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# A Matkowski–Sutô type equation

By PÁL BURAI (Debrecen)

**Abstract.** In the present paper we deal with the following equation:

$$\varphi^{-1}\left(\alpha\varphi(x)+(1-\alpha)\varphi(y)\right)+\psi^{-1}\left((1-\alpha)\psi(x)+\alpha\psi(y)\right)=x+y,$$

where $\varphi$ and $\psi$ are strictly monotone and continuous functions on the same interval. We give the continuously differentiable solutions.

## 1. Introduction

In [6], Daróczy and Páles have solved the so-called Matkowski–Sutô problem without any regularity assumption. In order to formulate this problem precisely, we need the following definitions:

*Definition 1.* Let $I\subset\mathbb{R}$ be a nonempty open interval and let $\mathcal{CM}(I)$ denote the class of all continuous, strictly monotone functions defined on $I$.

*Definition 2.* A continuous function $M: I^2\to I$ is called a mean on $I$ if

$$\min\{x,y\}\le M(x,y)\le\max\{x,y\}$$

for all $x,y\in I$.

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*Definition 3.* A mean $M : I^2 \to I$ is called quasi-arithmetic on $I$ if there exists $\varphi \in \mathcal{CM}(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) =: A_\varphi(x,y)$$

for every $x, y \in I$.

*Definition 4.* Let $p \in \mathbb{R}$ be a real constant and $I \subset \mathbb{R}$ a nonempty interval. Let us define the following function on $I$:

$$\chi_p(x) := \begin{cases} x & \text{if } p = 0 \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in I).$$

*Definition 5.* Let $\varphi, \Phi \in \mathcal{CM}(I)$. We say that $\varphi$ and $\Phi$ are equivalent on $I$, if there exist real constants $a$, $b$ $(a \neq 0)$ so that

$$\varphi(x) = a\Phi(x) + b \tag{1}$$

for every $x \in I$. We write $\varphi \sim \psi$ on $I$ or, $\varphi(x) \sim \psi(x)$ if $x \in I$.

Daróczy and Páles have determined all those functions $\varphi, \psi \in \mathcal{CM}(I)$ for which the functional equation

$$A_\varphi(x,y) + A_\psi(x,y) = x + y \tag{2}$$

holds for every $x, y \in I$. In [6], they have proved the following:

**Theorem 1.** *If $\varphi, \psi \in \mathcal{CM}(I)$ satisfy the functional equation* (2) *for every $x, y \in I$, then there exists $p \in \mathbb{R}$ such that*

$$\varphi \sim \chi_p, \qquad \psi \sim \chi_{-p} \quad \text{on} \quad I.$$

In the present paper we are examining a functional equation similar to (2). We need the following definition to formulate our problem.

*Definition 6.* The mean $M : I^2 \to I$ is called weighted-quasi-arithmetic on $I$ if there exist $\varphi \in \mathcal{CM}(I)$ and $\alpha \in \,]0,1[$ such that

$$M(x,y) = \varphi^{-1}\big(\alpha\varphi(x) + (1-\alpha)\varphi(y)\big) =: A_\varphi(x,y;\alpha)$$

for every $x, y \in I$.

If $\alpha = \frac{1}{2}$, we get the class of quasi-arithmetic means.

We give the continuously differentiable solutions of the following Matkowski–Sutô type equation:

$$A_\varphi(x, y; \alpha) + A_\psi(x, y; 1 - \alpha) = x + y, \quad x, y \in I. \tag{3}$$

The proof consists of two main parts. After the extendability of the solutions is verified (by a very similar method as in [4], more information in [2], [3], [6]), we solve the equation (3), assuming that the unknown functions are continuously differentiable on a subinterval of $I$, and we extend our solutions applying the extension theorem.

Surprisingly, the continuously differentiable solutions of the equations (2) and (3) are the same. Thus we "know" the solutions ahead, so we prove the extension theorem first.

## 2. The extension theorem

The following lemma allows us to work with functions which are equivalent to each other.

**Lemma 1.** *Assume that $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of the functional equation* (3). *If $\varphi \sim \Phi$ and $\psi \sim \Psi$ on $I$, then $\Phi$, $\Psi$ are also solutions of the functional equation* (3).

Proof. Easy calculation. □

Because of the previous lemma, we can assume without loss of generality that $\varphi$ and $\psi$ are strictly monotone increasing, furthermore

$$\alpha < 1 - \alpha.$$

If $\frac{1}{2} < \alpha < 1$, let $\beta := 1 - \alpha$ and interchange $\varphi$ with $\psi$, then we get

$$A_\varphi(x, y; \beta) + A_\psi(x, y; 1 - \beta) = x + y, \qquad x, y \in I,$$

where $0 < \beta < \frac{1}{2}$.

