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On a generalization of contact metric manifolds

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Abstract. We consider a metric f-structure on a manifold M of dimension $2n+s$ and suppose that its kernel is parallelizable. We give a sufficient condition for such a structure to be an $\mathcal{S}$-structure, that is a generalization of the Sasakian structure. Then we prove some identities for the Ricci operator. We exhibit also some examples.

1. Introduction

In recent years there is a very interest in contact geometry. Within the subject of contact geometry there is also the class of contact metric geometry and its generalizations, [3], [5]. A certain class of such manifolds is of our interest in the present paper.

Let (M,g) be a Riemannian manifold equipped with a metric f-structure, i.e. an endomorphism φ of the tangent bundle such that $\varphi^3+\varphi=0$ and which is compatible with g; the compatibility means that for each $X,Y \in TM$ we have $g(\varphi(X),Y)=-g(X,\varphi(Y))$, cf. [19]. Such manifolds are a natural generalization of almost Hermitian manifolds (case when φ is an isomorphism of TM). Moreover we assume that the kernel of φ is parallelizable, i.e. there exist global vector fields $\xi_1,\dots,\xi_s$ spanning $\ker\varphi$. Such manifolds are necessarily of dimension $2n+s$ where $2n$ is the rank of φ (supposed to be constant). The study of such manifolds

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was started by D. E. Blair, S. I. Goldberg, K. Yano, cf. [2], [13], [14], [6]. Let $\eta^1, \dots, \eta^s$ be the dual 1-forms of $\xi_1, \dots, \xi_s$. According to the definitions of [12], the set consisting of M with the geometric structures $(\varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$, g a compatible metric, is called an *almost $\mathcal{S}$-structure* if $d\eta^k = F$ for all $k = 1, \dots, s$ where F is the Sasaki 2-form defined by g and φ. Moreover, when F and the 1-forms $\eta^1, \dots, \eta^s$ are closed such a structure is called an almost $\mathcal{C}$-structure. Examples may be constructed using the suspension method or the pull-back of toroidal bundles, cf. [2], [9], [8], [10].

In the present paper we give a sufficient condition for an almost $\mathcal{S}$-structure to be an $\mathcal{S}$-structure. We prove that if $\xi_1, \dots, \xi_s$ are Killing vector fields, η^i are invariant with respect to ξ_j $(i, j = 1, \dots, s)$ and the covariant derivative of φ satisfies certain known identity then the structure is an $\mathcal{S}$-structure, cf. Proposition 3.1. Then we prove the existence of a metric f-structure with parallelizable kernel under some geometric conditions on the curvature tensor of M, cf. Proposition 3.2. These conditions were originally studied by D. E. Blair. Moreover, we study the Ricci operator of the almost $\mathcal{S}$-manifolds and obtain conditions for ξ_i to be Killing. These theorems generalize some results of [15], [4]. Then we consider the warped product of two manifolds and we show a natural method of constructing metric f-structures with parallelizable kernel. Finally, in the last section we provide new examples of such structures.

2. Preliminaries

Let M be a $(2n + s)$-dimensional manifold equipped with an *f.pk-structure*, i.e. an *f-structure φ with parallelizable kernel.* This means that there exist s global vector fields $\xi_1, \dots, \xi_s$ and 1-forms $\eta^1, \dots, \eta^s$ on M satisfying the following conditions

$$\varphi(\xi_i) = 0\ , \quad \eta^i \circ \varphi = 0, \quad \varphi^2 = -I + \sum_{j=1}^{s} \eta^j \otimes \xi_j, \quad \eta^i(\xi_j) = \delta^i_j$$

for all $i, j = 1, \dots, s$. We denote by $\mathcal{X}(M)$ the module of differentiable vector fields on M. On such a manifold there always exists a *compatible*

Riemannian metric g, in the sense that for each $X, Y \in \mathcal{X}(M)$ $g(X,Y) = g(\varphi(X), \varphi(Y)) + \sum_{j=1}^{s} \eta^j(X)\eta^j(Y)$. We fix such a metric on M; then the structure obtained is called *metric f.pk-structure.* Let F be the Sasaki form of φ defined by $F(X,Y) := g(X, \varphi Y)$ for $X, Y \in \mathcal{X}(M)$. We denote by $\mathcal{D}$ the bundle $\mathrm{Im}\varphi$ which is the orthogonal complement of the bundle $\ker\varphi = \langle \xi_1, \dots, \xi_s \rangle$.

