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Solution of a regularity problem in equality of Cauchy means

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Abstract. In a recent paper [1] L. Losonczi solved the equality problem for the Cauchy means assuming seven times differentiability of the involved functions and some algebraic condition. In the present paper we show that this strong regularity assumption can be made without any loss of generality. The algebraic condition can also be reduced. This is a solution of Problem 7 in [2].

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and let $f, g : I \to \mathbb{R}$ be differentiable functions such that $g' \neq 0$ and $\frac{f'}{g'}$ is invertible. Then, by the Cauchy mean value theorem, the function $D_{f,g} : I^2 \to I$,

$$D_{f,g} := \begin{cases} \left(\dfrac{f'}{g'}\right)^{-1} \left(\dfrac{f(x) - f(y)}{g(x) - g(y)}\right), & x \neq y \\ x, & x = y \end{cases}$$

is correctly defined and it is called a *Cauchy mean* generated by f and g. In a recent paper L. Losonczi [1] determined all families of functions $f_1, g_1, f_2, g_2 : I \to \mathbb{R}$ satisfying the equation $D_{f_1,g_1} = D_{f_2,g_2}$ under the assumption that these functions are seven times continuously differentiable

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and satisfy also an algebraic condition which says that a certain expression should either be identically zero, or it should be different from zero everywhere. In the present paper we show that this strong regularity can be assumed without any loss of generality. Moreover the algebraic condition can also be reduced (Remark 2). This is a solution of the Problem 7 presented in a survey paper by Zs. Páles [2].

2. Main result

Remark 1. Let $I \subset \mathbb{R}$ be an interval and suppose that $f, g : I \to \mathbb{R}$ are differentiable and $g' \neq 0$. If $\frac{f'}{g'} : I \to \mathbb{R}$ is one-to-one, then it is strictly monotonic and continuous.

Proof. The assumption of g implies that g is invertible in the interval $J := g(I)$, the function g^{-1} is differentiable and $(g^{-1})'(x) = 1/g'(g^{-1}(x))$ for $x \in J$. Put $h := f \circ g^{-1}$. Then h is differentiable and $h' = \frac{f'}{g'} \circ g^{-1}$ is one-to-one in J. If $\frac{f'}{g'}$ were not strictly monotonic then we would find $x, y, z \in J$, $x < y < z$ such that either

$$h'(y) < h'(x) < h'(z) \quad \text{or} \quad h'(z) < h'(x) < h'(y).$$

The first case cannot happen because $h'(x) \in (h'(y), h'(z))$ whence, by the Darboux property of derivative, $h'(x) = h'(u)$ for some $u \in (y, z)$, which contradicts the assumed injectivity of the function $\frac{f'}{g'}$. For the same reason the second case also cannot happen. Thus the function $\frac{f'}{g'}$ is strictly monotonic and, by the Darboux property of the derivative h', it must be continuous. □

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval, let $f_1, g_1, f_2, g_2 : I \to \mathbb{R}$ be differentiable functions and $g_1' \neq 0 \neq g_2'$ in I. Suppose that $\frac{f_1'}{g_1'}$ and $\frac{f_2'}{g_2'}$ ore one-to-one. Then*

$$D_{f_1,g_2} = D_{f_2,g_2}$$

if, and only if,

$$\gamma\left(\frac{f(x) - f(y)}{x - y}\right) = \frac{g(x) - g(y)}{h(x) - h(y)}, \quad x, y \in J,\ x \neq y, \tag{1}$$

where

$$J := g_1(I), \quad \gamma := \frac{f_2'}{g_2'} \circ \left(\frac{f_1'}{g_1'}\right)^{-1},$$

$$f := f_1 \circ g_1^{-1}, \quad g := f_2 \circ g_1^{-1}, \quad h := g_2 \circ g_1^{-1}.$$

Moreover, the functions f, g, h and γ are of the class C^∞ except for a nowhere dense set in J.

Proof. Since the iff part is easy (cf. [1]), we shall prove the "moreover" statement.

