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Finsler conformal transformations and the curvature invariances

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Abstract. This article studies the global conformal transformations f on a Finsler space (M,F), which satisfy $f^*F = e^{c(x)}F$, where $F := F(x,y)$ is a Finsler metric on M and $x \in M$, $y \in T_xM \setminus \{0\}$. We obtain the relations between some important geometric quantities of F and their correspondences respectively, including Riemann curvatures, Ricci curvatures, Landsberg curvatures, mean Landsberg curvatures and **S**-curvatures. Then, we discuss the properties of those conformal transformations on (M,F) which preserve Ricci curvature, Landsberg curvature, mean Landsberg curvature and **S**-curvature respectively.

1. Introduction

Let F be a Finsler metric on an n-dimensional manifold M. For a non-zero vector $y \in T_xM$, F induces an inner product g_y on T_xM by

$$g_y(u,v) := g_{ij}(x,y)u^iv^j = \frac{1}{2}[F^2]_{y^iy^j}u^iv^j.$$

For two arbitrary non-zero vectors $v, y \in T_xM$, the angle $\theta(y,v)$ between y and v is defined by

$$\cos\theta(y,v) := y_iv^i/F(x,y)\sqrt{g_{ij}(x,y)v^iv^j}, \tag{1}$$

where $y_i := g_{ij}(x,y)y^j$. It should be remarked that this notion of angle is not symmetric, that is, the angle $\theta(y,v)$ between y and v is different from the angle

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$\theta(v, y)$ between v and y generally. According to the above notion of angle, we have the following

Definition 1.1. Let F and $\bar{F}$ be two Finsler metrics on an n-dimensional manifold M. If the angle $\theta(y, v)$ with respect to F is equal to the angle $\bar{\theta}(y, v)$ with respect to $\bar{F}$ for any vectors $y, v \in T_xM \setminus \{0\}$ and any $x \in M$, then F is called conformal to $\bar{F}$ and the transformation $F \to \bar{F}$ of the metric is called a *conformal transformation.*

From the definition above, we can prove the following fundamental theorem.

Theorem 1.1 ([AIM]). *Let F and $\bar{F}$ be two Finsler metrics on an n-dimensional manifold M. Then F is conformal to $\bar{F}$ if and only if there exists a scalar function $c(x)$ such that*

$$\bar{F}(x, y) = e^{c(x)} F(x, y). \tag{2}$$

The scalar function $c(x)$ is called the conformal factor.

From (2), we can easily obtain the following

Lemma 1.1 ([Ma]). *Let F and $\bar{F}$ be two Finsler metrics on an n-dimensional manifold M. If $\bar{F}(x, y) = e^{c(x)} F(x, y)$, then*

(a) $\bar{g}_{ij}(x, y) = e^{2c(x)} g_{ij}(x, y)$, $\bar{g}^{ij}(x, y) = e^{-2c(x)} g^{ij}(x, y)$, *where* $(g^{ij}) = (g_{ij})^{-1}$.

(b) $\bar{h}_{ij}(x, y) = e^{2c(x)} h_{ij}(x, y)$, *where* $h_{ij} := g_{ij} - F_{y^i} F_{y^j}$ *is called the* angular metric tensor *of* F.

(c) $\bar{y}_k = e^{2c(x)} y_k$.

(d) $\bar{C}_{ijk} = e^{2c(x)} C_{ijk}(x, y)$, *where* C_{ijk} *is the Cartan torsion of* F.

(e) $\bar{C}^j_{ik}(x, y) = C^j_{ik}(x, y)$, $\bar{I}_k(x, y) = I_k(x, y)$, *where* $C^j_{ik} := g^{jl} C_{lik}$ *and* $I_k := g^{ij} C_{ijk}$ *is the mean Cartan torsion of* F.

