Publ. Math. Debrecen
64/3-4 (2004), 437–450

Complex rotundity of Musielak–Orlicz function spaces equipped with the Orlicz norm

By CUIXIA HAO (Edmonton), LIFANG LIU (Xiamen) and
TINGFU WANG†

Abstract. The criteria for complex extreme points, complex rotundity, complex locally uniformly rotund points, complex local uniform rotundity and complex uniform rotundity in complex Musielak–Orlicz function spaces equipped with the Orlicz norm are given.

0. Introduction

Many mathematicians worked on rotundity properties in real Banach spaces ([2], [6], [7]) because these properties are very important in geometry of Banach spaces and its applications. In the recent years, many mathematicians have developed the investigations concerning the geometric theory of complex Banach spaces, because its applications are irreplaceable by the geometric theory of real Banach spaces. Let D be a domain (an open connected subset) in the complex plane and let f be a complex-valued analytic function on D. Then the classical maximum modulus principle

Mathematics Subject Classification: 46B20, 46E30.
Key words and phrases: Musielak–Orlicz space, condition delta two, Orlicz norm, complex rotundity, complex locally uniformly rotund points, complex local uniform rotundity, complex uniform rotundity.
The first author was supported by Heilongjiang Educational Committee under Grant No. 9541034.
The second author was supported by the Center Program of Xiamen University under Grant No. X01103.

says that either $|f(z)|$ has no maximum on D or $|f(z)|$ is a constant on D. If f is analytic and has values in a complex Banach space, it is well known that the theorem still holds. However, the strong form of the maximum modulus theorem, where if $|f(z)|$ is constant then $f(z)$ is also constant, is no longer true in general. In 1967, E. Thorp and R. Whitley (see [11]) first investigated the structure of complex extreme points and showed that the strong form of the maximum modulus principle holds for a complex Banach space X if and only if each point of norm one is a "complex extreme point" of the unit sphere of X. In 1975, J. Globevnik (see [5]) investigated complex rotundity and complex uniform rotundity, and pointed out that $L_1[0,1]$ is complex uniformly rotund (real space $L_1[0,1]$ is not even rotund). Many mathematicians discussed complex rotundity in general Banach spaces (see [3], [4], [8]–[10] and [13]). It is well known that into the class of Musielak–Orlicz spaces include a lot of classical spaces such as L_p $(1 \leq p \leq \infty)$, Orlicz spaces etc. At the end of 1980's, C. Wu and H. Sun discussed complex extreme points, complex rotundity and complex uniform rotundity in Orlicz spaces (see [14]–[17]). Next T. Wang and Y. Teng (see [12]) introduced the concepts of complex locally uniformly rotund points and complex local uniform rotundity, and obtained criteria for them in Musielak–Orlicz spaces. But the above discussion was proceeded in the case of the Luxemburg norm. For the Orlicz norm, only one result on complex extreme points of Musielak–Orlicz sequence spaces was given by C. Wu and H. Sun (see [14]) in 1991. In this paper, we discuss complex rotundity, complex locally uniformly rotund points, complex local uniform rotundity and complex uniform rotundity in Musielak–Orlicz function spaces equipped with the Orlicz norm. The conclusions that we get seem to be clear and they differ a lot from the corresponding results concerning the Luxemburg norm.

Let $\mathcal{N}$ denote the set of natural numbers, $\mathcal{R}$ and $\mathcal{C}$ denote the sets of real and complex numbers, respectively. Let $(X, \|\cdot\|)$ be a complex Banach space and $S(X)$ be the unit sphere of X.

Let (T, Σ, μ) be a nonatomic, complete and σ-finite measure space and L^0 (resp. L^c) be the space of all (equivalence classes of) Σ-measurable real (resp. complex) functions defined on T. In the whole paper the equality of two functions of variable t (resp. two sequences with n) is understood in the sense "for μ-a.e. $t \in T$" (resp. "for all $n \in \mathbb{N}$"). Similarly, "for $t \in A$" means "for μ-a.e. $t \in A$", where $A \in \Sigma$.

A point $x \in S(X)$ is called a complex extreme point if for any $y \in X$ with $y \neq 0$ the inequality $\max_{|\lambda|\leq 1} \|x + \lambda y\| > 1$ holds.

A complex Banach space X is called complex rotund (CR for short) if every point $x \in S(X)$ is a complex extreme point.

A point $x \in S(X)$ is called a complex locally uniformly rotund point (C-LUR point for short) if for any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that for all $y \in X$ satisfying $\|y\| > \varepsilon$, the inequality $\max_{|\lambda|\leq 1} \|x + \lambda y\| \geq 1 + \delta$ holds.