Put

$$F(x,y) := \frac{f(x)-f(y)}{x-y}, \quad K(x,y) := \frac{g(x)-g(y)}{h(x)-h(y)}, \qquad x,y \in J,\ x \neq y,$$

and

$$J_f := \left\{ \frac{f(x)-f(y)}{x-y} : x,y \in J,\ x \neq y \right\}.$$

The function K is well defined because $h' \neq 0$ in I. The strict monotonicity of f' and $\frac{g'}{h'}$ (cf. Remark 1), and the Cauchy mean-value theorem imply that

$$f'(x) - \frac{f(x)-f(y)}{x-y} \neq 0, \quad x,y \in J,\ x \neq y,$$

$$g'(x) - h'(x)K(x,y) \neq 0, \quad x,y \in J,\ x \neq y.$$

Take an arbitrary $u_0 \in J_f$. Then $u_0 = F(x_0,y_0)$ for some $x_0, y_0 \in J$, $x_0 \neq y_0$. The functions

$$\varphi(x) := F(x,y_0), \qquad \psi(x) := K(x,y_0)$$

are differentiable in a neighbourhood of x_0 and, by the above inequalities, $\varphi'(x_0) \neq 0 \neq \psi'(x_0)$. From (1) we have $\gamma = \psi \circ \varphi^{-1}$ in a neighbourhood of u_0. It follows that γ is differentiable at the point u_0 and $\gamma'(u_0) \neq 0$. Consequently, γ is differentiable in the set J_f and $\gamma' \neq 0$ in J_f.

Differentiating both sides of (1) first with respect to x and then with respect to y we get, for all $x, y \in J$, $x \neq y$,

$$\begin{aligned} &\gamma'\left(\frac{f(x)-f(y)}{x-y}\right)\frac{f'(x)(x-y)-f(x)+f(y)}{(x-y)^2} \\ &\qquad = \frac{g'(x)[h(x)-h(y)]-h'(x)[g(x)-g(y)]}{[h(x)-h(y)]^2} \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\gamma'\left(\frac{f(x)-f(y)}{x-y}\right)\frac{-f'(y)(x-y)+f(x)-f(y)}{(x-y)^2} \\ &\qquad = \frac{-g'(y)[h(x)-h(y)]+h'(y)[g(x)-g(y)]}{[h(x)-h(y)]^2}. \end{aligned} \tag{3}$$

Since $\gamma' \neq 0$ in J_f, dividing by sides of (2) and (3) we obtain, for all $x, y \in J$, $x \neq y$,

$$\frac{f'(x)(x-y)-f(x)+f(y)}{f'(y)(x-y)-f(x)+f(y)} = \frac{g'(x)[h(x)-h(y)]-h'(x)[g(x)-g(y)]}{g'(y)[h(x)-h(y)]-h'(y)[g(x)-g(y)]},$$

which can be written in the form

$$\frac{f'(x)-\frac{f(x)-f(y)}{x-y}}{f'(y)-\frac{f(x)-f(y)}{x-y}} = \frac{g'(x)-h'(x)\frac{g(x)-g(y)}{h(x)-h(y)}}{g'(y)-h'(y)\frac{g(x)-g(y)}{h(x)-h(y)}}, \qquad x, y \in J,\ x \neq y,$$

or, equivalently,

$$\frac{f'(x)-F(x,y)}{f'(y)-F(x,y)} = \frac{g'(x)-h'(x)K(x,y)}{g'(y)-h'(y)K(x,y)}, \qquad x, y \in J,\ x \neq y. \tag{4}$$

This equation suggests the following general idea of the proof. Let us fix arbitrarily $x_0 \in J$. By putting $y = y_1$, y_2, $y_3 \neq x_0$ into this equation, one obtains a system of linear equations with respect to the unknown $f'(x)$, $g'(x)$, $h'(x)$. For the rest of the proof it would be sufficient to show that for every y_1, y_2, y_3 can be chosen so that this system of linear equations is uniquely solvable. Then, since F and K are in terms of f, g, h only, this results that f', g', h' enjoy the same regularity properties as F, K, i.e., as f, g, h. Hence f, g, h would be infinitely times differentiable in J.