**Lemma 2.** *Let $\varphi : [A, B] \to \mathbb{R}$ be a continuous and strictly monotone increasing function that satisfies*

$$\gamma := \frac{B - A}{\varphi(B) - \varphi(A)} \geq 1. \tag{4}$$

*Furthermore, suppose that the function*

$$f(t) := t - \alpha\varphi(t) \qquad t \in [A, B] \tag{5}$$

*satisfies the functional equation*

$$(1-\alpha)f(x) + \alpha f(y) = f(x + y - A_\varphi(x, y; \alpha)) \tag{6}$$

*for every* $x, y \in [A, B]$. *Then*

$$\varphi(x) = \frac{1}{\gamma}x - \frac{\sigma}{\gamma} \tag{7}$$

*holds for every* $x \in [A, B]$, *where*

$$\sigma = \frac{A\varphi(B) - B\varphi(A)}{\varphi(B) - \varphi(A)}.$$

Proof. If we write the functional equation (6) in detail, we get

$$\begin{aligned}(1-\alpha)x - (1-\alpha)\alpha\varphi(x) + \alpha y - \alpha^2\varphi(y)& \\ = x + y - A_\varphi(x, y; \alpha) - \alpha\varphi(x + y - A_\varphi(x, y; \alpha)).&\end{aligned} \tag{8}$$

Reduce (8), then we have

$$\alpha x + (1-\alpha)y + \alpha(1-\alpha)\varphi(x) + \alpha^2\varphi(y) = A_\varphi(x, y; \alpha) + \alpha\varphi\big(x + y - A_\varphi(x, y; \alpha)\big) \tag{9}$$

for every $x, y \in [A, B]$. By (9), with the substitutions

$$u := \varphi(x), \quad v := \varphi(y), \quad \varphi^{-1} := g,$$

we get

$$\begin{aligned}g\left(g(u) + \frac{1-\alpha}{\alpha}g(v) + (1-\alpha)u + \alpha v - \frac{1}{\alpha}g(\alpha u + (1-\alpha)v)\right)& \\ = g(u) + g(v) - g(\alpha u + (1-\alpha)v)&\end{aligned} \tag{10}$$

for all $u, v \in [\varphi(A), \varphi(B)]$.

Now let

$$b(u) := \frac{B - A}{\varphi(B) - \varphi(A)}u + \frac{A\varphi(B) - B\varphi(A)}{\varphi(B) - \varphi(A)} - g(u) \tag{11}$$

for $u \in [\varphi(A), \varphi(B)]$. It can readily be verified that $b(\varphi(B)) = b(\varphi(A)) = 0$.

(11) implies that

$$g(u) = \gamma u + \sigma - b(u) \qquad u \in [\varphi(A), \varphi(B)], \tag{12}$$

where $\gamma := \frac{B-A}{\varphi(B)-\varphi(A)} \geq 1$ and $\sigma = \frac{A\varphi(B)-B\varphi(A)}{\varphi(B)-\varphi(A)}$.

Using (10) and (12) we obtain that

$$\begin{aligned} g&\left((1-\alpha)u+\alpha v-b(u)-\frac{1-\alpha}{\alpha}b(v)+\frac{1}{\alpha}b(\alpha u+(1-\alpha)v)\right)\\ &=\gamma\left((1-\alpha)u+\alpha v-b(u)-\frac{1-\alpha}{\alpha}b(v)+\frac{1}{\alpha}b(\alpha u+(1-\alpha)v)\right)+\sigma\\ &\quad-b\left((1-\alpha)u+\alpha v-b(u)-\frac{1-\alpha}{\alpha}b(v)+\frac{1}{\alpha}b(\alpha u+(1-\alpha)v)\right)\\ &=\gamma u+\sigma-b(u)+\gamma v+\sigma-b(v)\\ &\quad-\gamma(\alpha u+(1-\alpha)v)-\sigma+b(\alpha u+(1-\alpha v)). \end{aligned}$$

Rearranging this equation, we have that

$$\begin{aligned} (1-\gamma)b(u)&+\left(1-\gamma\frac{1-\alpha}{\alpha}\right)b(v)+\left(\frac{\gamma}{\alpha}-1\right)b(\alpha u+(1-\alpha)v)\\ &=b\left((1-\alpha)u+\alpha v-b(u)-\frac{1-\alpha}{\alpha}b(v)+\frac{1}{\alpha}b(\alpha u+(1-\alpha)v)\right) \end{aligned}\tag{13}$$

for all $u, v \in [\varphi(A), \varphi(B)]$. We are going to prove that $b(u) = 0$ for every $u \in [\varphi(A), \varphi(B)]$.