Then the manifold M is equipped with the structure consisting of an f-structure φ, the complemented frame $\xi_1, \dots, \xi_s$, the 1-forms $\eta^1, \dots, \eta^s$, a compatible metric g and the Sasaki 2-form F.

We recall the definitions of certain types of metric f-structures which are used in the present paper, cf. [2], [7], [12]. If F and the 1-forms $\eta^1, \dots, \eta^s$ are closed the structure is called an *almost $\mathcal{C}$-structure* and the manifold is said to be an *almost $\mathcal{C}$-manifold.* If $F = d\eta^1 = \dots = d\eta^s$ then the structure is called an *almost $\mathcal{S}$-structure* and M an *almost $\mathcal{S}$-manifold.* If F is closed and the structure is normal then we deal with a *$\mathcal{K}$-structure.* The *normality* means that $N := [\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0$ where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A normal almost $\mathcal{C}$-manifold is called a *$\mathcal{C}$-manifold* and a normal almost $\mathcal{S}$-manifold is called an *$\mathcal{S}$-manifold.* We also put $\bar{\eta} = \sum_{j=1}^{s} \eta^j$, $\bar{\xi} = \sum_{i=1}^{s} \xi_i$ and $h_i := (1/2)L_{\xi_i}\varphi$, cf. [7, (2.5)].

3. Some remarks on *f.pk*-manifolds

Lemma 3.1. *Let (M, g) be a Riemannian manifold of dimension $2n + s$, $\xi_1, \dots, \xi_s$ Killing orthonormal vector fields and $\eta^1, \dots, \eta^s$ their dual 1-forms. If $L_{\xi_i}\eta^j = 0$ for all $i, j = 1, \dots, s$ then $\nabla_{\xi_i}\xi_j = 0$.*

Proof. For each $i, j, k = 1, \dots, s$ we have $\eta^j([\xi_i, \xi_k]) = -(L_{\xi_i}\eta^j)(\xi_k) = 0$ so that $[\xi_i, \xi_k] \in \mathcal{D}$. Moreover, $0 = (L_{\xi_i}g)(X, \xi_j) = g(\nabla_X \xi_i, \xi_j) + g(\nabla_{\xi_j}\xi_i, X)$ for each $X \in \mathcal{D}$ and $i, j = 1, \dots, s$. Hence

$$g(\nabla_X \xi_i, \xi_j) = -g(\nabla_{\xi_j}\xi_i, X) \tag{3.1}$$

and $g(\nabla_{\xi_i}\xi_j, X) = -g(\nabla_X \xi_j, \xi_i) = g(\nabla_X \xi_i, \xi_j) = -g(\nabla_{\xi_j}\xi_i, X)$, so that

$$g(\nabla_{\xi_i}\xi_j + \nabla_{\xi_j}\xi_i, X) = 0. \tag{3.2}$$

On the other hand $g([\xi_i, X], \xi_j) = -(L_{\xi_i}\eta^j)X = 0$ while $0 = (L_{\xi_i}g)(X, \xi_j) = -g([\xi_i, \xi_j], X)$. Combining with (3.2) we obtain

$$g(\nabla_{\xi_i}\xi_j, X) = 0. \tag{3.3}$$

Since $[\xi_i, \xi_j] \in \mathcal{D}$ from (3.3) we get $[\xi_i, \xi_j] = 0$. Hence we have $g(\nabla_{\xi_i}\xi_j, \xi_k) = -g(\xi_j, \nabla_{\xi_i}\xi_k) = -g(\xi_j, \nabla_{\xi_k}\xi_i) = g(\xi_i, \nabla_{\xi_j}\xi_k) = -g(\nabla_{\xi_j}\xi_i, \xi_k) = -g(\nabla_{\xi_i}\xi_j, \xi_k)$ and then $g(\nabla_{\xi_i}\xi_j, \xi_k) = 0$. Using (3.3) we prove the claim. □

Remark 3.1. Under the same assumptions as in Lemma 3.1 from (3.1) we get that for all $X \in \mathcal{D}$ and $i, j = 1, \dots, s$ $g(\nabla_X\xi_i, \xi_j) = 0$. Moreover, $\nabla_X\xi_i \in \mathcal{D}$ so that $\eta^j([X, \xi_i]) = g(\nabla_X\xi_i, \xi_j) - g(\nabla_{\xi_i}X, \xi_j) = 0$.