A complex Banach space X is called complex locally uniformly rotund (C-LUR for short) if every point $x \in S(X)$ is a C-LUR point.

A complex Banach space X is called complex uniformly rotund (CUR for short) if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\max\limits_{|\lambda|\leq 1} \|x + \lambda y\| \geq 1 + \delta$ holds for all $x \in S(X)$ and $y \in X$ satisfying $\|y\| > \varepsilon$.

A function $M : T \times \mathcal{R} \to [0, +\infty]$ is said to be a Musielak–Orlicz function if M has the following properties:

(1) $M(\cdot, u) \in L^0$ for any $u \in \mathcal{R}$,

(2) $M(t, \cdot)$ is even, convex, continuous at zero and left continuous on $\mathcal{R}_+$ $(t \in T)$,

(3) $M(t, 0) = 0$, $\lim_{u\to\infty} M(t, u) = \infty$ and $M(t, u_t) < \infty$ for some $u_t \in (0, +\infty)$ $(t \in T)$.

N is called the complementary function of M if

$$N(t, v) = \sup_{u\geq 0}\{u|v| - M(t, u)\} \quad (t \in T,\ v \in \mathcal{R}).$$

Then N is also a Musielak–Orlicz function. For any $t \in T$, define

$$e(t) = \sup\{u \geq 0 : M(t, u) = 0\},$$

$$E(t) = \sup\{u \geq 0 : M(t, u) < \infty\},$$

$$A(t) = \sup\{v \geq 0 : N(t, v) < \infty\}.$$

Let $p_-(t,u)$ (resp. $p(t,u)$) denote the left (resp. the right) derivative of $M(t,u)$ at u, assuming $p(t,u) = -p_-(t,-u) = \infty$ for $u \geq E(t)$ and $p_-(t,u) = -p(t,-u) = \infty$ for $u > E(t)$. Let $q_-(t,v)$ (resp. $q(t,v)$) be the left (resp. the right) derivative of $N(t,v)$ at v, assuming $q(t,v) = -q_-(t,-v) = \infty$ for $v \geq A(t)$ and $q_-(t,v) = -q(t,-v) = \infty$ for $v > A(t)$. Then $q(t,v) = \sup\{u \geq 0 : p(t,u) \leq v\}$ and $N(t,v) = \int_0^{|v|} q(t,s)ds$ for any $v \in \mathcal{R}$ $(t \in T)$.

It is well known that there holds the Young inequality

$$|uv| \leq M(t,u) + N(t,v) \quad (t \in T,\ u,v \in \mathcal{R}).$$

Moreover, $|uv| = M(t,u) + N(t,v)$ if and only if $p_-(t,u) \leq v \leq p(t,u)$ or $q_-(t,v) \leq u \leq q(t,v)$.

Given a Musielak–Orlicz function M, we define the convex modular ρ_M: $L^c \to [0,+\infty]$ by

$$\rho_M(x) = \int_T M(t,|x(t)|)d\mu.$$

The linear space

$$\{x \in L^c : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

equipped with the Luxemburg norm

$$\|x\|_M = \inf\left\{\lambda > 0 : \rho_M\left(\frac{x}{\lambda}\right) \leq 1\right\}$$

or with the Orlicz norm

$$\|x\|_M^0 = \sup\{\langle |x|,|y|\rangle : \rho_N(y) \leq 1\}$$

is a complex Banach space, where $\langle |x|,|y|\rangle = \int_T |x(t)||y(t)|d\mu$. We denote it by L_M or L_M^0, respectively. These two norms are equivalent and the inequalities $\|x\|_M \leq \|x\|_M^0 \leq 2\|x\|_M$ hold for any $x \in L_M$. It is known that if there exists an Orlicz function M such that $M(t,u) = M(u)$ for any $t \in T$ and $u \in \mathcal{R}$, then L_M becomes an Orlicz space. It is also known that $\|x\|_M^0 = \inf_{k>0} \frac{1}{k}(1 + \rho_M(kx))$ for any $x \in L_M$, which is called the Amemiya–Orlicz formula for the Orlicz norm.