Assume that $b(u) \not\equiv 0$ on $[\varphi(A), \varphi(B)]$. Then there are two possible cases:

*1. case:*

$b$ has a positive value inside of the interval $[\varphi(A), \varphi(B)]$. $b$ is continuous, so there exists

$$0 < M := \max_{u\in[\varphi(A),\, \varphi(B)]} b(u).$$

Let

$$u_0 := \sup\{u \in [\varphi(A),\, \varphi(B)] \mid b(u) = M\}. \tag{14}$$

Then there exists $\varepsilon > 0$ such that

$$u_0 + (1-\alpha)\varepsilon,\ u_0 - \alpha\varepsilon \in [\varphi(A), \varphi(B)]$$

and

$$0 < b(u_0 + (1-\alpha)\varepsilon) < M \quad \text{and} \quad 0 < b(u_0 - \alpha\varepsilon) \leq M. \tag{15}$$

Substituting $u = u_0 + (1-\alpha)\varepsilon$ and $v = u_0 - \alpha\varepsilon$ into (13), we get

$$\begin{aligned} (1-\gamma)b(u_0+(1-\alpha)\varepsilon)&+\left(1-\gamma\frac{1-\alpha}{\alpha}\right)b(u_0-\alpha\varepsilon)+\left(\frac{\gamma}{\alpha}-1\right)b(u_0)\\ &=b\left(u_0-b(u_0+(1-\alpha)\varepsilon)-\frac{1-\alpha}{\alpha}b(u_0-\alpha\varepsilon)+\frac{1}{\alpha}b(u_0)\right). \end{aligned}\tag{16}$$

Let
$$v(\varepsilon) := \frac{1}{\alpha}\, b(u_0) - b(u_0 + (1-\alpha)\varepsilon) - \frac{1-\alpha}{\alpha}\, b(u_0 - \alpha\varepsilon). \tag{17}$$
Because of (15) and (17) we have
$$M > \alpha v(\varepsilon) = b(u_0) - \alpha b(u_0 + (1-\alpha)\varepsilon) - (1-\alpha)\, b(u_0 - \alpha\varepsilon) > 0. \tag{18}$$
Using (16) and (17) we get
$$b(u_0 + v(\varepsilon)) = (\gamma - 1)v(\varepsilon) + \frac{1-\alpha}{\alpha} b(u_0) + \frac{2\alpha - 1}{\alpha} b(u_0 - \alpha\varepsilon). \tag{19}$$
Then by (4), (15), (18) and (19), we can write
$$\begin{aligned}(1-\alpha)M &= \alpha b(u_0 + v(\varepsilon)) + \alpha(1-\gamma)v(\varepsilon) + (1-2\alpha)b(u_0 - \alpha\varepsilon)\\ &< \alpha M + \alpha(1-\gamma)v(\varepsilon) + (1-2\alpha)M = (1-\alpha)M + \alpha(1-\gamma)v(\varepsilon).\end{aligned}$$
Simplifying the previous inequality, we obtain that
$$0 < \alpha(1-\gamma)v(\varepsilon),$$
which is a contradiction because of (4) and (18).

*2. case:*

$b$ has a negative value in the interval $[\varphi(A), \varphi(B)]$; $b$ is continuous, so there exists
$$0 > m := \min_{u \in [\varphi(A),\, \varphi(B)]} b(u).$$
Let
$$u_0 := \inf\{u \in [\varphi(A), \varphi(B)] \mid b(u) = m\}. \tag{20}$$
Then there exists $\varepsilon > 0$ such that
$$u_0 + (1-\alpha)\varepsilon,\ u_0 - \alpha\varepsilon \in [\varphi(A), \varphi(B)]$$
and
$$0 > b(u_0 + (1-\alpha)\varepsilon) \geq m \quad \text{and} \quad 0 > b(u_0 - \alpha\varepsilon) > m. \tag{21}$$
Similarly to the first case, we obtain the following:
$$\begin{aligned}(1-\alpha)m &= \alpha b(u_0 + v(\varepsilon)) + \alpha(1-\gamma)v(\varepsilon) + (1-2\alpha)b(u_0 - \alpha\varepsilon)\\ &> \alpha m + (1-2\alpha)m + \alpha(1-\gamma)v(\varepsilon),\end{aligned} \tag{22}$$

where we have used (21) and the definition of $v(\varepsilon)$ is the same as in (17), furthermore

$$m < \alpha v(\varepsilon) < 0. \tag{23}$$

Using (22) and (23) we get

$$0 > \alpha(1-\gamma)v(\varepsilon),$$

which leads to a contradiction because of (4) and (23). Thus $b(u) = 0$ for every $u \in [\varphi(A), \varphi(B)]$. For this reason

$$\varphi^{-1} = g(u) = \gamma u + \sigma \qquad u \in [\varphi(A), \varphi(B)],$$

that is

$$\varphi(x) = \frac{1}{\gamma}x - \frac{\sigma}{\gamma} \qquad x \in [A, B]. \qquad \square$$