It turns out however that it is not so easy to show that the respective system is uniquely solvable. Therefore we are forced to consider some special cases, and our result is weaker than one could expect.

Let us fix $y_1 \in J$. Putting here $y := y_1$, we obtain

$$h'(x) = \frac{1}{K(x,y_1)} \left\{ [h'(y_1)K(x,y_1) - g'(y_1)] \frac{f'(x) - F(x,y_1)}{f'(y_1) - F(x,y_1)} + g'(x) \right\},$$

for all $x \in J$, $x \neq y_1$, whence

$$h'(x) = H_f(x)f'(x) + \frac{g'(x)}{K(x,y_1)} + H(x), \qquad x \in J,\ x \neq y_1, \tag{5}$$

where

$$H_f(x) := \frac{h'(y_1)K(x,y_1) - g'(y_1)}{K(x,y_1)[f'(y_1) - F(x,y_1)]}, \qquad x \neq y_1;$$

$$H(x) := \frac{g'(y_1) - h'(y_1)K(x,y_1)}{K(x,y_1)[f'(y_1) - F(x,y_1)]} F(x,y_1), \qquad x \neq y_1,$$

and the functions H_f, H are continuously differentiable in $J \backslash \{y_1\}$.

Replacing $h'(x)$ in (4) by the right-hand side of (5) we get

$$\frac{f'(x) - F(x,y)}{f'(y) - F(x,y)} = \frac{g'(x) - \left[H_f(x)f'(x) + \frac{g'(x)}{K(x,y_1)} + H(x)\right] K(x,y)}{g'(y) - h'(y)K(x,y)}, \tag{6}$$

for all $x, y \in J\backslash\{y_1\}$, $x \neq y$. Setting here a fixed $y := y_2 \neq y_1$, we obtain

$$g'(x) = G_f(x)f'(x) + G(x), \qquad x \in J\backslash\{y_1, y_2\}, \tag{7}$$

where

$$G_f(x) := \frac{K(x,y_1)[g'(y_2) - h'(y_2)K(x,y_2)]}{[K(x,y_1) - K(x,y_2)][f'(y_2) - F(x,y_2)]} + \frac{H_f(x)K(x,y_1)K(x,y_2)]}{K(x,y_1) - K(x,y_2)]},$$

$$G(x) := \frac{H(x)K(x,y_1)K(x,y_2)]}{K(x,y_1) - K(x,y_2)]}$$

$$- \frac{K(x,y_1)[g'(y_2) - h'(y_2)K(x,y_2)]F(x,y_2)}{[K(x,y_1) - K(x,y_2)][f'(y_2) - F(x,y_2)]},$$

and the functions G_f and G are continuously differentiable in $J\backslash\{y_1, y_2\}$. Replacing $g'(x)$ in (6) by the right-hand side of (7) we obtain

$$\frac{f'(x) - F(x,y)}{f'(y) - F(x,y)} \tag{8}$$

$$= \frac{\left[\left(1-\frac{K(x,y)}{K(x,y_2)}\right) G_f(x) - H_f(x)K(x,y)\right] f'(x) + \left(1-\frac{K(x,y)}{K(x,y_2)}\right) G(x) - H(x)K(x,y)}{g'(y) - h'(y)K(x,y)}$$

for all $x, y \in J\backslash\{y_1, y_2\}$, $x \neq y$.

For the simplicity of notations put

$$F = F(x,y), \quad F_1 = F(x,y_1), \quad F_2 = F(x,y_2),$$

$$K = K(x,y), \quad K_1 = K(x,y_1), \quad K_2 = K(x,y_2).$$

From (8) we have, for all $x, y \in J\backslash\{y_1, y_2\}$, $x \neq y$,

$$\begin{aligned} &f'(x)\left\{g'(y) - h'(y)K - \left[f'(y) - F\right]\left[\left(1 - \frac{K}{K_1}\right)G_f(x) - H_f(x)K\right]\right\} \\ &= F\left[g'(y) - h'(y)K\right] + \left[f'(y) - F\right]\left[\left(1 - \frac{K}{K_1}\right)G(x) - H(x)K\right]. \end{aligned} \tag{9}$$