We say that for any $T_0 \in \Sigma$, M satisfies condition $\mathbf{\Delta}_2(T_0)$ ($M \in \mathbf{\Delta}_2(T_0)$ for short) if for any $h > 1$, there exist $k > 1$ and a nonnegative function $\delta \in L^0$ with $\int_{T_0} \delta(t)d\mu < \infty$ such that $M(t, hu) \leq kM(t, u) + \delta(t)$ ($t \in T_0, u \in \mathcal{R}$). Given a Musielak–Orlicz function M, we define the functional $\xi_M : L_M \to \mathcal{R}_+$ by $\xi_M(x) = \inf\{\lambda > 0 : \rho_M(\frac{x}{\lambda}) < \infty\}$. For any $x \in L^0_M$, we define

$$k^*_x = \inf\{k \geq 0 : \rho_N(p \circ kx) \geq 1\}, \quad k^{**}_x = \sup\{k \geq 0 : \rho_N(p \circ kx) \leq 1\}.$$

It is known (see [14]) that $\|x\|^0_M = \frac{1}{k}(1 + \rho_M(kx))$ for a number $k \in (0, \infty)$ if and only if $k^*_x \leq k \leq k^{**}_x$. In the sequel for any $x \in L_M$ we denote by S_x the set $\{t \in T : x(t) \neq 0\}$. By χ we denote the characteristic function.

Lemma 0.1 (see [8]). *For any Musielak–Orlicz function M, there exists an ascending sequence $(T_k)^\infty_{k=1} \subset \Sigma$ which satisfies $\bigcup^\infty_{k=1} T_k = T$, $\mu(T_k) < \infty$ and $\sup\{M(t, \lambda) : t \in T_k\} < \infty$ for any $\lambda > 0$ and $k \in \mathcal{N}$.*

Lemma 0.2 (see [14], Theorem 1). *Let $0 \neq x \in L^0_M$ and A is the function defined on page 3. Then:*

(1) *If $\rho_N(A\chi_{S_x}) > 1$, then $\|x\|^0_M = \frac{1}{k}(1 + \rho_M(kx))$ for some $k \in (0, \infty)$ and it is the only possibility to attain the norm $\|x\|^0_M$.*

(2) *If $\rho_N(A\chi_{S_x}) \leq 1$, then $\|x\|^0_M = \langle |x|, A\rangle$ and if $\rho_N(A\chi_{S_x}) < 1$, then $\|x\|^0_M$ can not be attained in which way $\|x\|^0_M$ is then given.*

Since C is complex uniformly rotund, so we have the following

Lemma 0.3 (see [1], Proposition 5.17). *Let i be the complex number satisfying $i^2 = -1$. For any $\varepsilon > 0$ there exists $\delta \in (0, \frac{1}{2})$ such that if $u, v \in \mathcal{C}$ and $|v| \geq \frac{\varepsilon}{8} \max_{j\in I} |u + jv|$, then $|u| \leq \frac{1-2\delta}{4} \sum_{j\in I} |u + jv|$, where $I := \{\pm 1, \pm i\}$.*

1. Main results

Lemma 1.1. *If $\|(\frac{1}{4} \sum_{j\in I} |x + jy|)\|^0_M = \langle \frac{1}{4} \sum_{j\in I} |x + jy|, A\rangle$, then $\rho_N(A) \leq 1$.*

Proof. For any $t \in T$, $\frac{1}{4} \sum_{j\in I} |x(t)+jy(t)| \neq 0$. So, $S_{\frac{1}{4} \sum_{j \in I} |x+jy|} = T$. By Lemma 0.2, $\rho_N(A) \leq 1$. □

Theorem 1.2. *If $\rho_N(A) \le 1$, then L^0_M is* CUR.

Proof. Otherwise, there exist $\varepsilon > 0$ and two sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subset L^0_M$ with $\|x_n\|^0_M = 1$, $\|y_n\|^0_M \ge \varepsilon$ such that $\|x_n + \lambda y_n\|^0_M \le 1 + \frac{1}{n}$ ($|\lambda| \le 1$). Denote

$$E_n = \Big\{t \in T : |y_n(t)| \ge \frac{\varepsilon}{8} \max_{j\in I} |x_n(t) + jy_n(t)|\Big\}.$$

Then $\|y_n\chi_{T\setminus E_n}\|^0_M \le \frac{\varepsilon}{2}(1+\frac{1}{n}) < \frac{2\varepsilon}{3}$ ($n \ge 3$). Therefore $\|y_n\chi_{E_n}\|^0_M > \frac{\varepsilon}{3}$ ($n \ge 3$). If $t \in E_n$, then $|x_n(t)| < (1-2\delta)\frac{1}{4}\sum_{j\in I}|x_n(t)+jy_n(t)|$, where $\delta \in (0, \frac{1}{2})$.