From (e) in Lemma 1.1, we know that C^j_{ik} and the mean Cartan torsion I_k are invariant under conformal transformation. Further, write $\|\mathbf{I}\|^2 := g^{ij} I_i I_j$ and $\mathbf{T}(x, y) := F^2 \|\mathbf{I}\|^2$, then by Lemma 1.1 we have the following

Lemma 1.2 ([Ma]). $\mathbf{T}(x, y)$ *is conformally invariant.*

The conformal properties of a Finsler metric deserve extra attention. The Weyl theorem states that the projective and conformal properties of a Finsler metric determine the metric properties uniquely [SV]. In conformal geometry, we naturally want to know the relations between some important geometric quantities and their correspondences. At the same time, we also want to know that, if a conformal transformation preserves some geometric quantities, then what properties does it have? For example, in Riemann conformal geometry, an interesting problem is to study the so-called Liouville transformation, that is a conformal transformation satisfying $\overline{\mathbf{Ric}} = \mathbf{Ric}$ [KR]. In this article, we will discuss the problem above in Finsler conformal geometry for Riemann curvature, Ricci curvature, Landsberg curvature, mean Landsberg curvature and **S**-curvature.

2. Curvatures

Let F be a Finsler metric on an n-dimensional manifold M. The geodesics of F are characterized by

$$\frac{d^2c^i}{dt^2} + 2G^i(c(t), \dot{c}(t)) = 0,$$

where $G^i := \frac{1}{2}g^{il}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\}$ are called the *geodesic coefficients* of F. The *Riemann curvature* $\mathbf{R}_y = R^i_k dx^k \otimes \frac{\partial}{\partial x^i}|_p : T_pM \to T_pM$ is a family of linear transformations on tangent spaces, which is defined by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{3}$$

For a two-dimensional plane $P \subset T_pM$ and $y \in T_pM \setminus \{0\}$ such that $P = \mathrm{span}\{y, u\}$, the pair $\{P, y\}$ is called a *flag* in T_pM. The *flag curvature* $\mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{g_y(u, \mathbf{R}_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}. \tag{4}$$

We say that F is of *scalar curvature* if for any $y \in T_pM \setminus \{0\}$ the flag curvature $\mathbf{K}(P, y) = \lambda(y)$ is independent of P containing y. This is equivalent to the following system in a local coordinate system (x^i, y^i) in TM:

$$R^i_k = \lambda F^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}.$$

If λ is a constant, then F is said to be of *constant curvature.*

The trace of the Riemann curvature

$$\mathbf{Ric}(y) := (n-1)R(y) = R^m_m(y)$$

is called the *Ricci curvature* and $R(y) := [1/(n-1)]\mathbf{Ric}(y)$ is called the *Ricci-scalar.*

There are many interesting non-Riemannian quantities in Finsler geometry. For a non-zero vector $y \in T_pM$, the *mean Berwald curvature* $\mathbf{E}_y = E_{ij}dx^i \otimes dx^j : T_pM \otimes T_pM \to R$ is defined by

$$E_{ij} := \frac{1}{2}\frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^m}(x, y). \tag{5}$$

The mean Berwald curvature $\mathbf{E} = \{\mathbf{E}_y\}$ is a symmetric bilinear form on T_pM. We say that F has isotropic mean Berwald curvature if

$$\mathbf{E}_y(u, v) = (n + 1)cF^{-1}(x, y)h_y(u, v), \tag{6}$$

or equivalently,

$$E_{ij} = (n + 1)cF_{y^iy^j},$$

where $c = c(x)$ is a scalar function on M.

For a non-zero vector $y \in T_pM$, the *Landsberg curvature* $\mathbf{L}_y = L_{ijk}(x, y)dx^i \otimes dx^j \otimes dx^k$ is defined by

$$L_{ijk}(x, y) := -\frac{1}{2}y^m g_{ml}\frac{\partial^3 G^l}{\partial y^i \partial y^j \partial y^k}(x, y). \tag{7}$$

The following lemma is useful.

