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On a multivalued iterative equation

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Abstract. A second order iterative functional equation is considered for multifunctions. A result on the existence and uniqueness of solutions in some class of multifunctions is presented.

1. Introduction

In the theory of functional equations in a single variable an important role is played by equations with superpositions of the unknown function (cf. [2], [7]). Among them is the iterative equation of the form

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = g(x), \tag{$*$}$$

where g is a given function. This equation (and its various special cases) was considered by many authors and there is a large number of papers devoted to it. In particular results on the existence, uniqueness and stability of its solutions in several classes of functions can be found, e.g., in [5], [8], [9], [17]–[19]. Up to the authors' best knowledge, equation $(*)$ was not considered so far in the class of multifunctions, cf. however the papers

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[12], [13], [16]. Multivalued solutions of functional equations in several variables were investigated by several authors, cf., e.g., [10], [11], [14], [15].

The subject of this note is the second order multivalued iterative equation

$$\lambda_1 F(x) + \lambda_2 F^2(x) = G(x), \tag{1.1}$$

where G is a given multifunction, F is an unknown multifunction, and λ_1, λ_2 are real constants. Here F^2 stands for the second iterate of F, that is $F^2(x) := \cup\{F(y) : y \in F(x)\}$. We present a result on the existence and uniqueness of solutions in some class of upper semicontinuous multifunctions. As the upper semicontinuity for multifunctions is much weaker than the continuity for functions, the method used for continuous solutions and smooth solutions in [17], [18] has to be improved substantially.

2. A class of multifunctions

Let $I = [a, b]$ be a given interval and $cc(I)$ denote the family of all nonempty convex compact subsets of I. This family endowed with the Hausdorff distance defined by

$$h(A, B) = \max\{\sup\{d(a, B) : a \in A\},\ \sup\{d(b, A) : b \in B\}\}, \tag{2.2}$$

where $d(a, B) = \inf\{|a - b| : b \in B\}$, is a complete metric space (cf. e.g. [6], Cor. 4.3.12).

A multifunction $F : I \to cc(I)$ is *increasing* (resp. *strictly increasing*) if for every $x, y \in I$, $x < y$, we have $\max F(x) \le \min F(y)$ (resp. $\max F(x) < \min F(y)$) (cf. [1], Def. 3.5.1).

$F : I \to cc(I)$ is *upper semicontinuous* (abbreviated by *USC*) at a point $x_0 \in I$ if for every open set $V \subset \mathbf{R}$ with $F(x_0) \subset V$ there exists a neighbourhood U_{x_0} of x_0 such that $F(x) \subset V$ for every $x \in U_{x_0}$. F is USC on I if it is USC at every point in I.

Let $\mathcal{F}(I)$ be the family of all multifunctions $F : I \to cc(I)$ and let $\Phi(I)$ be its subfamily defined by

$$\Phi(I) = \{F \in \mathcal{F}(I) : \text{ is USC, increasing, } F(a) = \{a\},\ F(b) = \{b\}\}. \tag{2.3}$$

We endow $\Phi(I)$ with the metric

$$D(F_1, F_2) = \sup\{h(F_1(x), F_2(x)) : x \in I\}, \quad \forall F_1, F_2 \in \Phi(I). \tag{2.4}$$

Lemma 1. *The metric space* $(\Phi(I), D)$ *is complete.*

Proof. Let (F_n) be a Cauchy sequence in $\Phi(I)$. Then for every fixed $x \in I$, $(F_n(x))$ is a Cauchy sequence in $cc(I)$. Since $(cc(I), h)$ is complete, there exists $\lim_{n\to\infty} F_n(x) =: F(x) \in cc(I)$. Then $F_n \to F$ in the sense of the metric D. To see this, fix arbitrarily $\varepsilon > 0$. Since (F_n) is a Cauchy sequence, there exists an $n_0 \in \mathbf{N}$ such that $D(F_n, F_k) \le \varepsilon$ for all $n, k \ge n_0$. Hence $h(F_n(x), F_k(x)) \le \varepsilon$, $\forall x \in I$. Letting $k \to \infty$, we obtain $h(F_n(x), F(x)) \le \varepsilon$, $\forall x \in I$, which means that $D(F_n, F) \le \varepsilon$.