By the assumption that $\rho_N(A) \le 1$ and by Lemma 0.2, we have

$$\begin{aligned}
1 = \|x_n\|^0_M &= \big\langle |x_n|, A\chi_{T\setminus E_n}\big\rangle + \langle |x_n|, A\chi_{E_n}\rangle \\
&\le \Big\langle \frac{1}{4}\sum_{j\in I}|x_n + jy_n|, A\chi_{T\setminus E_n}\Big\rangle + (1-2\delta)\Big\langle \frac{1}{4}\sum_{j\in I}|x_n+jy_n|, A\chi_{E_n}\Big\rangle \\
&= \Big\langle \frac{1}{4}\sum_{j\in I}|x_n + jy_n|, A\Big\rangle - 2\delta\Big\langle \frac{1}{4}\sum_{j\in I}|x_n+jy_n|, A\chi_{E_n}\Big\rangle \\
&\le \bigg\|\bigg(\frac{1}{4}\sum_{j\in I}|x_n+jy_n|\bigg)\bigg\|^0_M - 2\delta\,\langle |y_n|, A\chi_{E_n}\rangle \le 1 + \frac{1}{n} - 2\delta\|y_n\chi_{E_n}\|^0_M \\
&\le 1 + \frac{1}{n} - \frac{2\delta\varepsilon}{3} \le 1 - \frac{\delta\varepsilon}{3} \qquad (1)
\end{aligned}$$

for n large enough, which is a contradiction. □

Theorem 1.3. *A point $x \in S(L^0_M)$ is a complex extreme point if and only if for any $k \in (0,\infty)$ satisfying $\|x\|^0_M = \frac{1}{k}(1+\rho_M(kx)$ we have $\mu\{t \in T : k|x(t)| < e(t)\} = 0$.*

Proof. Necessity. If $\mu\{t \in T : k|x(t)| < e(t)\} > 0$, then there exist $T_0 \in \Sigma$ with $\mu T_0 > 0$ and $c > 0$ such that $k|x(t)| + c \le e(t)$ ($t \in T_0$). If $y = \frac{c}{k}\chi_{T_0}$, then $y \ne 0$ and

$$\|x+\lambda y\|^0_M \le \frac{1}{k}\big(1 + \rho_M(k(x+\lambda y))\big)$$

$$\le \frac{1}{k}(1+\rho_M(kx\chi_{T\setminus T_0})+\rho_M\big((k|x|+c)\chi_{T_0})\big)$$

$$= \frac{1}{k}(1+\rho_M(kx\chi_{T\setminus T_0}))$$

$$= \frac{1}{k}(1+\rho_M(kx)) = 1 \qquad (|\lambda|\le 1).$$

This means that x is not a complex extreme point.

Sufficiency. Assume that there exist $\varepsilon > 0$ and $y \in L^0_M$ satisfying $\|y\|^0_M > \varepsilon$ such that $\max_{|\lambda|\le 1}\|x+\lambda y\|^0_M \le 1$. By Lemma 0.3, there exists $\delta \in (0,\frac{1}{2})$ such that if $u, v \in \mathcal{C}$ and $|v| \ge \frac{\varepsilon}{8}\max_j |u+jv|$, then $|u| < (1-2\delta)\frac{1}{4}\sum_{j\in I}|u+jv|$. Let $E = \{t \in T : |y(t)| \ge \frac{\varepsilon}{8}\max_{j\in I}|x(t)+jy(t)|\}$. Then $\|y\chi_{T\setminus E}\|^0_M \le \frac{\varepsilon}{8}\|(\max_{j\in I}|x+jy|)\|^0_M \le \frac{\varepsilon}{8}\sum_{j\in I}\|x+jy\|^0_M \le \frac{\varepsilon}{2}$. Therefore $\|y\chi_E\|^0_M > \frac{\varepsilon}{2}$. If $t\in E$, then $|x(t)| < (1-2\delta)\frac{1}{4}\sum_{j\in I}|x(t)+jy(t)|$.