**Theorem 2.** *Let us assume that* $\varphi, \psi \in \mathcal{CM}(I)$, $0 < \alpha < \frac{1}{2}$ *and* (3) *is satisfied for all* $x, y \in I$. *If there exist a nonempty open interval* $K \subset I$ *and* $p \in \mathbb{R}$ *such that*

$$\varphi \sim \chi_p \quad \text{and } \psi \sim \chi_{-p} \quad \text{on } K, \tag{24}$$

*then*

$$\varphi \sim \chi_p \quad \text{and } \psi \sim \chi_{-p} \quad \text{on } I. \tag{25}$$

Proof. Because of the Lemma 1, we can assume that

$$\varphi = \chi_p \quad \text{and} \quad \psi = \chi_{-p} \tag{26}$$

on $K$. Moreover, we can also assume that the open interval $K$ is maximal, that is, there is no strictly larger interval where the above equalities are satisfied. We are going to prove that then $K$ must be identical with $I$.

Let $K = ]\,a, b[$ and suppose the contrary, that $K \neq I$. Then at least one of the endpoints of $K$ is an interior point of $I$. Without loss of generality we can assume that $a \in I$. We can also assume that $b < \infty$. If this is not true, let us choose $b^* < b$ and work with this value. Because of the strict monotonicity and continuity there exists $0 < \delta < b - a$ so that

$$\alpha\varphi(x) + (1-\alpha)\varphi(y) \in \varphi(K) \quad \text{and} \quad (1-\alpha)\psi(x) + \alpha\psi(y) \in \psi(K) \tag{27}$$

for all $x \in ]\,a-\delta, a[\,$, $y \in ]\,b-\delta, b[\,$.

*1. case:* $p \neq 0$.

(24) implies

$$\varphi^{-1}(t) = \frac{1}{p}\log t, \quad \text{if } t \in \varphi(K) \subset \mathbb{R}_+ \tag{28}$$

and

$$\psi^{-1}(t) = -\frac{1}{p}\log t, \quad \text{if } t \in \psi(K) \subset \mathbb{R}_+. \tag{29}$$

Using (24), (27), (28) and (29), from the functional equation (3) we obtain

$$\frac{1}{p}\log\left(\alpha\varphi(x) + (1-\alpha)e^{py}\right) - \frac{1}{p}\log\left((1-\alpha)\psi(x) + \alpha e^{-py}\right) = x + y \tag{30}$$

for every $x \in \,]\,a-\delta, a[\,$, $y \in \,]\,b-\delta, b[\,$. Rearranging the functional equation (30), we have

$$\alpha\varphi(x) - (1-\alpha)e^{px} = (1-\alpha)e^{py}\left(\psi(x)e^{px} - 1\right)$$

for all $x \in \,]\,a-\delta[\,$, $y \in \,]\,b-\delta, b[\,$. Therefore

$$\psi(x)e^{px} - 1 = 0, \quad \text{if } x \in \,]\,a-\delta, a[\,.$$

From this

$$\psi(x) = e^{-px}$$

for all $]\,a-\delta, b[\,$. For this reason

$$\varphi(x) = e^{px}$$

for all $]\,a-\delta, b[$ which is larger than $K$. This contradicts the maximality of $K$.

*2. case:* $p = 0$.

Then

$$\varphi(x) = \psi(x) = x \quad \text{if } x \in \,]\,a, b[\,. \tag{31}$$

Let $x \in \,]\,a-\delta, a[\,$, $y \in \,]\,b-\delta, b[\,$. Using (3), (27) and (31), we get

$$\alpha\varphi(x) + (1-\alpha)\psi(x) = x \tag{32}$$

for every $x \in \,]\,a-\delta, a[\,$. Then

$$\alpha\varphi(a-\delta) + (1-\alpha)\psi(a-\delta) = a - \delta.$$

Thus we can assume that

$$\varphi(a-\delta) \geq a - \delta. \tag{33}$$

Let $x, y \in [a-\delta, a]$, then (3) and (32) imply

$$(1-\alpha)\frac{x - \alpha\varphi(x)}{1-\alpha} + \alpha\frac{y - \alpha\varphi(y)}{1-\alpha} = \psi\left(x + y - A_\varphi(x, y; \alpha)\right)$$

for every $x, y \in \,]\, a-\delta, a[\,$. Using the functional equation (32) again, we obtain

$$\alpha x+(1-\alpha)y+(1-\alpha)\alpha\varphi(x)+\alpha^2\varphi(y) = A_\varphi(x,y;\alpha)+\alpha\varphi(x+y-A_\varphi(x,y;\alpha)) \quad (34)$$

for all $x, y \in \,]\, a-\delta, a[\,$. Let

$$f(t) := t - \alpha\varphi(t) \quad t \in \,]\, a-\delta, a[\,. \quad (35)$$

Then we have from (34) and (35) that

$$(1-\alpha)f(x) + \alpha f(y) = f(x+y-A_\varphi(x,y;\alpha))$$

for every $x, y \in \,]\, a-\delta, a[\,$. Because of (33)