Lemma 2.1 ([AIM], [Sh]). *The Landsberg curvature coefficients L_{ijk} are given by the expressions*

$$L_{ijk} = -\frac{1}{2}g_{ij;k} = C_{ijk;m}y^m, \tag{8}$$

where “;” *denotes the Berwald covariant derivative determined by F.*

Further, the *mean Landsberg curvature* $\mathbf{J}_y = J_i(x, y)dx^i : T_pM \to R$ is defined by

$$J_i(x, y) := g^{jk}L_{ijk}.$$

It is easy to show that([Sh])

$$J_i = I_{i;k}y^k, \quad E_{ij} = \frac{1}{2}\{I_{j;i} + J_{i\cdot j}\}. \tag{9}$$

Express the volume form of F by

$$dV_F = \sigma(x)dx^1 \cdots dx^n.$$

For a non-zero vector $y \in T_pM$, the *S-curvature* $\mathbf{S}(y)$ is defined by

$$\mathbf{S}(y) := \frac{\partial G^i}{\partial y^i}(x,y) - \frac{y^i}{\sigma(x)}\frac{\partial \sigma}{\partial x^i}(x). \tag{10}$$

From the definition, we have

$$E_{ij} = \frac{1}{2}\mathbf{S}_{y^i y^j}. \tag{11}$$

We say that F is of isotropic S-curvature if

$$\mathbf{S} = (n+1)cF, \tag{12}$$

where $c = c(x)$ is a scalar function on M.

3. $\overline{\mathbf{Ric}} = \mathbf{Ric}$

Let $\bar{F}$ and F be two Finsler metrics on an n-dimensional manifold M. We have a relation between the geodesic coefficients $\bar{G}^i$ and G^i as follows:

$$\bar{G}^i = G^i + \frac{\bar{F}_{;k}y^k}{2\bar{F}}y^i + \frac{\bar{F}}{2}\bar{g}^{il}\left\{\bar{F}_{;k\cdot l}y^k - \bar{F}_{;l}\right\}. \tag{13}$$

If $\bar{F} = e^{c(x)}F$, then $\bar{F}_{;k} = e^{c(x)}c_kF$, where $c_k := \partial c/\partial x^k$. From Lemma 1.1 we have

$$\begin{aligned}\bar{G}^i &= G^i + \frac{1}{2}(c_ky^k)y^i + \frac{F}{2}g^{il}\{(c_ky^k)F_{y^l} - c_lF\}\\ &= G^i + (c_ky^k)y^i - \frac{F^2}{2}c^i,\end{aligned}$$

where $c^i = g^{il}c_l$. Write

$$\bar{G}^i = G^i + Py^i - Q^i, \tag{14}$$

where

$$P := c_ky^k, \quad Q^i := \frac{F^2}{2}c^i.$$

From (14) we have

$$\bar{G}^i_j = G^i_j + P_jy^i + P\delta^i_j - Q^i_j, \tag{15}$$

$$\bar{G}^i_{jk} = G^i_{jk} + P_j\delta^i_k + P_k\delta^i_j - Q^i_{jk}, \tag{16}$$

where $G^i_j := \partial G^i/\partial y^j$, $G^i_{jk} := \partial G^i_j/\partial y^k$ and $P_j := \partial P/\partial y^j$, $Q^i_j := \partial Q^i/\partial y^j$, $Q^i_{jk} := \partial Q^i_j/\partial y^k$, etc. Substituting (14), (15), (16) into (3) and using the homogeneities of P and Q^i, we have

$$\bar{R}^i_k = R^i_k + \Xi\delta^i_k + \tau_k y^i - 2Q^i_{;k} + y^j Q^i_{k;j} - 2P_j Q^j \delta^i_k + 2Q^j Q^i_{jk} - Q^i_j Q^j_k, \qquad (17)$$

where

$$\Xi := P^2 - P_{;r}y^r, \quad \tau_k := 3(P_{;k} - PP_k) + \Xi_{\cdot k} + P_j Q^j_k.$$

It is easy to see that

$$\frac{\partial g^{ij}}{\partial y^k} = -2g^{ir}g^{js}C_{krs}, \quad \frac{\partial c^i}{\partial y^j} = -2c^r C^i_{jr}. \qquad (18)$$