If $\|(\frac{1}{4}\sum_{j\in I}|x+jy|)\|^0_M = \langle \frac{1}{4}\sum_{j\in I}|x+jy|, A\rangle$, then L^0_M is CUR by Theorem 1.2. If $\|(\frac{1}{4}\sum_{j\in I}|x+jy|)\|^0_M = \frac{1}{k}(1+\rho_M(\frac{k}{4}\sum_{j\in I}|x+jy|))$. In this case, $\|x\|^0_M = 1$ and

$$1 \ge \left\|\left(\frac{1}{4}\sum_{j\in I}|x+jy|\right)\right\|^0_M = \frac{1}{k}\left(1+\rho_M\left(\frac{k}{4}\sum_{j\in I}|x+jy|\right)\right)$$

$$\ge \frac{1}{k}(1+\rho_M(kx)) \ge \|x\|^0_M.$$

So $\|x\|^0_M = \frac{1}{k}(1+\rho_M(kx))$. By the condition that if $k\in(0,\infty)$ satisfying $\|x\|^0_M = \frac{1}{k}(1+\rho_M(kx)$ we have $\mu\{t\in T : k|x(t)| < e(t)\} = 0$, we get $k|x(t)| \ge e(t)$ $(t\in T)$. Therefore,

$$1 = \|x\|^0_M = \frac{1}{k}\left(1+\rho_M(kx\chi_{T\setminus E})+\rho_M(kx\chi_E)\right)$$

$$< \frac{1}{k}\left(1+\rho_M\left(\frac{k}{4}\sum_{j\in I}|x+jy|\chi_{T\setminus E}\right)+(1-2\delta)\rho_M\left(\frac{k}{4}\sum_{j\in I}|x+jy|\chi_E\right)\right)$$

$$= \frac{1}{k}\left(1+\rho_M\left(\frac{k}{4}\sum_{j\in I}|x+jy|\right)-2\delta\rho_M\left(\frac{k}{4}\sum_{j\in I}|x+jy|\chi_E\right)\right)$$

$$\leq \frac{1}{k}\left(1+\rho_M\left(\frac{k}{4}\sum_{j\in I}|x+jy|\right)\right)-\frac{2\delta}{k}\rho_M\left(\frac{kx}{1-2\delta}\chi_E\right)$$

$$\leq \left\|\left(\frac{1}{4}\sum_{j\in I}|x+jy|\right)\right\|_M^0-\frac{2\delta}{k}\rho_M\left(\frac{kx}{1-2\delta}\chi_E\right)<1. \tag{2}$$

This is a contradiction, which finishes the proof. □

Theorem 1.4. *The space* L_M^0 *is* CR *if and only if* $e(t)=0$ *for* μ*-a.e.* $t\in T$ *or* $\rho_N(A)\leq 1$.

Proof. The proof of Sufficiency is trivial by Theorems 1.2 and 1.3.

Necessity. Otherwise, there exists $T_0\in\Sigma$ such that $\mu T_0>0$, $e(t)>0$ for $t\in T_0$ and $\rho_N(A\chi_{T\setminus T_0})>1$. Take $x\in L_M^0$ such that $S_x=T\setminus T_0$. By Lemma 0.2, there exists $k\in(0,\infty)$ such that $\|x\|_M^0=\frac{1}{k}(1+\rho_M(kx))$. However, $k|x(t)|=0<e(t)$ $(t\in T_0)$. By Theorem 1.3, x is not a complex extreme point. □

Theorem 1.5. *If* $x\in S(L_M^0)$. *Then* x *is a* C-LUR *point if and only if for* $k\in(0,\infty)$ *satisfying* $\|x\|_M^0=\frac{1}{k}(1+\rho_M(kx))$, *there holds:*

(1) $\mu\{t\in T: k|x(t)|<e(t)\}=0$,

(2) *If there exist* $s\in(0,1)$ *and* $T_0\in\Sigma$ *with* $\mu T_0>0$ *satisfying* $\rho_M(\frac{kx}{1-s}\chi_{T_0})<\infty$, *then* $M\in\mathbf{\Delta}_2(T_0)$.

Proof. Necessity. Since "C-LUR ⇒ CR", the necessity of condition (1) is trivial.

If (2) does not hold, then there exist $s\in(0,1)$ and $T_0\in\Sigma$ with $\mu T_0>0$ satisfying $\rho_M(\frac{kx}{1-s}\chi_{T_0})<\infty$, but $M\notin\mathbf{\Delta}_2(T_0)$. There exists $z\in L_M^0$ with $S_z=T_0$ such that $\rho_M(z)\leq 1$ and $\xi_M(z)=1$. Define y_n with $y_n=\frac{s}{k}z\chi_{T\setminus T_n}$, where the sequence $(T_n)_{n=1}^\infty$ satisfies Lemma 0.1. Then $\|y_n\|_M^0=\frac{s}{k}\|z\chi_{T\setminus T_n}\|_M^0\geq\frac{s}{k}\xi_M(z)=\frac{s}{k}>0$. But