Equation (1.10) of [7] is an interesting result about $\mathcal{S}$-manifolds; with our notation this equation may be rewritten as

$$(\nabla_X\varphi)(Y) = g(\varphi(X), \varphi(Y))\bar{\xi} + \bar{\eta}(Y)\varphi^2(X) \tag{3.4}$$

for each $X, Y \in \mathcal{X}(M)$. In the following proposition we prove a sufficient condition for an *f.pk*-manifold to be $\mathcal{S}$-manifold. This may be considered as the inverse of the above result of D. E. Blair, cf. [2], [7].

Proposition 3.1. *Let* $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ *be an f.pk-manifold. If* $\xi_1, \dots, \xi_s$ *are Killing,* $L_{\xi_i}\eta^j = 0$ *for each* $i, j = 1, \dots, s$ *and* (3.4) *holds then* M *is an* $\mathcal{S}$*-manifold.*

Proof. Applying equation (3.4) to a vector field X and ξ_k ($k = 1, \dots, s$) we get $\varphi(\nabla_X\xi_k) = X - \sum_{i=1}^s \eta^i(X)\xi_i$ and $\varphi(X) = -\nabla_X\xi_k + \sum_{i=1}^s \eta^i(\nabla_X\xi_k)\xi_i$. Therefore, if $X, Y \in \mathcal{D}$ then

$$\begin{aligned} d\eta^k(X, Y) &= -\frac{1}{2}\eta^k([X, Y]) \\ &= -\frac{1}{2}\left[-g(Y, \nabla_X\xi_k) + g(X, \nabla_Y\xi_k)\right] \\ &= -\frac{1}{2}\left[g(Y, \varphi(X)) - g(X, \varphi(Y))\right] \\ &= g(X, \varphi(Y)) = F(X, Y). \end{aligned}$$

Furthermore, due to Lemma 3.1 and Remark 3.1 we get $F(\xi_i, X) = 0 = d\eta^k(\xi_i, X)$ and also $F(\xi_i, \xi_j) = 0 = d\eta^k(\xi_i, \xi_j)$ for each $i, j, k = 1, \dots, s$ and each $X \in \mathcal{D}$. We conclude that $F = d\eta^1 = \cdots = d\eta^s$. To complete the proof we need to obtain the normality of the structure. If $X, Y \in \mathcal{D}$ then from equation (3.4) we have

$$\begin{aligned}[][\varphi,\varphi](X,Y) &= (\nabla_{\varphi(X)}\varphi)(Y) - (\nabla_{\varphi(Y)}\varphi)(X) - \varphi((\nabla_X\varphi)(Y)) \\ &\quad + \varphi((\nabla_Y\varphi)(X)) \\ &= \big[g(\varphi^2(X),\varphi(Y)) - g(\varphi(X),\varphi^2(Y))\big]\bar{\xi} \\ &= -2\sum_{i=1}^{s} F(X,Y)\xi_i = -2\sum_{i=1}^{s} d\eta^i(X,Y)\xi_i,\end{aligned}$$

that is $N(X,Y) = 0$. Then we observe that for each $X \in \mathcal{D}$ and $i = 1,\dots,s$ we have $[\varphi,\varphi](X,\xi_i) = 0$. Therefore $N(X,\xi_i) = 2\sum_{j=1}^{s} d\eta^j(X,\xi_i)\xi_j = 0$ due to Lemma 3.1 and Remark 3.1. Finally, again from Lemma 3.1, for each $i,j = 1,\dots,s$ we have $N(\xi_i,\xi_j) = \varphi^2[\xi_i,\xi_j] + 2\sum_{k=1}^{s} d\eta^k(\xi_i,\xi_j)\xi_k = 0$. □

Proposition 3.2. *Let (M,g) be a Riemannian manifold, $\xi_1,\dots,\xi_s$ orthonormal Killing vector fields, $\bar{\xi}$ their sum and $\eta^1,\dots,\eta^s$ their dual 1-forms. We suppose that $R_{X\bar{\xi}}\bar{\xi} = X$ for each $X \in \mathcal{D}$ and $L_{\xi_i}\eta^j = 0$ for each $i,j = 1,\dots,s$. Then M has a natural f.pk-structure φ and its Sasaki form F satisfies*

$$F = \sum_{i=1}^{s} d\eta^i. \tag{3.5}$$

Here D denotes the orthogonal bundle to $\xi_1,\dots,\xi_s$.