From (18) we get

$$P_j Q^j_k = y_k \|\nabla c\|^2_F - F^2 c^i c^j C_{ijk},$$

where $\|\nabla c\|^2_F := c_j c^j = g^{ij}c_i c_j$. Similarly, $P_j Q^j = \frac{F^2}{2}\|\nabla c\|^2_F$. Hence we have

$$\bar{R}^i_k = R^i_k + \Xi\delta^i_k + \tau_k y^i - 2Q^i_{;k} + y^j Q^i_{k;j} + 2Q^j Q^i_{jk} - Q^i_j Q^j_k - F^2\|\nabla c\|^2_F \delta^i_k \qquad (19)$$

and

$$\Xi = P^2 - \Phi, \quad \tau_k = 3(P_{;k} - PP_k) + \Xi_{\cdot k} + y_k\|\nabla c\|^2_F - F^2 c^i c^j C_{ijk}, \qquad (20)$$

where $\Phi := c_{i;j}y^i y^j$. Further, it is easy to see that

$$g^{ij}_{;k} = 2g^{ir}L^j_{rk}$$

and

$$\begin{aligned} c^i_{;k} &= 2c^r L^i_{rk} + g^{ir}c_{r;k}, \\ Q^i_{;k} &= \frac{F^2}{2}c^i_{;k}, \\ Q^i_j &= y_j c^i - F^2 c^r C^i_{jr}, \\ Q^i_{j;k} &= y_j c^i_{;k} - F^2 c^r_{;k} C^i_{jr} - F^2 c^r C^i_{jr;k}, \\ Q^i_{jk} &= g_{jk}c^i - 2y_j c^r C^i_{rk} - 2y_k c^r C^i_{rj} + 2F^2 c^r C^s_{kr} C^i_{js} - F^2 c^r C^i_{jr\cdot k}. \end{aligned} \qquad (21)$$

Now

$$\begin{aligned} y^j Q^i_{k;j} &= y_k c^i_{;0} - F^2 c^r_{;0} C^i_{kr} - F^2 c^r C^i_{kr;0}, \\ 2Q^j Q^i_{jk} &= F^2(c_k c^i - 2c_0 c^r C^i_{kr} - 2y_k c^j c^r C^i_{jr} \\ &\quad + 2F^2 c^j c^r C^s_{kr} C^i_{js} - F^2 c^j c^r C^i_{jr\cdot k}), \\ Q^i_j Q^j_k &= c_0 c^i y_k - F^2 y_k c^j c^r C^i_{jr} + F^4 c^r c^s C^j_{kr} C^i_{js}. \end{aligned} \qquad (22)$$

From (19) and (20) we get

$$\begin{aligned}\overline{\mathbf{Ric}}(y) &= \mathbf{Ric}(y) + (n-1)\left(\Xi - F^2\|\nabla c\|_F^2\right)\\ &\quad - 2Q^k_{;k} + y^j Q^k_{k;j} + 2Q^j Q^k_{jk} - Q^k_j Q^j_k. \end{aligned} \tag{23}$$

From (21) and (22) we have

$$\begin{aligned}\overline{\mathbf{Ric}}(y) &= \mathbf{Ric}(y) + (n-2)\left(\Xi - F^2\|\nabla c\|_F^2\right)\\ &\quad - 2F^2(c^r J_r) - F^2 g^{ij} c_{i;j} - F^2(c^r I_r)_{;0} - 2F^2 c_0 (c^r I_r)\\ &\quad + 2F^4 I_r c^j c^k C^r_{jk} - F^4 c^j c^k I_{j\cdot k} - F^4 c^j c^k C^s_{jr} C^r_{ks}. \end{aligned} \tag{24}$$