$$\begin{aligned}\|x+\lambda y_n\|_M^0&\leq\frac{1}{k}(1+\rho_M(k(x+\lambda y_n)))\\&\leq\frac{1}{k}\left(1+\rho_M(kx)+\int_{T\setminus T_n}M\left(t,(1-s)\frac{k|x(t)|}{1-s}+s|z(t)|\right)d\mu\right)\end{aligned}$$

$$\leq \|x\|_M^0 + \frac{1-s}{k}\rho_M\left(\frac{kx\chi_{T\setminus T_n}}{1-s}\right) + \frac{s}{k}\rho_M\left(z\chi_{T\setminus T_n}\right)$$
$$\to \|x\|_M^0 = 1.$$

This shows that x is not a C-LUR point.

Sufficiency. Assume that there exist $\varepsilon > 0$ and $(y_n)_{n=1}^{\infty} \subset L_M^0$ with $\|y_n\|_M^0 > \varepsilon$ satisfying

$$\|x + \lambda y_n\|_M^0 \leq 1 + \frac{1}{n} \ (\forall\ |\lambda| \leq 1).$$

Denote

$$E_n = \left\{t \in T : |y_n(t)| \geq \frac{\varepsilon}{8}\max_{j\in I}|x(t) + jy_n(t)|\right\}.$$

Then $\|y_n\chi_{T\setminus E_n}\|_M^0 \leq \frac{\varepsilon}{2}(1 + \frac{1}{n}) < \frac{2\varepsilon}{3}$ $(n \geq 3)$. Therefore $\|y_n\chi_{E_n}\|_M^0 > \frac{\varepsilon}{3}$ $(n \geq 3)$. If $t \in E_n$, then

$$|x(t)| < (1-2\delta)\frac{1}{4}\sum_{j\in I}|x(t) + jy_n(t)|,$$

where $\delta \in (0, \frac{1}{2})$. By Theorem 1.2, it is sufficient to discuss two cases.

Case I. $\|(\frac{1}{4}\sum_{j\in I}|x+jy_n|)\|_M^0 = \frac{1}{k_n}(1+\rho_M(\frac{k_n}{4}\sum_{j\in I}|x+jy_n|))$ $(n \in \mathbb{N})$ and $k_n \to \infty$. In virtue of (2), we obtain

$$\begin{aligned}1 = \|x\|_M^0 &\leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}\rho_M\left(\frac{k_n}{4}\sum_{j\in I}|x + jy_n|\chi_{E_n}\right)\\ &\leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}\rho_M(k_n y_n\chi_{E_n}) \leq 1 + \frac{1}{n} - 2\delta\|y_n\chi_{E_n}\|_M^0 + \frac{2\delta}{k_n}\\ &\leq 1 - \frac{\delta\varepsilon}{3} \qquad (3)\end{aligned}$$

for n large enough, which is a contradiction.

Case II. $\|(\frac{1}{4}\sum_{j\in I}|x + jy_n|)\|_M^0 = \frac{1}{k_n}\big(1 + \rho_M(\frac{k_n}{4}\sum_{j\in I}|x + jy_n|)\big)$ $(n \in \mathbb{N})$ and $k_n \to k < \infty$. From

$$1 + \frac{1}{n} \geq \left\|\left(\frac{1}{4}\sum_{j\in I}|x + jy_n|\right)\right\|_M^0 = \frac{1}{k_n}\left(1 + \rho_M\left(\frac{k_n}{4}\sum_{j\in I}|x + jy_n|\right)\right)$$

$$\geq \frac{1}{k_n}\big(1+\rho_M(k_n x)\big) \geq \|x\|_M^0 = 1,$$

taking $n \to \infty$, we get $1 = \|x\|_M^0 = \frac{1}{k}(1+\rho_M(kx))$.

II-1. $\inf_n \rho_M(\frac{kx}{1-\delta}\chi_{E_n}) = a > 0$. Then in virtue of (2), we get for n large enough,

$$\|x\|_M^0 \leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}\rho_M\left(\frac{kx}{1-\delta}\chi_{E_n}\right) \leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}a \leq 1 - \frac{\delta a}{k}.$$

This is a contradiction.