$$\gamma := \frac{a-(a-\delta)}{\varphi(a)-\varphi(a-\delta)} \geq \frac{\delta}{a-(a-\delta)} = 1,$$

consequently the assumptions of Lemma 2 are satisfied, thus

$$\varphi(x) = \frac{1}{\gamma}x - \frac{\sigma}{\gamma} \qquad x \in [a-\delta, a] \quad (36)$$

and

$$\psi(x) = \frac{x-\alpha\varphi(x)}{1-\alpha} = \frac{(1-\frac{\alpha}{\gamma})x + \alpha\frac{\sigma}{\gamma}}{1-\alpha} \quad x \in [a-\delta, a]. \quad (37)$$

Since $\varphi(a) = a$, from (36) we obtain

$$\sigma = (1-\gamma)a. \quad (38)$$

So

$$\varphi(x) = \frac{1}{\gamma}x - \frac{1-\gamma}{\gamma}a \qquad x \in [A,B] \quad (39)$$

$$\psi(x) = \frac{\gamma-\alpha}{\gamma(1-\alpha)}x + \frac{(1-\gamma)\alpha}{\gamma(1-\alpha)}a \qquad x \in [A,B] \quad (40)$$

$$\varphi^{-1}(x) = \gamma x + (1-\gamma)a \qquad x \in \varphi([A,B]) \quad (41)$$

$$\psi^{-1}(x) = \frac{\gamma(1-\alpha)}{\gamma-\alpha}x - \frac{(1-\gamma)\alpha}{\gamma-\alpha}a \qquad x \in \psi([A,B]). \quad (42)$$

Let $y \in [a,b]$ and $x \in [a-\delta, a]$, then by (3), (39), (40), (41) and (42), we get

$$\left(\gamma - \alpha\gamma + \frac{\gamma(1-\alpha)}{\gamma-\alpha}\alpha - 1\right)y + \left(1-\gamma-\alpha+\alpha\gamma - \frac{1-\gamma}{\gamma-\alpha}\alpha^2\right)a = 0 \quad (43)$$

for every $y \in [a,b]$. Thus

$$\gamma - \alpha\gamma + \frac{\gamma(1-\alpha)}{\gamma-\alpha}\alpha - 1 = 0. \quad (44)$$

Using (4) and (44), we have

$$\gamma = 1. \quad (45)$$

□

## 3. Continuously differentiable solutions

**Lemma 3.** *Let $K \subset \mathbb{R}$ be a nonempty open interval, $f, g : K \to \mathbb{R}_+$ continuous functions, and $\alpha \in \,]0,1[\, \setminus \{\frac{1}{2}\}$. If*

$$\alpha^2 f(u)g(v) - (1-\alpha)^2 f(v)g(u) = \big(\alpha g(v) - (1-\alpha)g(u)\big) f\big(\alpha u + (1-\alpha)v\big) \tag{46}$$

*holds for every $u, v \in K$ and $f$ is continuously differentiable on $J \subset K$, then $g$ is continuously differentiable on $J$ too.*

Proof. Let $v_0$ be fixed and $u$ an arbitrary value in $J$. Then from (46) we get

$$\Big[(1-\alpha) f\big((\alpha u + (1-\alpha)v_0\big) - (1-\alpha)^2 f(v_0)\Big] g(u) \\ = \Big[\alpha f\big(\alpha u + (1-\alpha)v_0\big) - \alpha^2 f(u)\Big] g(v_0).$$

If $(1-\alpha) f((\alpha u + (1-\alpha)v_0) - (1-\alpha)^2 f(v_0) = 0$ on some interval where $v_0$ is fixed, then $f$ is constant, thus $g$ is also constant on the same interval. So we can assume that there exists a fixed value $v_0$ in $J$ so that $(1-\alpha) f((\alpha u + (1-\alpha)v_0) - (1-\alpha)^2 f(v_0) \neq 0$, $(u \in J)$. Then

$$g(u) = \frac{\Big(\alpha f\big((\alpha u + (1-\alpha)v_0\big) - (1-\alpha)^2 f(u)\Big) g(v_0)}{(1-\alpha) f\big((\alpha u + (1-\alpha)v_0\big) - (1-\alpha)^2 f(v_0)}.$$

Because of the differentiability of $f$ there exists $g'(u)$ for every $u \in J$. Using the assumption of the lemma, $g'$ is continuous. □

**Theorem 3.** *Let $K \subset \mathbb{R}$ be a nonempty open interval, $f, g : K \to \mathbb{R}_+$ continuous functions, $\alpha \in \,]0,1[\, \setminus \{\frac{1}{2}\}$. If (46) is satisfied, then there exists a nonempty interval $J \subset K$ such that $f$ and $g$ are continuously differentiable on $J$.*

Proof. Because of the previous lemma it is sufficient to show that $f$ is continuously differentiable on a nonempty subinterval of $K$.