Proof. We put $\varphi(X) = -\nabla_X\bar{\xi}$ for each $X \in \mathcal{X}(M)$. Then from Lemma 3.1 it follows that $\varphi(\xi_i) = 0$ for $i = 1,\dots,s$. Since $\bar{\xi}$ is Killing then it is also an affine vector field so $\nabla_{\bar{\xi}}\bar{\xi} = 0$, $\nabla_{\nabla_X\bar{\xi}}\bar{\xi} = -R_{X\bar{\xi}}\bar{\xi} = -X$ for each $X \in \mathcal{D}$. Hence we have $\varphi^2(X) = -X$. It follows that $\varphi^2 = -I + \sum_{i=1}^{s}\eta^i\otimes\xi_i$. From Remark 3.1 we immediately get that $\eta^i\circ\varphi = 0$. To prove that g is a compatible metric, we take $X, Y \in \mathcal{D}$ and get

$$\begin{aligned} g(\nabla_X\bar{\xi},\nabla_Y\bar{\xi}) &= -g(X, R_{\bar{\xi}Y}\bar{\xi}) - g(\nabla_{[\bar{\xi},Y]}\bar{\xi}, X) \\ &\quad + \bar{\xi}(g(X,\nabla_Y\bar{\xi})) + g([X,\bar{\xi}],\nabla_Y\bar{\xi})\end{aligned}$$

$$= g(X,Y) - g(\nabla_{[\bar\xi,Y]}\bar\xi, X) + g([\bar\xi, \nabla_Y\bar\xi], X)$$
$$= g(R_{\bar\xi\bar\xi}Y, X) + g(X,Y) = g(X,Y)$$

where we use the relations $\nabla_{\bar\xi}\bar\xi = 0$ and $\nabla_{\nabla_X\bar\xi}\bar\xi = -X$. On the other hand $g(\varphi(\xi_i), \varphi(X)) = 0 = g(\xi_i, X)$ for each $i = 1, \dots, s$, $X \in \mathcal{D}$. Moreover $g(\varphi(\xi_i), \varphi(\xi_j)) = 0 = g(\xi_i, \xi_j) - \sum_{k=1}^{s} \eta^k(\xi_i)\eta^k(\xi_j)$ for $i, j = 1, \dots, s$ so that g is a compatible metric. Finally, from (3.1) it follows that $F(X, \xi_i) = 0 = d\eta^i(X, \xi_j)$ for each $i, j = 1, \dots, s$ and $X \in \mathcal{X}(M)$. On the other hand, since ξ_i is Killing then for each $X, Y \in \mathcal{D}$ and $i = 1, \dots, s$

$$d\eta^i(X,Y) = -\frac{1}{2}\eta^i([X,Y]) = \frac{1}{2}\big[g(Y, \nabla_X\xi_i) - g(X, \nabla_Y\xi_i)\big] = g(Y, \nabla_X\xi_i)$$

so that we obtain (3.5). □

Remark 3.2. Suppose that $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ is an almost $\mathcal{S}$-manifold. We put

$$\widetilde{g} := sg - (s-1)\sum_{i=1}^{s} \eta^i \otimes \eta^i.$$

Then $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, \widetilde{g})$ is an $f.pk$-structure which verifies (3.5).

4. Properties of the Ricci operator on almost $\mathcal{S}$-manifolds

In this section we study some properties of the curvature of metric $f.pk$-manifolds. We give conditions under which the vector fields ξ_i are Killing.

Proposition 4.1. *Let $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ be an almost $\mathcal{S}$-manifold and fix $i \in \{1, \dots, s\}$. Then ξ_i is Killing if and only if the sectional curvatures of all planes generated by ξ_i and any $X \in \mathcal{D}$ are 1.*

Proof. If ξ_i is Killing then from Proposition 2.4 of [12], Theorem 2.6 and equation (2.3) of [7] we get $\nabla_X\xi_i = -\varphi(X)$ for each $X \in \mathcal{X}(M)$. Let $X \in \mathcal{D}$ with $\|X\| = 1$. Then using equations (2.3) and (2.4) of [7] we get $R_{X\xi_i}\xi_i = X$ so that $g(R_{X\xi_i}\xi_i, X) = g(X, X) = 1$. Conversely, if $X \in \mathcal{D}$