Now we will show that $F \in \Phi(I)$. Of course $F(a) = \{a\}$ and $F(b) = \{b\}$. To prove that F is USC, fix an $x_0 \in I$ and take an open set V containing $F(x_0)$. Since $F(x_0)$ is compact, there exists an $\varepsilon > 0$ such that $F(x_0) + (-\varepsilon, \varepsilon) \subset V$. Using the fact that $F_n \to F$, we can find an $n_0 \in \mathbf{N}$ such that $D(F_{n_0}, F) < \varepsilon/3$. Hence $h(F_{n_0}(x), F(x)) < \varepsilon/3$, $\forall x \in I$. Consequently,

$$\begin{aligned} F(x) &\subset F_{n_0}(x) + (-\varepsilon/3, \varepsilon/3) \quad \text{and} \\ F_{n_0}(x_0) &\subset F(x_0) + (-\varepsilon/3, \varepsilon/3). \end{aligned} \tag{2.5}$$

Since F_{n_0} is USC at x_0, there exists a neighbourhood U_{x_0} of x_0 such that

$$F_{n_0}(x) \subset F_{n_0}(x_0) + (-\varepsilon/3, \varepsilon/3)\ , \forall x \in U_{x_0}. \tag{2.6}$$

Using (2.5) and (2.6) we get $F(x) \subset F(x_0) + (-\varepsilon, \varepsilon), \forall x \in U_{x_0}$, which proves that F is USC at x_0.

Finally we will show that F is increasing. On the contrary, suppose that there exist $x_1, x_2 \in I$, $x_1 < x_2$, such that $\sup F(x_1) > \inf F(x_2)$. Put $\varepsilon := \sup F(x_1) - \inf F(x_2)$. Since $F_n(x_1) \to F(x_1)$ and $F_n(x_2) \to F(x_2)$, we can find an $n_0 \in \mathbf{N}$ such that $F(x_1) \subset F_{n_0}(x_1) + (-\varepsilon/2, \varepsilon/2)$ and $F(x_2) \subset F_{n_0}(x_2) + (-\varepsilon/2, \varepsilon/2)$. Hence

$$\begin{aligned} \sup F(x_1) &< \sup F_{n_0}(x_1) + \varepsilon/2 \quad \text{and} \\ \inf F(x_2) &> \inf F_{n_0}(x_2) - \varepsilon/2. \end{aligned} \tag{2.7}$$

Consequently, using (2.7) and the definition of ε, we obtain

$$\sup F_{n_0}(x_1) > \sup F(x_1) - \varepsilon/2 = \inf F(x_2) + \varepsilon/2 > \inf F_{n_0}(x_2),$$

which contradicts the fact that F_{n_0} is increasing. $\square$

Lemma 2. *If $F, G \in \Phi(I)$ and $F(x) \subset G(x)$ for all $x \in I$, then $F = G$.*

Proof. Suppose, contrary to our claim, that $F(x_0) \neq G(x_0)$ for some $x_0 \in I$. Take a point $y_0 \in G(x_0) \setminus F(x_0)$. Since $F(x_0)$ is a compact interval, we have

$$y_0 < \min F(x_0) \quad \text{or} \quad y_0 > \max F(x_0).$$

Assume that the first case occurs (the proof in the second case is analogous). Put $\varepsilon := \min F(x_0) - y_0$. Since F is USC at x_0, there exists a neighbourhood U_{x_0} of x_0 such that

$$F(x) \subset F(x_0) + (-\varepsilon, \varepsilon), \quad \forall x \in U_{x_0}. \tag{2.8}$$

By the monotonicity of G we have

$$\max G(x) \leq \min G(x_0) \leq y_0, \quad \forall x < x_0. \tag{2.9}$$

Using (2.8), the definition of ε and (2.9), we obtain for every $x \in U_{x_0}$, $x < x_0$

$$\max G(x) \leq y_0 = \min F(x_0) - \varepsilon < \min F(x).$$

This contradicts the fact that $F(x) \subset G(x)$ and completes the proof. □

3. The result

Theorem 1. *Let $G \in \Phi(I)$, $\lambda_1 > \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$. Then equation* (1.1) *has a unique solution $F \in \Phi(I)$.*