Let us interchange $u$ and $v$ in (46). Then we obtain the following linear system of equations for the unknown pair $(g(u), g(v))$:

$$\begin{aligned} &\Big[\alpha^2 f(u) - \alpha f\big(\alpha u + (1-\alpha)v\big)\Big] g(v) \\ &\qquad + \Big[(1-\alpha) f\big(\alpha u + (1-\alpha)v\big) - (1-\alpha)^2 f(v)\Big] g(u) = 0 \\ &\Big[(1-\alpha) f\big(\alpha v + (1-\alpha)u\big) - (1-\alpha)^2 f(u)\Big] g(v) \\ &\qquad + \Big[\alpha^2 f(v) - \alpha f\big(\alpha v + (1-\alpha)u\big)\Big] g(u) = 0. \end{aligned} \tag{47}$$

(47) is a homogeneous linear system of equations for every fixed pair $(u, v)$. Because of the positivity of $g(u)$ and $g(v)$ the determinant of the system is equal to zero. Thus we obtain a functional equation, in which there is only one unknown function:

$$\begin{aligned}\Big(\alpha^2 f(u) - \alpha f\big(\alpha u + (1-\alpha)v\big)\Big)\Big(\alpha^2 f(v) - \alpha f\big(\alpha v + (1-\alpha)u\big)\Big)\\ = \Big((1-\alpha) f\big(\alpha u + (1-\alpha)v\big) - (1-\alpha)^2 f(v)\Big)\\ \cdot \Big((1-\alpha) f\big(\alpha v + (1-\alpha)u\big) - (1-\alpha)^2 f(u)\Big).\end{aligned} \tag{48}$$

Rearranging the previous equation we get

$$\begin{aligned}\Big[\Big(\alpha^4 - (1-\alpha)^4\Big) f(u) - \alpha^3 f\big(\alpha u + (1-\alpha)v\big)\\ + (1-\alpha)^3 f\big(\alpha v + (1-\alpha)u\big)\Big] f(v)\\ = \alpha^3 f\big(\alpha v + (1-\alpha)u\big) f(u) - (1-\alpha)^3 f\big(\alpha u + (1-\alpha)v\big)\\ + \Big((1-\alpha)^3 - \alpha^3\Big) f\big(\alpha v + (1-\alpha)u\big) f\big(\alpha u + (1-\alpha)v\big).\end{aligned} \tag{49}$$

Let

$$\begin{aligned}F(u,v) := \Big(\alpha^4 - (1-\alpha)^4\Big) f(u) - \alpha^3 f\big(\alpha u + (1-\alpha)v\big)\\ + (1-\alpha)^3 f\big(\alpha v + (1-\alpha)u\big), \qquad u, v \in K,\end{aligned}$$

and

$$N := \{(u,v) \in K^2 \mid F(u,v) = 0\}. \tag{50}$$

Then $F$ is continuous on $K^2$, hence $N$ is a closed subset of $K^2$. Furthermore

$$F(u,u) = \alpha(1-\alpha)(1-2\alpha) f(u) \neq 0. \tag{51}$$

So $(u,u)$ cannot be an accumulation point of $N$ for all $u \in K$. Thus for every $u_0 \in K$ there exists $\varepsilon_0 > 0$ so that

$$F(u,v) \neq 0 \qquad (u,v) \in G\big((u_0,u_0), \varepsilon_0\big), \tag{52}$$

namely there is $J \subset K$, such that

$$F(u,v) \neq 0 \qquad u, v \in J. \tag{53}$$

Using (49) and (53) we obtain

$$\begin{aligned} f(v) = \Big[&\alpha^3 f\big(\alpha v + (1-\alpha)u\big) f(u) - (1-\alpha)^3 f\big(\alpha u + (1-\alpha)v\big) \\ &+ \big((1-\alpha)^3 - \alpha^3\big) f\big(\alpha v + (1-\alpha)u\big) f\big(\alpha u + (1-\alpha)v\big)\Big] \\ &\cdot \Big[\big(\alpha^4 - (1-\alpha)^4\big) f(u) - \alpha^3 f\big(\alpha u + (1-\alpha)v\big) \\ &+ (1-\alpha)^3 f\big(\alpha v + (1-\alpha)u\big)\Big]^{-1} \end{aligned} \tag{54}$$

for every $u, v \in J$. Let us now apply Járai's theorem [8, Theorem 11.6] to (54) with the following casting:

$$Z \subset \mathbb{R} \text{ an open set}, \quad T = Y = K, \quad \mathbb{R}^k = \mathbb{R}^s = \mathbb{R}$$
$$D = K^2, \quad v = t, \ u = y$$
$$g_1(t,y) = y, \quad g_2(t,y) = \alpha t + (1-\alpha)y, \quad g_3(t,y) = \alpha y + (1-\alpha)t$$
$$h(z_1, z_2, z_3) = \frac{\alpha^3 z_1 z_2 - (1-\alpha)^3 z_3 + ((1-\alpha)^3 - \alpha)^3 z_2 z_3}{((1-\alpha)^4 - \alpha^4) z_1 - \alpha^3 z_3 + (1-\alpha)^3 z_2}.$$

