 = (x_1^{-1}x_2 \cdot y_2)(x_1^{-1}, x_2, y_2).$$

Let $a = (x_1^{-1}, x_2, y_2)$. Then $a^m = a^n = 1$ using R_9. As

$$(m, n) = 1, \quad a = 1.$$

So $y_1 = x_1^{-1}x_2 \cdot y_2 \therefore y_1y_2^{-1} = x_1^{-1}x_2 \in H \cap K$.

Since the order of an element divides the order of a diassociative loop [1, p. 92, Theorem 1.2], $H \cap K = \{1\}$.

So $y_1 = y_2$ and $x_1 = x_2$.

Lemma 4. *Let L be an F loop and p a prime, $p \nmid |L_a|$. Then*

(a) *x is a p-element $\implies x \in N$*

(b) *L/N is a group $\implies p \nmid |L/N|$*

Proof. Let $|x| = p^\alpha$ and $y, z \in L$. Then $(x, y, z)^{p^\alpha} = \left(x^{p^\alpha}, y, z\right) = (1, y, z) = 1$. But $(x, y, z)^{|L_a|} = 1$. Since $(p, |L_a|) = 1$, $(x, y, z) = 1$ and $x \in N$. Suppose L/N is a group and $p \mid |L/N|$. Let gN be an element of order p in L/N. Then $g^p \in N$ and $g \notin N$. Let $|g| = p^\beta m$ with $(p, m) = 1$. Then g^m is a p-element and hence $g^m \in N$. Since $(p, m) = 1$, $g \in N$. This is a contradiction. Hence $p \nmid |L/N|$.

Lemma 5. *Let L be an F loop of order $8p^\alpha$ with $\alpha \leq 3$ if $p = 3$ and $\alpha \leq 4$ if $p \geq 5$. Then L is a group.*

Proof. By [1, p. 92, Theorem 1.2], the order of each element of L divides $8p^\alpha$. If each of the elements of L has order a power of 2, then $|H|$ would be a power of 2 by [9, p. 415, Theorem]. On the other hand, if each of the elements of L has order a power of p, then $|H|$ would be a power of p by [8, p. 395, Theorem 1]. So, there exists 2-elements as well p-elements in L.

Case 1: Suppose $\exists$ a p-element y such that $y \notin N$. By Lemma 2, $\exists$ a nonassociative subloop P_p of order p^m with $m \geq \alpha + 1$.

1.1: Suppose $\exists$ 2-element x such that $x \in N$. By Lemma 2, $\exists$ a nonassociative subloop P_2 of order 2^k with $k \geq 4$. By Lemma 3, $|P_2 P_p| = 2^k p^m \geq 2^4 p^{\alpha+1} > 8p^\alpha = |L|$, a contradiction.

1.2: Suppose all 2-elements lie in N. Suppose $2 \mid |L/N|$. As L/N is group by Theorem 1, there exists $g \in L - N$ such that $g^2 \in N$. Let $|g| = 2^a p^b$. Then g^{p^b} is a 2-element. Thus $g^{p^b} \in N$. Since also $g^2 \in N$, we have $g \in N$, a contradiction. So $2 \nmid |L/N|$.

Then $2^3 \mid |N|$. Let P_2 be a Sylow 2-subgroup of the group N. Then $|P_2| = 8$. By Lemma 3, $|P_2 P_p| \geq 8p^{\alpha+1} > |L|$ a contradiction.

Case 2: Suppose all p-elements of L lie in N. We can similarly show that $p \nmid |L/N|$. Then $p^\alpha \mid |N|$. So, letting P_p be a Sylow p-subgroup of N, we have $|P_p| = p^\alpha$.

2.1: Suppose $\exists$ a 2-element x such that $x \notin N$.
By Lemma 2, $\exists$ a nonassociative subloop P_2 of order 2^k with $k \geq 4$.
By Lemma 3, $|P_2 P_p| = 2^k p^\alpha \geq 16p^\alpha > 8p^\alpha = |L|$, a contradiction.

2.2: Suppose all 2-elements of L lie in N.
Then clearly all elements of L lie in N. Hence $L = N$ is a group.

Lemma 6. *Let L be an F loop of order $2^3 \cdot 3 \cdot 5$. Then L is a group.*

Proof. *Case 1:* Suppose L has an element w of order 5.

1.1: Suppose $w \notin N$. By Lemma 2, $\exists$ a subloop P_5 with $|P_5| = 5^\alpha$, $\alpha \geq 5$. So $|P_5| \geq 5^5 > 2^3 \cdot 3 \cdot 5 = |L|$, a contradiction.