II-2. $\inf_n \rho_M(\frac{kx}{1-\delta}\chi_{E_n}) = 0$. Passing to a subsequence of $(E_n)_{n=1}^{\infty}$ if necessary we can assume that

$$\sum_{n=1}^{\infty}\rho_M\left(\frac{kx}{1-\delta}\chi_{E_n}\right) < \infty.$$

Denote $E = \bigcup_{n=1}^{\infty} E_n$. Then $\rho_M(\frac{kx}{1-\delta}\chi_E) < \infty$. By the assumption, we have $M \in \mathbf{\Delta}_2(E)$. So there exist $K > 1$ and a nonnegative function $\delta \in L^0$ with $\int_E \delta(t)d\mu < \infty$ satisfying

$$M\left(t, \frac{12}{\varepsilon}u\right) \leq KM(t,u) + \delta(t) \ \ (t \in E).$$

Take $\eta > 0$ such that if $\Omega \in E \cap \Sigma$ and $\mu\Omega < \eta$, then $\int_\Omega \delta(t)d\mu < \frac{1}{2}$. Take $D > 0$ large enough such that $\mu\{t \in E : M(t,1) > D\} < \frac{\eta}{3}$ and $\frac{\varepsilon}{D} < 1$. Let

$$F = \{t \in E : M(t,1) \leq D\}, \quad z = \frac{\varepsilon^2}{12D}\chi_F.$$

Without loss of generality, we may assume that $\varepsilon\mu E \leq 1$. Then

$$\begin{aligned}\|z\|_M^0 &\leq \frac{\varepsilon}{12}\left(1+\rho_M\left(\frac{12}{\varepsilon}z\right)\right) = \frac{\varepsilon}{12}\left(1+\int_F M\left(t, \frac{\varepsilon^2}{12D}\cdot\frac{12}{\varepsilon}\right)d\mu\right)\\ &\leq \frac{\varepsilon}{12}\left(1+\frac{\varepsilon}{D}\int_F M(t,1)d\mu\right) \leq \frac{\varepsilon}{12}\left(1+\frac{\varepsilon}{D}\cdot D\cdot\mu F\right) < \frac{\varepsilon}{6}.\end{aligned}$$

Thus for any $y \in L_M^0$, we have

$$\|y\chi_{\{t\in E: M(t,1)\leq D, |y(t)|<\frac{\varepsilon^2}{12D}\}}\|_M^0 \leq \|z\|_M^0 < \frac{\varepsilon}{6}.$$

Combining this with $\|y_n\chi_{E_n}\|_M^0 > \frac{\varepsilon}{3}$ $(n \geq 3)$ and defining

$$F_n = \left\{t \in E_n : M(t,1) > D \text{ or } |y_n(t)| \geq \frac{\varepsilon^2}{12D}\right\},$$

we get $\|y_n\chi_{F_n}\|_M^0 > \frac{\varepsilon}{6}$ $(n \geq 3)$.

II-2-1. Without loss of generality, assume that $\mu F_n < \eta$ $(n \in \mathbb{N})$. Notice that $\|\frac{12}{\varepsilon}y_n\chi_{F_n}\|_M \geq \|\frac{6}{\varepsilon}y_n\chi_{F_n}\|_M^0 > 1$, $F_n \subset E_n \subset E$ and $k_n \to k > 1$. For n large enough, there hold the inequalities

$$\begin{aligned} 1 &\leq \rho_M\left(\frac{12}{\varepsilon}y_n\chi_{F_n}\right) \leq \rho_M\left(\frac{12}{\varepsilon}k_ny_n\chi_{F_n}\right) \\ &\leq K\rho_M(k_ny_n\chi_{F_n}) + \int_{F_n}\delta(t)d\mu \leq K\rho_M(k_ny_n\chi_{F_n}) + \frac{1}{2}. \end{aligned}$$

So $\rho_M(k_ny_n\chi_{F_n}) \geq \frac{1}{2K}$. Then in virtue of (3),

$$\begin{aligned} 1 = \|x\|_M^0 &\leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}\rho_M(k_ny_n\chi_{E_n}) \leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}\rho_M(k_ny_n\chi_{F_n}) \\ &\leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}\cdot\frac{1}{2K} \leq 1 - \frac{\delta}{2Kk}. \end{aligned}$$

This is a contradiction.