Then $h$ and $g_i$ $(i = 1, 2, 3)$ fulfil the assumptions of the above mentioned theorem of Járai, since $h$ is two times continuously differentiable, $g_i$'s derivatives with respect to $y$ do not vanish and $f$ is continuous.

We have to determine only the compact set $C \subset K$. It can be assumed that $K$ is bounded. Let $K := \,]\,a, b\,[$ and

$$C := [(1-\alpha+\alpha^2)a + \alpha(1-\alpha)b\,,\ \alpha(1-\alpha)a + (1-\alpha+\alpha^2)b].$$

Because of Járai's theorem mentioned above, $f$ is a locally Lipschitz function on $K$.

Applying another theorem of Járai [8, Theorem 14.2] to (54) with similar casting as above, with $f_i := f$ a locally Lipschitz function, the previously mentioned $f$ is differentiable almost everywhere. Using [8, Theorem 14.2] we obtain that there exists a nonempty, open subinterval of $K$, where $f$ is continuously differentiable. □

**Theorem 4.** *Let $J \subset \mathbb{R}$ be a nonempty open interval, $f, g : J \to \mathbb{R}_+$ continuously differentiable functions, $\alpha \in \,]\,0, 1\,[\, \setminus \{\frac{1}{2}\}$. If (46) is satisfied, then there exists $c \in \mathbb{R}$, $c \neq 0$ such that*

$$f(u)g(u) = c \qquad u \in J. \tag{55}$$

Proof. Differentiate the functional equation (46) with respect to $u$ and substitute $u = v$, then we obtain the following functional equation:

$$\alpha^2 f'(u)g(u) - (1-\alpha)^2 f(u)g'(u) = -(1-\alpha)g'(u)f(u) + (2\alpha - 1)\alpha g(u)f'(u).$$

Let us rearrange the previous equation

$$f'(u)g(u) + f(u)g'(u) = 0,$$

then

$$\big(f(u)g(u)\big)' = 0 \quad u \in J.$$

Consequently

$$f(u)g(u) = c \neq 0 \quad \text{because } f(u)g(u) > 0 \quad u \in J. \qquad \square$$

**Theorem 5.** *Let $K \subset \mathbb{R}$ be a nonempty open interval, $f, g : K \to \mathbb{R}_+$ continuous functions, $\alpha \in \,]0,1[\, \setminus \{\frac{1}{2}\}$. If (46) is satisfied then there exists a nonempty open interval $J \subset K$ such that*

$$f(u) = Au + B, \qquad g(u) = \frac{c}{Au + B}, \quad u \in J, \tag{56}$$

*where $A$, $B$, $c$ are real constants and $Ac \neq 0$.*

Proof. Because of Theorem 2. there exists a nonempty open subinterval $J \subset K$ such that $f$ and $g$ are continuously differentiable on $J$. Using now Theorem 3. we can effect the substitution $g(u) = \frac{c}{f(u)}$ $(u \in J)$ in the functional equation (46), and we obtain

$$\Big(\alpha f(u) - (1-\alpha)f(v)\Big)\Big(\alpha f(u) + (1-\alpha)f(v) - f(\alpha u + (1-\alpha)v)\Big) = 0,$$

for every $u, v \in J$. Let

$$F(u,v) := \alpha f(u) - (1-\alpha)f(v), \quad u, v \in J,$$

then $F(u,u) = (2\alpha - 1)f(u) \neq 0$. Similarly as in the proof of Theorem 3 we see that there exists $J^* \subset J$ so that

$$F(u,v) \neq 0$$

for every $u, v \in J^*$. Thus

$$\alpha f(u) + (1-\alpha)f(v) - f(\alpha u + (1-\alpha)v) = 0, \quad u, v \in J^*.$$

From this equation we obtain that $f$ is Jensen-convex and Jensen-concave at the same time (see [5] and [10]). According to the assumptions of the theorem, $f$ is continuous on $J^*$, so $f$ is also affine on $J^*$ (see [1, Part I, Chapter 2.1.4, Theorem 1, p. 46], [9, Part III, Chapter 2, Theorem 2, p. 316]). $\square$

We can give now the continuously differentiable solutions.