with $\|X\| = 1$ then also $\varphi(X) \in \mathcal{D}$ and $\|\varphi(X)\| = 1$, so that $g(R_{X\xi_i}\xi_i, X) = g(R_{\varphi(X)\xi_i}\xi_i, \varphi(X)) = 1$. From the first equation in the proof of Theorem 3.8 in [7] it follows that $R_{\xi_i X}\xi_i - \varphi(R_{\xi_i\varphi(X)}\xi_i) = 2(h_i^2(X) + \varphi^2(X))$. This implies $-2 = g(R_{\xi_i X}\xi_i - \varphi(R_{\xi_i\varphi(X)}\xi_i), X) = 2g(h_i^2(X) - X, X) = 2(g(h_i(X), h_i(X)) - 1)$ so that $g(h_i(X), h_i(X)) = 0$. This means $h_i = 0$ that is ξ_i is Killing, cf. Theorem 2.6 in [7]. □

Proposition 4.2. *Let $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ be an almost $\mathcal{S}$-manifold and fix $i \in \{1, \dots, s\}$. Then ξ_i is Killing if and only if for each $X \in \mathcal{X}(M)$ we have*

$$R_{X\xi_i}\xi_i = X - \sum_{j=1}^{s} \eta^j(X)\xi_j. \tag{4.1}$$

Proof. We argue as in the proof of Proposition 4.1: since ξ_i is Killing then $\nabla_X\xi_i = -\varphi(X)$. Equation (2.3) of [7] and Lemma 3.1 imply $R_{X\xi_i}\xi_i = \nabla_{\xi_i}(\varphi(X)) + \varphi([X, \xi_i]) = \varphi(\nabla_X\xi_i) = \varphi^2(X)$, that is (4.1) holds. The converse follows from Proposition 4.1. □

Proposition 4.3. *Let $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ be an almost $\mathcal{S}$-manifold and suppose that $\xi_1, \dots, \xi_s$ are Killing. Then for each $i = 1, \dots, s$ we have*

$$Q(\xi_i) = 2n\bar{\xi} \tag{4.2}$$

where Q is the Ricci operator.

Proof. Since each ξ_i is Killing, hence affine, then $R_{X\xi_i}Y = \nabla_X\nabla_Y\xi_i - \nabla_{\nabla_X Y}\xi_i$. From Proposition 2.4 of [12] it follows that $R_{\xi_i X}Y = (\nabla_X\varphi)(Y)$. Then, with respect to a φ-basis $\{X_1, \dots, X_n, \varphi(X_1), \dots, \varphi(X_n), \xi_1, \dots, \xi_s\}$ and from (2.2) of [7] we have

$$\begin{aligned} g(Q(\xi_i), \xi_j) &= \sum_{\alpha=1}^{2n} g(R_{\xi_i X_\alpha}X_\alpha, \xi_j) = \sum_{\alpha=1}^{2n} g((\nabla_{X_\alpha}\varphi)(X_\alpha), \xi_j) \\ &= \sum_{\alpha=1}^{2n} g(\varphi(X_\alpha), \varphi(X_\alpha)) = \sum_{\alpha=1}^{2n} g(X_\alpha, X_\alpha) = 2n, \end{aligned}$$

for each $i, j = 1, \dots, s$. Analogously, using (2.2) of [7] and Corollary 2.1(a) of [12] we get $g(Q(\xi_i), X_\beta) = 0$ and $g(Q(\xi_i), \varphi(X_\beta)) = 0$ for each $i = 1, \dots, s$ and $\beta = 1, \dots, n$. Then $Q(\xi_i) = \sum_{j=1}^{s} g(Q(\xi_i), \xi_j)\xi_j = 2n\bar{\xi}$. □

Lemma 4.1. *Let* $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ *be an* $\mathcal{S}$*-manifold. For each* $i = 1, \dots, s$ *and* $X, Y \in \mathcal{D}$ *we have*

$$R_{XY}\xi_i = 0, \quad R_{X\xi_i}\xi_i = X. \tag{4.3}$$

Proof. The statement follows immediately from (3.6) of [7]. □

Proposition 4.4. *If* $(M, \varphi, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ *is an* $\mathcal{S}$*-manifold then*

$$Q \circ \varphi = \varphi \circ Q. \tag{4.4}$$

Proof. From (4.3) and Lemma 2.2.b) of [2] considering $X, Y \in \mathcal{D}$ and a φ-basis $\{X_1, \dots, X_n, X_{n+1} = \varphi(X_1), \dots, X_{2n} = \varphi(X_n), \xi_1, \dots, \xi_s\}$, we have