By (19) we have the following

Theorem 3.1. *Let F and $\bar F$ be two Finsler metrics on an n-dimensional manifold M. If $\bar F(x,y) = e^{c(x)}F(x,y)$, then $\bar R^i_k = R^i_k$ if and only if the following equation holds:*

$$\Xi\delta^i_k + \tau_k y^i - 2Q^i_{;k} + y^j Q^i_{k;j} + 2Q^j Q^i_{jk} - Q^i_j Q^j_k - F^2\|\nabla c\|_F^2 \delta^i_k = 0. \tag{25}$$

Further, when (25) holds, then F is of scalar curvature $\lambda(y)$ if and only if $\bar F$ is of scalar curvature $\bar\lambda(y)$ and

$$\bar\lambda = \lambda / e^{2c(x)}.$$

Proof. We only need to prove the second conclusion. From $\bar R^i_k = R^i_k$ and Lemma 1.1(c), we can see that, if F is of scalar curvature $\lambda(y)$, then

$$\begin{aligned}\bar R^i_k &= \lambda F^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}\\ &= \lambda e^{-2c(x)}\bar F^2\{\delta^i_k - \bar F^{-1}\bar F_{y^k}y^i\}.\end{aligned}$$

Clearly, $\bar F$ is of scalar curvature $\bar\lambda(y) = \lambda(y)e^{-2c(x)}$. The converse holds obviously. □

From Theorem 3.1, we have the following

Corollary 3.1. *Let F and $\bar F$ be two Finsler metrics on an n-dimensional manifold M. If $\bar F = e^c F(x,y)$ where $c = \text{constant}(\neq 0)$ (that is, the conformal transformation is a homothety), then F is of scalar curvature $\lambda(y)$ if and only if $\bar F$ is of scalar curvature $\bar\lambda(y)$ and $\bar\lambda = \lambda/e^{2c}$.*

Proof. As $c = \text{constant}$, (25) becomes trivial. So, the Corollary follows from Theorem 3.1. □

By (23) we have the following

Theorem 3.2. *Let F and $\bar{F}$ be two Finsler metrics on an n-dimensional manifold M. If $\bar{F}(x,y) = e^{c(x)}F(x,y)$, then $\overline{\mathbf{Ric}}(y) = \mathbf{Ric}(y)$ if and only if the following equation holds:*

$$\Xi = F^2\|\nabla c\|_F^2 + (2Q^k_{;k} - y^j Q^k_{k;j} - 2Q^j Q^k_{jk} + Q^k_j Q^j_k)/(n-1). \tag{26}$$

In particular, if $c(x) = \text{constant}$, then $\overline{\mathbf{Ric}} = \mathbf{Ric}$.

Remark 3.1. In Riemann conformal geometry, a conformal transformation satisfying $\overline{\mathbf{Ric}}(y) = \mathbf{Ric}(y)$ is called a *Liouville transformation*. A globally defined Liouville transformation is a homothety [KR]. A natural problem arises: in Finsler conformal geometry, is this statement still true? This problem is still open.

If $c(x) = \text{constant}$ then, by (14), the conformal transformation $\bar{F} = e^c F$ preserves the geodesics. Inversely, is a conformal transformation a homothety if it preserves the geodesics? We have the following

Theorem 3.3. *Let F and $\bar{F}$ be two Finsler metrics on an n-dimensional manifold M. If a conformal transformation $\bar{F} = e^{c(x)}F$ preserves the geodesics, then it must be a homothety, that is $c = \text{constant}$.*

Proof. Since the conformal transformation $\bar{F} = e^{c(x)}F$ preserves the geodesics, we have

$$\bar{G}^i = G^i + p(x,y)y^i, \quad p(x,\lambda y) = \lambda p(x,y) \quad \forall \lambda > 0. \tag{27}$$