1.2: Suppose $w \in N$. Then L/N is a group by Lemma 5 and Theorem 1. So $L_a \subset N$.

(a) If $2 \nmid |L_a|$, then by Lemma 4, $|L/N| \mid 3$. So $L/N = \langle \bar{x} \rangle$ or $L = N\langle x \rangle$ for some $x \in L$. So L is a group by diassociativity.

(b) If $3 \nmid |L_a|$, then by Lemma 4, any 3-element, if such exists, lies in N. Suppose $3 \mid |L/N|$. Let $\bar{g}$ be an element of order 3 in L/N. Then $g^3 \in N$ but $g \in N$. Let $|g| = 3^\alpha m$, (3,m)=1. Then $|g^m| \mid 3^\alpha$. So $g^m \in N$. This implies $g \in N$, a contradiction. So $3 \nmid |L/N|$ and $|L/N| \mid 2^3$. If $|L/N| = 2^3$, then $2 \nmid |N|$. As $L_a \subset N$, $2 \nmid |L_a|$. By Lemma 4, $2 \nmid |L/N|$, a contradiction. Hence $|L/N| \leq 2^2$. So $L/N = \langle \bar{x}, \bar{y} \rangle$ or $L = N\langle x, y \rangle$ for some $x, y \in L$. Thus L is a group by diassociativity.

(c) We can assume $6 \mid |L_a|$. So $30 \mid |N|$. $|L/N| \mid 2^2$. Again L is a group by diassociativity.

Case 2: Suppose L has no element of order 5. Clearly $5 \nmid |N|$.
L must have an element u of order 3. Otherwise, L would be a 2-loop of order a power of two.

2.1: Suppose $u \notin N$. By Lemma 2, $\exists$ a subloop P_3 such that $|P_3| = 3^m$, $m \geq 4$. Let $v \in L - P_3$. Then by R_7, $|\langle v, P_3 \rangle| \geq 2 \cdot 3^m \geq 2 \cdot 3^4 > |L|$, a contradiction.

2.2: Suppose $u \in N$. So $|L/N| = 2^\alpha 5$, $\alpha \leq 3$. By Lemma 5, Theorem 1 and R_{10}, L/N is a group. Let $\bar{x}$ be an element of order 5 in L/N, ie. $x \in L - N$ and $x^5 \in N$. Then $x^{5|N|} = 1$ or $\left(x^{|N|}\right)^5 = 1$. As L has no element of order 5, $x^{|N|} = 1$. As $(5, |N|) = 1$, $x \in N$, a contradiction.

Lemma 7. *Let L be an F loop of order $2^3 \cdot 3^3 \cdot 5$. Then L is a group.*

PROOF. *Case 1:* Suppose L has an element x of order 5.

1.1: Suppose $x \notin N$. By Lemma 2, $\exists$ a subloop P_5 with $|P_5| \geq 5^5 > 2^3 \cdot 3^3 \cdot 5 = |L|$, a contradiction.

1.2: Suppose $x \in N$. So $|L/N| \mid 2^3 \cdot 3^3$. By Lemma 5, L/N is a group. Thus $L_a \subset N$.

1.2(a) If there exist both 2-elements and 3-elements in $L - N$, then by Lemma 2, $\exists$ subloops P_2 and P_3 with $|P_2| \geq 2^4$ and $|P_3| \geq 3^4$. By Lemma 3, $|P_2P_3| \geq 2^4 \cdot 3^4 > 2^3 3^3 5 = |L|$, a contradiction.

1.2(b) If all the 2-elements lie in N, then $|L/N| \mid 3^3$. If $|L/N| \mid 3^2$, then L is a group by disassociativity. So we assume $|L/N| = 3^3$. Then $3 \nmid |N|$. In other words, $3 \nmid |L_a|$. As in the case 1.2(b), we can use Lemma 4 to show that $3 \nmid |L/N|$. This is a contradiction.

1.2(c) If all the 3-elements lie in N, we obtain a contradiction by a similar argument.

Case 2: Suppose L has no element of order 5. Clearly L has p-elements for $p = 2$ and $p = 3$.

2.1: Suppose it has a 2-element as well as a 3-element lying in $L-N$. By Lemma 2, $\exists$ subloops P_2 and P_3 with orders 2^α and 3^β respectively, α, $\beta \geq 4$. By Lemma 3, $|P_2P_3| = 2^\alpha 3^\beta \geq 2^4 3^4 > |L|$, a contradiction.