$$\begin{aligned} g(Q(\varphi(X)), \varphi(Y)) &= \sum_{\alpha=1}^{2n} g(R_{\varphi(X)X_\alpha}X_\alpha, \varphi(Y)) + \sum_{i=1}^{s} g(R_{\varphi(X)\xi_i}\xi_i, \varphi(Y)) \\ &= \sum_{\alpha=1}^{2n} g(R_{\varphi^2(X)\varphi(X_\alpha)}\varphi(X_\alpha), \varphi^2(Y)) + \sum_{i=1}^{s} g(\varphi(X), \varphi(Y)) \\ &= \sum_{\alpha=1}^{2n} g(R_{X\varphi(X_\alpha)}\varphi(X_\alpha), Y) + \sum_{i=1}^{s} g(R_{X\xi_i}\xi_i, Y) \\ &= g(Q(X), Y) = g(\varphi(Q(X)), \varphi(Y)). \end{aligned}$$

Thus (4.4) holds on $\mathcal{D}$. From (4.2) it follows that (4.4) is true on $\mathcal{D}^\perp$. □

5. *f.pk*-structures on warped products

In this section we consider the warped product of *f.pk*-manifolds. We show how to construct new examples of *f.pk*-manifolds.

Let (B, g_1), (V, g_2) be Riemannian manifolds and $\rho : B \to \mathbb{R}$ be a strictly positive function. Let $\widetilde{M} := B \times V$, $\pi_1 : \widetilde{M} \to B$ and $\pi_2 : \widetilde{M} \to V$ be the natural projections. Then $\widetilde{M}$ carries a structure of *warped manifold with the warping function* ρ i.e. the product manifold equipped with the metric tensor $\widetilde{g} = \pi_1^* g_1 + \rho^2 \pi_2^* g_2$. The geometry of the warped product manifolds has been intensively studied because of its applications in the theoretical physics. The bibliography on the subject is very vast, we only suggest here [1], [16].

The $C^\infty(B)$-module $\mathcal{X}(B)$ of vector fields on B is naturally immersed in the $C^\infty(\widetilde{M})$-module $\mathcal{X}(\widetilde{M})$ of vector fields on $\widetilde{M}$. Analogically, the $C^\infty(V)$-module $\mathcal{X}(V)$ of vector fields on V is naturally immersed in the $C^\infty(\widetilde{M})$-module $\mathcal{X}(\widetilde{M})$ of vector fields on $\widetilde{M}$. Hence, with a slight abuse of notation, we denote the elements of $\mathcal{X}(B)$ and $\mathcal{X}(V)$ as vector fields on $\widetilde{M}$ with the same letters.

Let $(B, \varphi_1, \xi_1, \ldots, \xi_s, \eta^1, \ldots, \eta^s, g_1)$, $(V, \varphi_2, \zeta_1, \ldots, \zeta_t, \theta^1, \ldots, \theta^t, g_2)$ be two $f.pk$-manifolds. We consider the following structures on the warped product $\widetilde{M}$

$$\widetilde{\xi}_i := \begin{cases} \xi_i & \text{if } i = 1, \ldots, s \\ \frac{1}{\rho}\zeta_{i-s} & \text{if } i = s+1, \ldots, s+t \end{cases} \tag{5.1}$$

$$\widetilde{\eta}^i := \begin{cases} \pi_1^*\eta^i & \text{if } i = 1, \ldots, s \\ \rho\pi_2^*\theta^{i-s} & \text{if } i = s+1, \ldots, s+t \end{cases} \tag{5.2}$$

and $\widetilde{\varphi} := \varphi_1 \oplus \varphi_2$.

The following two lemmas are direct consequence of the definitions of the warped product and the structures on it. The proofs are straightforward and we omit them here.

Lemma 5.1. *$(\widetilde{M}, \widetilde{\varphi}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_{s+t}, \widetilde{\eta}^1, \ldots, \widetilde{\eta}^{s+t}, \widetilde{g})$ is a metric $f.pk$-manifold.*

Then we denote by $\widetilde{F}$ the Sasaki form associated with $\widetilde{g}$ and $\widetilde{\varphi}$.

Lemma 5.2. *For each $i = 1, \ldots, s$ and for each $h = s+1, \ldots, s+t$ one has*

$$d\widetilde{\eta}^i = \pi_1^* d\eta^i$$

$$d\widetilde{\eta}^h = d\rho \wedge \pi_2^*\theta^{h-s} + \rho\pi_2^* d\theta^{h-s}$$

$$\widetilde{F} = \pi_1^* F_1 + \rho^2 \pi_2^* F_2,$$

where F_1 and F_2 are the fundamental forms associated with φ_1 and φ_2.