Proof. Define the mapping $L : \Phi(I) \to \mathcal{F}(I)$ by

$$LF(x) = \lambda_1 x + \lambda_2 F(x), \quad \forall x \in I, \tag{3.10}$$

where $F \in \Phi(I)$. Clearly, LF is USC and $LF(a) = \{a\}$, $LF(b) = \{b\}$. Moreover, for any $x_2 > x_1$ in I, we have $\min F(x_2) - \max F(x_1) \geq 0$ since F is increasing. Therefore

$$\min LF(x_2) - \max LF(x_1) = \lambda_1 x_2 - \lambda_1 x_1 + \lambda_2 \min F(x_2) - \lambda_2 \max F(x_1)$$

$$\geq \lambda_1(x_2 - x_1) > 0, \tag{3.11}$$

for $\lambda_1 > 0$ and $\lambda_2 \geq 0$. This means that LF is strictly increasing and, consequently,

$$LF(x) \cap LF(y) = \emptyset, \quad \forall x \neq y. \tag{3.12}$$

Thus, $LF \in \Phi(I)$. Furthermore,

$$LF(I) := \cup_{x\in I} LF(x) = I \tag{3.13}$$

because $a, b \in LF(I)$ and $LF(I)$ is connected as the image of a connected set by an USC multifunction with connected values (cf. [4], Prop. 2.24).

By (3.12), the multifunction $(LF)^{-1}$, defined by $(LF)^{-1}(y) = \{x \in I : y \in LF(x)\}$ for each $y \in I$, is single-valued. Moreover, it is also increasing (cf. [1], p. 105) and USC (cf. [1], Prop. 1.4.8). Consequently, being single-valued, it is continuous. Define the mapping $\mathcal{T} : \Phi(I) \to \mathcal{F}(I)$ by

$$\mathcal{T}F(x) = (LF)^{-1}(G(x)), \quad \forall F \in \Phi(I),\ \forall x \in I. \tag{3.14}$$

For every $F \in \Phi(I)$, $\mathcal{T}F$ has values in $cc(I)$ as continuous images of compact intervals. $\mathcal{T}F$ is also USC as a composition of USC multifunctions (cf. e.g. [4], Prop. 2.56 or [3], Prop. 14.10), increasing and $\mathcal{T}F(a) = \{a\}$, $\mathcal{T}F(b) = \{b\}$. Therefore $\mathcal{T}F : \Phi(I) \to \Phi(I)$. Moreover,

$$(LF)^{-1}(y_2) - (LF)^{-1}(y_1) \leq \frac{1}{\lambda_1}(y_2 - y_1), \tag{3.15}$$

for any $y_2 > y_1$ in I. In fact, let $x_j = (LF)^{-1}(y_j)$ (where $j = 1, 2$) since $(LF)^{-1}$ is single-valued. Then $y_j \in LF(x_j)$ and therefore $\min LF(x_2) \leq y_2$ and $\max LF(x_1) \geq y_1$. From (3.11) we see that

$$(LF)^{-1}(y_2) - (LF)^{-1}(y_1) \leq \frac{1}{\lambda_1}(\min LF(x_2) - \max LF(x_1)) \leq \frac{1}{\lambda_1}(y_2 - y_1),$$

which proves (3.15). Thus, for $F_1, F_2 \in \Phi(I)$, we obtain by (3.15) and (2.2) that

$$\sup_{y\in I} |(LF_1)^{-1}(y) - (LF_2)^{-1}(y)|$$

$$= \sup_{y\in I} |(LF_1)^{-1}(y) - (LF_1)^{-1}(LF_1((LF_2)^{-1}(y)))| \le \frac{1}{\lambda_1} \sup_{y\in I} d(y, LF_1((LF_2)^{-1}(y))) \le \frac{1}{\lambda_1} \sup_{x\in I} h(LF_2(x), LF_1(x)), \tag{3.16}$$

where we note that $LF_2(I) = I$ and that $d(y, LF_1((LF_2)^{-1}(y))) = d(y, LF_1(x)) \le h(LF_2(x), LF_1(x))$ because $x = (LF_2)^{-1}(y)$ is single-valued and $y \in LF_2(x)$. For every $z \in I$ and $y_\ell \in G(z)$,