2.2: Suppose all the 2-elements of L lie in N. It can be seen easily that $2 \nmid |L/N|$. Clearly $5 \nmid |N|$. So $|L/N| = 3^\gamma 5$, $\gamma \leq 3$.
Applying R_{10}, L/N is a group. Let $\bar{x}$ be an element of order 5. Then $x \in L-N$ and $x^5 \in N$. So $x^{5|N|} = 1$ or $\left(x^{|N|}\right)^5 = 1$. Since L has no element of order 5, $x^{|N|} = 1$. But $(5, |N|) = 1$. So $x \in N$, a contradiction.

2.3: If all the 3-elements of L lie in N, a contradiction arises in a similar way by applying Lemma 5.

Lemma 8. *Let L be a nonassociative F loop of order $2^3p^\alpha q^\beta$; p and q distinct primes with $p < q$; $\alpha \leq 3$ if $p = 3$ and $\alpha \leq 4$ if $p \geq 5$; $\beta \leq 4$. Then L is nonsimple.*

Proof. Suppose L is simple. *By* R_5, L is isomorphic to one of the $M(r^n)$. But $|M(r^n)| = 2^3p^\alpha q^\beta$ with p, q, α, β as specified if and only if $n = 1$, $r = 2$ or 3, but $|M(2)| = 2^3 \cdot 3 \cdot 5$ and $|M(3)| = 2^3 \cdot 3^3 \cdot 5$. By Lemma 6 and Lemma 7, we have a contradiction.

Lemma 9. *Let L be an F loop of order $2^3p^\alpha q^\beta$ defined as above. Then L has p-elements (as well as q-elements).*

Proof. If L is associative, then the result follows by Sylow theory. Suppose L is not associative. By Lemma 8, L is not simple. Let $L_1 \triangleleft \cdot L$. Suppose $2 \mid |L_1|$. Then L/L_1 is a group by Theorem 1 and R_{10}. Suppose $2 \nmid |L_1|$. If L/L_1 is nonassociative, then L/L_1 is nonsimple by Lemma 8. But this contradicts the maximality of L_1. In any case, L/L_1 is a group. In fact, it is a simple group. Moreover if L_1 is nonassociative, then $|L_1| = 2^3p^{\alpha_1}q^\beta$ by Theorem 1 and R_{10}. So $|L/L_1| = p^{\alpha_0}q^{\beta_0}$. But a simple group of this odd order is isomorphic to C_p or C_q.

Now suppose L_1 is nonassociative. By a similar argument, we have $L_2 \triangleleft \cdot L_1$ with L_1/L_2 a simple group. Continuing, we have a series of subgroups

$$L_{m+1} \triangleleft \cdot L_m \triangleleft \cdot \cdots L_2 \triangleleft \cdot L_1 \triangleleft \cdot L_0 = L \text{ such that}$$

(a) L_i/L_{i+1} is a simple cyclic group for $0 \leq i \leq m$
(b) L_i is nonassociative for $0 \leq i \leq m$
(c) L_{m+1} is a group.

If $p \mid |L_{m+1}|$, we are through by (c).
Otherwise, let j be the smallest integer such that $p \nmid |L_{j+1}|$ but $p \mid |L_j|$. Then $|L_j/L_{j+1}| = p$ by (a).

Let $x \in L_j - L_{j+1}$. Then $x^p \in L_{j+1}$. Write $|L_{j+1}| = \ell$. $(x^p)^\ell = 1$ ie. $(x^\ell)^p = 1$. If $x^\ell \neq 1$, then we are through. If $x^\ell = 1$, as $(p, \ell) = 1$, $x \in L_{j+1}$, a contradiction.

Lemma 10. *Let L be an F loop or order $2^3p^\alpha q^\beta$ defined as above. Suppose $L_a \subset N$ with $|L_a| = 2^k m$, $k \geq 1$ and m odd. Then L is a group.*

Proof. *Case 1*: $m = 1$

As $pq \nmid |L_a|$, $pq \nmid |L/N|$ by Lemma 4. So $|L/N| \mid 2^2$. Thus $L/N = \langle \bar{x}, \bar{y} \rangle$ or $L = N\langle x, y\rangle$ for some $x, y \in N$. So L is a group by diassociativity.

Case 2: $m > 1$

By R_8, $L_a \subset C_L(N)$. So $L_a \subset Z(N)$, the centre of N. Let K be a subloop of order 2^k in L_a. As L_a is an abelian group and $L_a \triangleleft L$, $K \triangleleft L$. L/K is a group by Theorem 1 and R_{10}. Thus $L_a \subset K$. Hence $|L_a| = |K| = 2^k$, a contradiction.