From the above Lemmas we get the following corollaries.

Corollary 5.1. *Suppose that B and V are almost $\mathcal{C}$-manifolds. Then $\widetilde{M}$ is an almost $\mathcal{C}$-manifold if and only if ρ is constant.*

Corollary 5.2. *If B and V are almost $\mathcal{S}$-manifolds then $\widetilde{M}$ is never an almost $\mathcal{S}$-manifold.*

Remark 5.1. From a direct computation we get

$$N_{\widetilde{\varphi}} = N_{\varphi_1} + N_{\varphi_2} + 2\sum_{l=1}^{t} d\rho \wedge \pi_2^* \theta^l$$

so that the structure $\widetilde{\varphi}$ is normal if and only if φ_1 and φ_2 are normal structures and ρ is constant.

Corollary 5.3. *The manifold $\widetilde{M}$ is a $\mathcal{K}$-manifold if and only if ρ is constant and B, V are $\mathcal{K}$-manifolds.*

In the following two corollaries we consider the particular case when (B, g_1, φ_1) is an almost Hermitian manifold and $\dim V = t$. Hence V is parallelizable by $\zeta_1, \ldots, \zeta_t$. From Lemma 5.1 it follows:

Corollary 5.4. *The structure $(\widetilde{M}, \widetilde{\varphi}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_t, \widetilde{\eta}^1, \ldots, \widetilde{\eta}^t, \widetilde{g})$ defined by (5.1) and (5.2) is an $f.pk$-manifold. Moreover, $d\widetilde{\eta}^i = d\rho \wedge \pi_2^*\theta^i + \rho\pi_2^* d\theta^i$ for each $i = 1, \ldots, s$ and $\widetilde{F} = \pi_1^*\Phi$ where Φ is the Kähler form associated with φ_1 and g_1.*

Corollary 5.5. *The $f.pk$-structure on $\widetilde{M}$ is never almost $\mathcal{S}$. It is an almost $\mathcal{C}$ if and only if ρ is constant, B is symplectic and $\theta^1, \ldots, \theta^s$ are closed. Moreover, for each $i, j = 1, \ldots, t$ and each $X, Y \in \mathcal{X}(B)$ one has*

$$N_{\widetilde{\varphi}}(\widetilde{\xi}_i, \widetilde{\xi}_j) = -[\zeta_i, \zeta_j]$$

$$N_{\widetilde{\varphi}}(X, \widetilde{\xi}_i) = X(\ln \rho)\zeta_i$$

$$N_{\widetilde{\varphi}}(X, Y) = [\varphi_1, \varphi_1](X, Y).$$

It follows that $\widetilde{\varphi}$ is normal if and only if φ_1 is integrable, $[\zeta_i, \zeta_j] = 0$ and ρ is constant.

In the following two corollaries we assume that $\dim B = s$ and hence B is parallelizable by $\xi_1, \ldots, \xi_s$. Moreover we assume that (V, g_2, φ_2) is an almost Hermitian manifold. Then as an immediate consequence of Lemma 5.1 we get the following result.

Corollary 5.6. *If $(\widetilde{M}, \widetilde{\varphi}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_s, \widetilde{\eta}^1, \ldots, \widetilde{\eta}^s, \widetilde{g})$ is a metric $f.pk$-manifold then $d\widetilde{\eta}^i = \pi_1^* d\eta^i$ for all $i = 1, \ldots, s$ and $\widetilde{F} = \rho^2 \pi_2^* \Phi$, where Φ is the Kähler form associated with φ_2 and g_2.*

Hence we have the following property.

Corollary 5.7. *The $f.pk$-structure on $\widetilde{M}$ is never an almost $\mathcal{S}$-manifold. Then $\widetilde{M}$ is an almost $\mathcal{C}$-manifold if and only if $\eta^1, \ldots, \eta^s$ are closed, V is symplectic and ρ is constant. Moreover, for each $i, j = 1, \ldots, s$ and $X, Y \in \mathcal{X}(V)$ we have*

$$N_{\widetilde{\varphi}}(\widetilde{\xi}_i, \widetilde{\xi}_j) = \sum_{k=1}^{s} \eta^k [\xi_j, \xi_i] \xi_k$$

$$N_{\widetilde{\varphi}}(X, \widetilde{\xi}_i) = 0$$

$$N_{\widetilde{\varphi}}(X, Y) = [\varphi_2, \varphi_2](X, Y).$$

Hence, if φ_2 is integrable and $[\xi_i, \xi_j] = 0$ for each $i, j = 1, \ldots, s$ then the $f.pk$-structure is normal.