$$\begin{aligned} d((LF_1)^{-1}(y_\ell), (LF_2)^{-1}(G(z))) &= \inf_{y\in G(z)} |(LF_1)^{-1}(y_\ell) - (LF_2)^{-1}(y)| \\ &\le |(LF_1)^{-1}(y_\ell) - (LF_2)^{-1}(y_\ell)| + \inf_{y\in G(z)} |(LF_2)^{-1}(y_\ell) - (LF_2)^{-1}(y)| \\ &\le \frac{1}{\lambda_1} \sup_{x\in I} h(LF_2(x), LF_1(x)) + \frac{1}{\lambda_1} \inf_{y\in G(z)} |y_\ell - y| \\ &= \frac{1}{\lambda_1} \sup_{x\in I} h(LF_2(x), LF_1(x)) \end{aligned} \tag{3.17}$$

by (3.15) and (3.16). Similarly we get

$$d((LF_2)^{-1}(y_\ell), (LF_1)^{-1}(G(z))) \le \frac{1}{\lambda_1} \sup_{x\in I} h(LF_2(x), LF_1(x)). \tag{3.18}$$

By (3.17) and (3.18) we see that for every $F_1, F_2 \in \Phi(I)$,

$$\begin{aligned} D(\mathcal{T}F_1, \mathcal{T}F_2) &= \sup_{z\in I} h((LF_1)^{-1}(G(z)), (LF_2)^{-1}(G(z))) \\ &\le \frac{1}{\lambda_1} \sup_{x\in I} h(LF_2(x), LF_1(x)). \end{aligned} \tag{3.19}$$

By the definition of L and the known properties of Hausdorff metric, we obtain for every $x \in I$

$$\begin{aligned} h(LF_1(x), LF_2(x)) &= h(\lambda_1 x + \lambda_2 F_1(x), \lambda_1 x + \lambda_2 F_2(x)) \\ &= \lambda_2 h(F_1(x), F_2(x)). \end{aligned} \tag{3.20}$$

From (3.19) and (3.20) we obtain that

$$\begin{aligned} D(\mathcal{T}F_1, \mathcal{T}F_2) &\le \frac{1}{\lambda_1} \sup_{x\in I} h(LF_1(x), LF_2(x)) \\ &\le \frac{\lambda_2}{\lambda_1} \sup_{x\in I} h(F_1(x), F_2(x)) \le \frac{\lambda_2}{\lambda_1} D(F_1, F_2), \end{aligned} \tag{3.21}$$

which implies that $\mathcal{T}$ is a contraction because $\lambda_2 < \lambda_1$. Therefore, by the

Banach's fixed point principle, $\mathcal{T}$ has a unique fixed point F in $\Phi(I)$, i.e.

$$(LF)^{-1}(G(x)) = F(x), \quad \forall x \in I. \tag{3.22}$$

Hence we obtain

$$\lambda_1 F(x) + \lambda_2 F^2(x) = (LF)(F(x)) = LF(LF)^{-1}(G(x)) \supset G(x), \quad \forall x \in I.$$

Since G, $\lambda_1 F + \lambda_2 F^2 \in \Phi(I)$, using Lemma 2 we get

$$\lambda_1 F(x) + \lambda_2 F^2(x) = G(x), \quad \forall x \in I.$$

This completes the proof. □

For example, the multifunction

$$G(x) = \begin{cases} 2x, & x \in \left[0, \frac{1}{4}\right], \\ \frac{1}{2}, & x \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ \left[\frac{1}{2}, \frac{3}{4}\right], & x = \frac{1}{2}, \\ \frac{1}{2}x + \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

is in the class $\Phi(I)$ where $I = [0, 1]$. With this G in (1.1) our theorem can be applied.

As another example, $G(x) = \sqrt[3]{x}$, being a continuous single-valued function on $I = [-1, 1]$, is not differentiable at $x = 0$. Thus G does not satisfy the Lipschitzian condition and Theorems in [17] and [18] do not work for this G. However, $G \in \Phi(I)$. So our theorem can be applied.

Since we do not require the Lipschitz condition, in this paper some techniques used in [17] and [18] cannot be employed to generalize our result to equation $(*)$ (of the n-th order). Discussing the general n-th iterative equation will be the subject of our next work.

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