If for some $x = x_0$ there is $y \in J\backslash\{y_1, y_2\}$, $y \neq x_0$ such that

$$g'(y) - h'(y)K - \left[f'(y) - F\right]\left[\left(1 - \frac{K}{K_1}\right)G_f(x) - H_f(x)K\right] \neq 0 \tag{10}$$

at the point (x_0, y), then, by the continuity of the function on the left-hand side, there is a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \subset J\backslash\{y_1, y_2, y\}$,

$$g'(y) - h'(y)K - \left[f'(y) - F\right]\left[\left(1 - \frac{K}{K_1}\right)G_f(x) - H_f(x)K\right] \neq 0.$$

From (9) we get

$$f'(x) = \frac{F\left[g'(y) - h'(y)K\right] + \left[f'(y) - F\right]\left[\left(1 - \frac{K}{K_1}\right)G(x) - H(x)K\right]}{g'(y) - h'(y)K - \left[f'(y) - F\right]\left[\left(1 - \frac{K}{K_1}\right)G_f(x) - H_f(x)K\right]} \tag{11}$$

for all $x \in (x_0-\delta, x_0+\delta)$. Since the function on the right-hand side of this formula is continuously differentiable with respect to x in $(x_0-\delta, x_0+\delta)$, we infer that f is twice continuously differentiable in $(x_0-\delta, x_0+\delta)$. Now the formulas (8) and (6) imply that g and h are twice differentiable in $(x_0-\delta, x_0+\delta)$.

Suppose that for an $x \in J$ there is no $y \in J\backslash\{y_1, y_2, x_0\}$ such that (10) holds true. Then either in every neighbourhood of x there is a point x_0 such (10) holds for all $y \in J\backslash\{y_1, y_2\}$, $y \neq x_0$, and we can argue as previously, or there is a neighbourhood J_0 of x_0 such that

$$g'(y) - h'(y)K - \big(f'(y) - F\big)\left[\left(1 - \frac{K}{K_1(x)}\right)G_f(x) - H_f(x)K\right] = 0,$$

for all $x \in J_0$, $y \in J\backslash\{y_1, y_2, x\}$. Replacing here y by x and x by z and taking into account the symmetry of F and K we can write this equation in the form

$$\begin{aligned} g'(x) - Kh'(x) + \left\{\left[\frac{G_f(z)}{K_1(z)} + H_f(z)\right]K - G_f(z)\right\}f'(x) \\ = \left\{\left[\frac{G_f(z)}{K_1(z)} + H_f(z)\right]K - G_f(z)\right\}F \end{aligned} \tag{12}$$

for all $x, z \in J_0\backslash\{y_1, y_2\}$, $x \neq z$. Setting here $z = z_1, z_2, z_3 \in J_0$, $z_1 \neq z_2 \neq z_3 \neq z_1$, we obtain the system of linear equations

$$g'(x) - M_i(x)h'(x) + (a_iM_i(x) + b_i)f'(x) = N_i(x), \quad i = 1, 2, 3, \tag{13}$$

with respect to g', h', f', where

$$a_i := \frac{G_f(z_i)}{K_1(z_i)} + H_f(z_i), \quad b_i := -G_f(z_i), \quad M_i(x) := K(x, z_i), \tag{14}$$

$$N_i(x) := \left\{\left[\frac{G_f(z_i)}{K_1(z_i)} + H_f(z_i)\right]K(x, z_i) - G_f(z_i)\right\}F(x, z_i). \tag{15}$$

Now either there are $z_1, z_2, z_3 \in J_0$ and a nonempty open subinterval $J_1 \subset J_0$ such that the determinant

$$\begin{vmatrix} 1 & -M_1(x) & a_1M_1(x) + b_1 \\ 1 & -M_2(x) & a_2M_2(x) + b_2 \\ 1 & -M_3(x) & a_3M_3(x) + b_3 \end{vmatrix} \neq 0$$

for all $x \in J_1 \subset J_0 \backslash \{y_1, y_2, z_1, z_2, z_3\}$, or for all the choices of z_1, z_2, z_3 this determinant is equal zero. Since the functions M_i and N_i, $i = 1, 2, 3$, are of the class C^1 in J_1, in the first case the functions f', g', h' satisfying system (13) must be also continuously differentiable in J_1. Since