6. Examples

Example 6.1. Let (V_0, g_0, J_0) be a Kähler manifold with the fundamental 2-form Ω_0 such that $\Omega_0 = d\omega_0$. We consider the product $N = M_0 \times \mathbb{R}^s$, $s \geq 1$, which can be thought as the total space of a principal bundle with structure group $\mathbb{R}^s$. The 1-forms $\eta^i = \pi^* \omega_0 + dt_i$, $i = 1, \ldots, s$, $(t_1, \ldots, t_s$ are natural coordinates on $\mathbb{R}^s)$ verify $d\eta^i = \pi^* \Omega_0$ and $(\eta^1, \ldots, \eta^s)$ is a connection form on N. Hence, from [2] we obtain that N admits an $\mathcal{S}$-structure, whose metric tensor is $g = \pi^* g_0 + \sum_{i=1}^{s} \eta^i \otimes \eta^i$ and the Sasaki form is $F = \pi^* \Omega_0$. If we modify the metric as $\widetilde{g} = s\pi^* g_0 + \sum_{i=1}^{s} \eta^i \otimes \eta^i$, then we have $\widetilde{F} = sF$. So we obtain an $f.pk$-manifold which satisfies condition (3.5).

The previous construction of N can be modified by substituting $\mathbb{R}^s$ with the s-dimensional torus $(S^1)^s$. Then the construction goes in the same way and we obtain the desired $f.pk$-structure on the toroidal bundle $M_0 \times (S^1)^s$.

The following example is an application of Corollary 5.6

Example 6.2. We consider $B := \mathbb{R}^s$ and $V := \mathbb{C}^n$ both with their canonical Riemannian flat structures and a warping function $\rho : \mathbb{R}^s \to \mathbb{R}$ such that $\rho(t_1, \ldots, t_s) = e^{t_1+\cdots+t_s}$. We take $\xi_1 = \frac{\partial}{\partial t_1}, \ldots, \xi_s = \frac{\partial}{\partial t_s}$ the canonical vector fields on $\mathbb{R}^s$ immersed in the warped product $\widetilde{M} = B \times_\rho V$. Using the construction from Lemma 5.1 we obtain that $\widetilde{M}$ is a $f.pk$-manifold. We observe that $\xi_1, \ldots, \xi_s$ are not Killing since for each $X, Y \in \mathcal{X}(V)$ and $i = 1, \ldots, s$ we have $(L_{\xi_i}\widetilde{g})(X, Y) = 2e^{2(t_1+\cdots+t_s)}\langle X, Y\rangle$. Moreover the metric $\widetilde{g}$ is not flat; for instance, for each $X \in \mathcal{X}(V)$ and $i, j = 1, \ldots, s$ we have $R_{X\xi_i}\xi_j = \frac{H^\rho(\xi_i,\xi_j)}{\rho}X = X$ where H^ρ denotes the Hessian of ρ, cf. [16].

Example 6.3. We consider a Kähler manifold (M_0, J_0, g_0) such that its fundamental form is exact, namely $\Omega_0 = d\omega_0$. We put $M = M_0 \times \mathbb{R}^s$, $s = p + q$, and

$$\eta_i := \begin{cases} \pi^*\omega_0 + dt_i & \text{for } i = 1, \ldots, p \\ dt_i & \text{for } i = p+1, \ldots, s, \end{cases}$$

where $t_1, \ldots, t_s$ are natural coordinates in $\mathbb{R}^s$. Then M with the metric $g = \pi^* g_0 + \sum_{i=1}^s \eta^i \otimes \eta^i$ is a metric $f.pk$-structure such that $d\eta^i = F$ for all $i = 1, \ldots, p$ and $d\eta_i = 0$ for all $i = p+1, \ldots, s$. This is an example of metric $f.pk$-manifold such that $d\eta^i = 0$ for some $i \in \{1, \ldots, s\}$ and $d\eta^j = F$ for the other values of the index. Such manifolds were firstly studied in [17], [18]. Moreover, for $p = 1$ and $q = s - 1$ this is an example of a metric $f.pk$-manifold satisfying identity (3.5).

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