From (14) and (27) we get

$$py^i = c_0 y^i - Q^i, \tag{28}$$

where $c_0 = c_i y^i$. Contracting (28) with y_i yields

$$pF^2 = c_0 F^2 - \frac{F^2}{2}c_0 = \frac{F^2}{2}c_0.$$

Hence, $p = \frac{1}{2}c_0$, and then $Q^i = \frac{1}{2}c_0 y^i$. Further, we have

$$\frac{1}{2}c_0 y^i = \frac{F^2}{2}c^i. \tag{29}$$

Contracting (29) with c_i yields $c_0^2 = F^2\|\nabla c\|^2$. From this, we have $\|\nabla c\|^2 = 0$. Or else, we know that $Rank(g_{ij}) \leq 1$, which is a contradiction. Furthermore, $c_r = 0$ because (g^{ij}) is positive definite, which implies that $c = \text{constant}$. □

4. The Landsberg curvatures

By Lemma 2.1, the Landsberg curvature coefficients $\bar{L}_{ijk}$ of $\bar{F}$ are given by

$$\bar{L}_{ijk} = -\frac{1}{2}\bar{g}_{ij|k},$$

where "|" denotes the Berwald covariant derivative determined by $\bar{F}$. If $\bar{F} = e^{c(x)}F$, $\bar{g}_{ij} = e^{2c(x)}g_{ij}$, we have

$$\begin{aligned}\bar{L}_{ijk} &= -\frac{1}{2}(e^{2c(x)}g_{ij})_{|k} \\ &= -e^{2c(x)}c_k g_{ij} - \frac{1}{2}e^{2c(x)}g_{ij|k}. \end{aligned} \tag{30}$$

From (14), (15) and (16) we have

$$\begin{aligned} g_{ij|k} &= \frac{\partial g_{ij}}{\partial x^k} - 2\bar{G}^r_k C_{ijr} - g_{ir}\bar{G}^r_{jk} - g_{rj}\bar{G}^r_{ik} \\ &= \frac{\partial g_{ij}}{\partial x^k} - 2(G^r_k C_{ijr} + PC_{ijk} - Q^r_k C_{ijr}) - (g_{ir}G^r_{jk} + P_j g_{ik} \\ &\quad + P_k g_{ij} - g_{ir}Q^r_{jk}) - (g_{rj}G^r_{ik} + P_i g_{jk} + P_k g_{ij} - g_{rj}Q^r_{ik}) \\ &= g_{ij;k} - 2(PC_{ijk} - Q^r_k C_{ijr} + P_k g_{ij}) - (P_j g_{ik} + P_i g_{jk}) \\ &\quad + (g_{ir}Q^r_{jk} + g_{rj}Q^r_{ik}). \end{aligned}$$

Substituting these into (30), we get

$$\begin{aligned} \bar{L}_{ijk} &= e^{2c(x)}L_{ijk} + e^{2c(x)}(PC_{ijk} - Q^r_k C_{ijr}) \\ &\quad + \frac{1}{2}e^{2c(x)}\left\{(P_j g_{ik} + P_i g_{jk}) - (g_{ir}Q^r_{jk} + g_{rj}Q^r_{ik})\right\}. \end{aligned} \tag{31}$$

Further, the mean Landsberg curvature $\bar{\mathbf{J}}$ is determined by

$$\bar{J}_i = \bar{g}^{jk}\bar{L}_{ijk} = J_i + (PI_i - Q^r_k C^k_{ir}) + \frac{n+1}{2}P_i - \frac{g_{ir}g^{jk}Q^r_{jk} + Q^k_{ik}}{2}. \tag{32}$$

By (21), we have

$$\begin{aligned} \bar{L}_{ijk} &= e^{2c(x)}L_{ijk} + e^{2c(x)}\{PC_{ijk} + c^r(y_i C_{rjk} + y_j C_{irk} + y_k C_{ijr}) \\ &\quad - F^2 c^r(C^s_{ir}C_{sjk} + C^s_{jr}C_{isk} + C^s_{kr}C_{ijs}) + F^2 c^s C_{ijk\cdot s}\}, \end{aligned} \tag{33}$$

$$\bar{J}_i = J_i + PI_i + F^2 c^r I_{i\cdot r} + y_i(c^r I_r) - F^2 c^s I_r C^r_{is}. \tag{34}$$

From (34) we obtain the following

Theorem 4.1. *Let F and $\bar{F}$ be two Finsler metrics on an n-dimensional manifold M. If $\bar{F}(x,y) = e^{c(x)}F(x,y)$, then the conformal transformation preserves the mean Landsberg curvature if and only if the conformal factor $c(x)$ satisfies the following equations:*

$$PI_i + F^2 c^r I_{i\cdot r} + y_i(c^r I_r) - F^2 c^s I_r C^r_{is} = 0. \tag{35}$$

In particular, if $c(x) = \text{constant}$, then $\bar{\mathbf{J}} = \mathbf{J}$.