Theorem 2. *Let L be an F loop of order $2^3p^\alpha q^\beta$ defined az above. Then L is a group.*

Proof. By Lemma 9, $\exists$ both p-elements and q-elements in L.

Case 1: Suppose $\exists$ both p-elements and q-elements in $L - N$.
By Lemma 2, $\exists$ nonassociative subloops P and Q such that $|P| \geq p^{\alpha+1}$ and $|Q| \geq q^{\beta+1}$.
By Lemma 3, $|PQ| \geq p^{\alpha+1}q^{\beta+1}2^3p^\alpha q^\beta = |L|$, a contradiction.

Case 2: Suppose all p-elements lie in N and some q-element lies in $L - N$. Then $p^\alpha \mid |N|$. Also, by Lemma 2, $\exists$ a q-subloop Q of L, such that $|Q| \geq q^{\beta+1}$. By Lemma 5 and R_{10}, L/N is a group and $L_a \subset N$. By Lemma 10, we may assume that $|L_a|$ is odd. As $2 \nmid |L_a|$, $2^3 \mid |N|$ by Lemma 4. So $|N| = 2^3p^\alpha$. Then, since $N \cap Q = \{1\}$, $|NQ| = \frac{|N|\,|Q|}{|N \cap Q|} \geq 2^3p^\alpha q^{\beta+1} > |L|$, a contradiction.

Case 3: Suppose all q-elements lie in N and some p-element lies in $L - N$. A contradiction arises just as in case 2.

Case 4: Suppose all p-elements and q-elements lie in N.
Then $|L/N| \mid 2^3$
If $|L/N| \leq 2^2$, then $L/N = \langle \bar{x}, \bar{y} \rangle$ or $L = N\langle x, y\rangle$ for some $x, y \in L$ and L is a group by diassociativity. If $|L/N| = 2^3$, then $2 \nmid |N|$. Therefore $2 \nmid |L_a|$ because $L_a \subset N$ and by Lemma 4, $2 \nmid |L/N|$, a contradiction.

References

[1] R. H. Bruck, A Survey of Binary System, *Springer Verlag*, 1971.
[2] R. H. Bruck, Contribution to the Theory of Loops, *Trans. Amer. Math. Soc.* **60** (1946), 245–253.
[3] Orion Chein, Moufang Loops of Small Order I, *Trans. Amer. Math. Soc.* **188** (1974), 31–51.
[4] Leong Fook, Moufang Loops of Order p^4, *Nanta Mathematica* **VII** (1974), 33–34.
[5] Leong Fook, The Devil and Angel of Loops, *Proceedings of the A.M.S.* **54** (1976), 32–34.
[6] Leong Fook and Teh Pang Eng, Which F Loops are Groups?, *Bulletin of Malaysian Math. Soc* **14** (1991), 79–82.

[7] LEONG FOOK and TEH PANG ENG, Moufang Loops of Order $2pq$, *Bulletin of Malaysian Math. Soc.* **15** (1992).
[8] G. GLAUBERMAN, On Loops of Odd Order II, *J. of Algebra* **8** (1968), 393–414.
[9] G. GLAUBERMAN and C.R.B. WRIGHT, Nilpotency of Finite Moufang 2-Loops, *J. of Algebra* **8** (1968), 415–417.
[10] M.W. LIEBECK, The Classification of Finite Simple Moufang Loops, *Math. Proc. Camb. Phil. Soc.* **102** (1987), 33–47.
[11] L. J. PAIGE, A Class of Simple Moufang Loops, *Proc. Amer. Math. Soc.* **7** (1956), 471–482.
[12] MARK PURTILL, On Moufang Loops of Order the Product of three Odd Primes, *J. of Algebra* **112** (1988), 122–128.
[13] MARK PURTILL, Moufang Loops of Even Order p^2q (*to appear*).
[14] TEH PANG ENG, The Orders of Nonassociative Simple Moufang Loops, Master thesis, *Universiti Sains Malaysia*, 1992.
[15] C.R.B. WRIGHT, Nilpotency conditions for Finite Loops, *Illinois J. Math.* **9** (1965), 399–409.

LEONG FOOK
UNIVERSITY SAINS MALAYSIA
11800 PENANG
MALAYSIA

TEH PANG ENG
UNIVERSITY SAINS MALAYSIA
11800 PENANG
MALAYSIA

(Received December 27, 1991; revised January 18, 1993)