**Theorem 6.** *Let $I \subset \mathbb{R}$ be a nonempty open interval, $\varphi, \psi \in \mathcal{CM}(I)$. Assume that $\varphi$ and $\psi$ are solutions of the functional equation* (3)*. If there exists a nonempty open interval $J \subset I$, such that $\varphi$ and $\psi$ are continuously differentiable on $J$, then there is a constant $p \in \mathbb{R}$ so that*

$$\varphi \sim \chi_p \quad \text{and} \qquad \psi \sim \chi_{-p} \tag{57}$$

*for every $x \in I$.*

Proof. Let us differentiate (3) with respect to $x$ and $y$ respectively. Then we obtain the following equations:

$$\frac{\alpha\varphi'(x)}{\varphi'\left(A_\varphi(x,y;\alpha)\right)} + \frac{(1-\alpha)\psi'(x)}{\psi'\left(A_\varphi(x,y;1-\alpha)\right)} = 1 \tag{58}$$

$$\frac{(1-\alpha)\varphi'(y)}{\varphi'\left(A_\varphi(x,y;\alpha)\right)} + \frac{\alpha\psi'(y)}{\psi'\left(A_\varphi(x,y;1-\alpha)\right)} = 1 \tag{59}$$

for every $x, y \in J$. Multiplying by $\alpha\psi'(y)$ the first equation and by $(1-\alpha)\psi'(x)$ the second one we get two more equations. Let us subtract the second of these equations from the first one. Thus we get

$$\frac{\alpha^2\varphi'(x)\psi'(y) - (1-\alpha)^2\varphi'(y)\psi'(x)}{\varphi'\left(A_\varphi(x,y;\alpha)\right)} = \alpha\psi'(y) - (1-\alpha)\psi'(x) \quad x, y \in J. \tag{60}$$

Using the substitutions in (60)

$$\varphi(x) := u, \quad \varphi(y) := v, \quad \varphi' \circ \varphi^{-1} := f, \quad \psi' \circ \varphi^{-1} := g,$$

we obtain that

$$\alpha^2 f(u)g(v) - (1-\alpha)^2 f(v)g(u) = \big(\alpha g(v) - (1-\alpha)g(u)\big) f\big(\alpha u + (1-\alpha)v\big) \tag{61}$$

for every $u, v \in K =: \varphi(J)$, where $f$ and $g$ are continuous functions on $K$. By the previous theorems there exist a nonempty open interval $J^* \subset K$ and $A$, $B$, $c$, $(Ac \neq 0)$ real constants so that

$$f(u) = Au + B, \qquad g(u) = \frac{c}{Au+B}, \quad u \in J^*. \tag{62}$$

Consequently

$$\varphi'(x) = A\varphi(x) + B \qquad x \in \varphi^{-1}(J^*) \subset I. \tag{63}$$

Then there exists $p$ such that

$$\varphi(x) \sim \chi_p(x) \qquad x \in \varphi^{-1}(J^*).$$

Using this and (62) we have

$$\psi(x) \sim \chi_{-p}(x) \qquad x \in \varphi^{-1}(J^*).$$

Therefore we obtain the solutions on a subinterval of $I$. We can extend them with the aid of the extension theorem. So the proof is complete. □

## References


[1] J. Aczél, Lectures on Functional Equations and Their Applications, Volume 19 of Mathematics in Science and Engineering, *Academic Press*, *New York – London*, 1966.

[2] P. Burai, Extension theorem for a functional equation, (submitted).

[3] Z. Daróczy, G. Hajdu and C. T. Ng, An extension theorem for a Matkowski–Sutô problem, *Colloq. Math.* **95**(2) (2003), 153–161.

[4] Z. Daróczy, Gy. Maksa and Zs. Páles, Extension theorems for the Matkowski–Sutô problem, *Demonstratio Math.* **33**(3) (2000), 547–556.

[5] Z. Daróczy and Zs. Páles, Convexity with given infinite weight sequences, *Stochastica* **11** (1987), 5–12.

[6] Z. Daróczy and Zs. Páles, Gauss-composition of means and the solution of the Matkowski–Sutô problem, *Publ. Math. Debrecen* **61**(1–2) (2002), 157–218.

[7] G. Hajdu, Investigations in the theory of functional equations, PhD Thesis, *Institute of Mathematics, University of Debrecen, Debrecen, Hungary*, 2003.

[8] A. Járai, Regularity properties of functional equations in several variables, Advances in mathematics, Vol. 8, *Springer*, 2005.

[9] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, *Warszawa, Krakow, Katowice, Panstwowe Wydawnictwo Naukowe, Uniwersytet Slaski*, 1985.

[10] N. Kuhn, A note on $t$-convex functions, General Inequalities 4, *Birkhäuser Verlag*, 1984, 269–276.



PÁL BURAI
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, P. O. BOX 12
HUNGARY

*E-mail:* buraip@math.klte.hu


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