5. *S*-curvature

Let $\bar{F}(x,y) = e^{c(x)}F(x,y)$. The Busemann–Hausdorff volume forms $d\mu_F$ and $d\mu_{\bar{F}}$ are defined by

$$d\mu_F := \sigma_F(x)\omega^1 \wedge \cdots \wedge \omega^n,$$

$$d\mu_{\bar{F}} := \sigma_{\bar{F}}(x)\omega^1 \wedge \cdots \wedge \omega^n,$$

where

$$\sigma_F(x) := \frac{\omega_n}{\text{EuclideanVol}(B^n_x)}, \quad \sigma_{\bar{F}}(x) := \frac{\omega_n}{\text{EuclideanVol}(\bar{B}^n_x)}$$

and

$$B^n_x := \left\{(y^i) \in R^n, F(y^i e_i) < 1\right\},$$

$$\bar{B}^n_x := \left\{(y^i) \in R^n, \bar{F}(y^i e_i) < 1\right\}.$$

We have

$$\text{EuclideanVol}(\bar{B}^n_x) = \int_{\bar{B}^n_x} dy^1 \cdots dy^n. \tag{36}$$

Pay attention to $\bar{B}^n_x = \{(y^i) \in R^n, F(e^{c(x)}y^i e_i) < 1\}$. Let $z^i = e^{c(x)}y^i$ in the integral (36), we get

$$\text{EuclideanVol}(\bar{B}^n_x) = \int_{B^n_x} e^{-nc(x)} dz^1 \cdots dz^n = e^{-nc(x)}\text{EuclideanVol}(B^n_x).$$

Hence, we have

$$\sigma_{\bar{F}}(x) = e^{nc(x)}\sigma_F(x). \tag{37}$$

By (15) and $Q^i_j = y_j c^i - F^2 c^r C^i_{jr}$, we obtain

$$\begin{aligned}\bar{G}^m_m &= G^m_m + (n+1)P - Q^m_m \\ &= G^m_m + (n+1)P - (y_m c^m - F^2 c^r I_r) \\ &= G^m_m + nP + F^2 c^r I_r.\end{aligned}$$

On the other hand, we have

$$\frac{y^m}{\sigma_{\bar{F}}(x)}\frac{\partial \sigma_{\bar{F}}}{\partial x^m} = nc_m y^m + \frac{y^m}{\sigma_F(x)}\frac{\partial \sigma_F}{\partial x^m} = nP + \frac{y^m}{\sigma_F(x)}\frac{\partial \sigma_F}{\partial x^m}.$$

Therefore

$$\bar{\mathbf{S}}(y) = \frac{\partial \bar{G}^m}{\partial y^m} - \frac{y^m}{\sigma_{\bar{F}}(x)}\frac{\partial \sigma_{\bar{F}}}{\partial x^m} = \mathbf{S}(y) + F^2 c^r I_r. \tag{38}$$

Theorem 5.1. *Let F and $\bar{F}$ be two Finsler metrics on an n-dimensional manifold M. If $\bar{F}(x,y) = e^{c(x)}F(x,y)$, then $\bar{\mathbf{S}} = \mathbf{S}$ if and only if $c^r I_r = 0$, that is, the gradient vector ∇c of the conformal factor $c(x)$ is orthogonal to the covariant vector field I_i with respect to the dual metric F^* of F. In particular, if* $c = \text{constant}$, *then* $\bar{\mathbf{S}} = \mathbf{S}$.

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