$$M_i(x) = \frac{g(x) - \gamma_i}{h(x) - \eta_i}, \quad i = 1, 2, 3; \quad x \in J_1,$$

where $\gamma_i := g(z_i)$, $\eta_i := h(z_i)$, in the second case we have

$$\begin{vmatrix} 1 & -\dfrac{g(x) - \gamma_1}{h(x) - \eta_1} & a_1 \dfrac{g(x) - \gamma_1}{h(x) - \eta_1} + b_1 \\[2ex] 1 & -\dfrac{g(x) - \gamma_2}{h(x) - \eta_2} & a_2 \dfrac{g(x) - \gamma_2}{h(x) - \eta_2} + b_2 \\[2ex] 1 & -\dfrac{g(x) - \gamma_3}{h(x) - \eta_3} & a_3 \dfrac{g(x) - \gamma_3}{h(x) - \eta_3} + b_3 \end{vmatrix} = 0, \qquad x \in J_1.$$

Put $\varphi := g \circ h^{-1}$. Taking here $x = h^{-1}(u)$ for $u \in h(J)$, we get

$$\begin{vmatrix} 1 & -\dfrac{\varphi(u) - \gamma_1}{u - \eta_1} & a_1 \dfrac{\varphi(u) - \gamma_1}{u - \eta_1} + b_1 \\[2ex] 1 & -\dfrac{\varphi(u) - \gamma_2}{u - \eta_2} & a_2 \dfrac{\varphi(u) - \gamma_2}{u - \eta_2} + b_2 \\[2ex] 1 & -\dfrac{\varphi(u) - \gamma_3}{u - \eta_3} & a_3 \dfrac{\varphi(u) - \gamma_3}{u - \eta_3} + b_3 \end{vmatrix} = 0, \qquad u \in h(J_1),$$

which reduces to the equation

$$\frac{(au + b)[\varphi(u)]^2 - (cu + d)\varphi(u) + pu^2 + qu + r}{(u - \eta_1)(u - \eta_2)(u - \eta_3)} = 0, \quad u \in h(J_1), \tag{16}$$

where a, b, c, d, p, q, r are some real coefficients. Careful calculations show that, for instance,

$$a = f'(y_1)h'(y_2),$$

which proves the these coefficients do not vanish simultaneously. From (16) we infer that φ, being of the form,

$$\varphi(u) = \frac{(cu + d) \pm \sqrt{Au^3 + Bu^2 + Cu + D}}{2(au + b)}$$

is regular in $h(J_1)$ except for at most one point. Since $g = \varphi \circ h$, the function g' continuously differentiable in J_1(except for at most one point) iff so is h'. Now the continuous differentiability of f', g', h' is an easy consequence of the formulas (5) and (7).

Thus we have shown that every neighbourhood of any point $x_0 \in J$ contains a non-empty open interval on which the f', g', h' are continuously differentiable and, consequently, f, g, h are twice differentiable in J except for a nowhere dense set. Now the induction completes the proof. $\square$

Remark 2. To avoid the regularity assumptions in paper [1] it is sufficient to determine the solutions of equation (1) on a maximal nonempty subinterval J_0 of J on which the functions f, g, h are of the class of C^∞, the existence of which is guaranteed by Theorem 1, and then just observe that $J = J_0$. This approach allows to reduce an algebraic condition concerning the Schwarzian derivative of some function assumed in [1].

Remark 3. Equation (4):

$$\frac{f'(x) - \frac{f(x)-f(y)}{x-y}}{f'(y) - \frac{f(x)-f(y)}{x-y}} = \frac{g'(x) - h'(x)\frac{g(x)-g(y)}{h(x)-h(y)}}{g'(y) - h'(y)\frac{g(x)-g(y)}{h(x)-h(y)}}, \qquad x, y \in J,\ x \neq y,$$

which appeared in the proof suggests a different method in solving the equality problem of Cauchy means than that applied in [1].

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