II-2-2. Without loss of generality, assume that $\mu F_n \geq \eta$ $(n \in \mathbb{N})$. Notice that for $t \in T$, if $e(t) > 0$, then $M(t, \frac{e(t)}{1-\delta}) > 0$. If $e(t) = 0$, then $M(t, \frac{\varepsilon^2}{12D}) > 0$. So, there exists $c > 0$ small enough such that

$$\begin{aligned} \mu\Big\{t \in F_n : e(t) > 0 \text{ and } M\left(t, \frac{e(t)}{1-\delta}\right) < c \\ \text{or} \quad e(t) = 0 \text{ and } M\left(t, \frac{\varepsilon^2}{12D}\right) < c\Big\} < \frac{\eta}{3}. \end{aligned}$$

Since $\mu\{t \in E : M(t,1) > D\} < \frac{\eta}{3}$, setting

$$\begin{aligned} H_n = \Big\{t \in F_n : M(t,1) \leq D,\ e(t) > 0 \Rightarrow M\left(t, \frac{e(t)}{1-\delta}\right) \geq c, \\ e(t) = 0 \Rightarrow M\left(t, \frac{\varepsilon^2}{12D}\right) \geq c\Big\}, \end{aligned}$$

we get $\mu H_n \geq \frac{\eta}{3}$. If $t \in H_n$ and $e(t) > 0$, we have for n large enough,

$$M\left(t, \frac{k_n}{4}\sum_{j\in I}|x(t)+jy_n(t)|\right) \geq M\left(t, \frac{k_n|x(t)|}{1-2\delta}\right)$$
$$\geq M\left(t, \frac{k|x(t)|}{1-\delta}\right) \geq M\left(t, \frac{e(t)}{1-\delta}\right) \geq c.$$

If $t \in H_n$ and $e(t) = 0$, for n large enough there hold the inequalities

$$M\left(t, \frac{k_n}{4}\sum_{j\in I}|x(t)+jy_n(t)|\right) \geq M\big(t, k_n|y_n(t)|\big) \geq M\big(t, |y_n(t)|\big)$$
$$\geq M\left(t, \frac{\varepsilon^2}{12D}\right) \geq c.$$

So $\rho_M(\frac{k_n}{4}\sum_{j\in I}|x+jy_n|\chi_{H_n}) \geq \frac{1}{3}c\eta$. Then in virtue of (3), we obtain

$$1 = \|x\|_M^0 \leq 1 + \frac{1}{n} - \frac{2\delta}{k_n}\cdot\frac{c\eta}{3} \to 1 - \frac{\delta c\eta}{3k}.$$

This is a contradiction, which finishes the proof. □

Theorem 1.6. *The following assertions are equivalent:*

(1) L_M^0 *is* CUR,

(2) L_M^0 *is* C-LUR,

(3) $\rho_N(A) \leq 1$ *or* $e(t) = 0$ *for* μ*-a.e.* $t \in T$ *and* $M \in \mathbf{\Delta}_2$.

Proof. The implications (1) ⇒ (2) and (2) ⇒ "$\rho_N(A) \leq 1$ or $e(t) = 0$ for μ-a.e. $t \in T$" are trivial by Theorem 1.4. Let L_M^0 is C-LUR and $\rho_N(A) > 1$ but $M \notin \mathbf{\Delta}_2$. There exists $z \in L_M^0$ such that $\rho_M(z) \leq 1$ and $\xi_M(z) = 1$. Take n_0 large enough such that $\rho_N(A\chi_{T_{n_0}}) > 1$, where the sequence $(T_n)_{n=1}^\infty$ is from Lemma 0.1. We can find $x \in S(L_M^0)$ with $S_x = T_{n_0}$. By Lemma 0.2, there exists $k > 0$ such that $\|x\|_M^0 = \frac{1}{k}(1 + \rho_M(kx))$. Define

$$y_n = \frac{1}{k} z\chi_{T\setminus T_n} \quad (n \in \mathbb{N}).$$

Then $\|y_n\|_M^0 = \frac{1}{k}\|z\chi_{T\setminus T_n}\|_M^0 \geq \frac{1}{k}\xi_M(z) = \frac{1}{k}$ $(n \in \mathbb{N})$. But if $n > n_0$, there holds

$$\|x + \lambda y_n\|_M^0 \leq \frac{1}{k}\big(1 + \rho_M(k(x+\lambda y_n))\big)$$

$$\leq \frac{1}{k}\big(1+\rho_M(kx\chi_{T_n})+\rho_M(z\chi_{T\setminus T_n})\big)$$

$$\to \frac{1}{k}(1+\rho_M(kx))=\|x\|_M^0=1.$$

This contradicts the fact that x is a C-LUR point.

(3) $\Rightarrow$ (1). The proof is similar to the proof of sufficiency of Theorem 1.5, so we omit it here. □

References

[1] S. Chen, Geometry of Orlicz spaces, Dissertationes Math. 356, *Warszawa*, 1996.

[2] S. Chen and H. Hudzik, On the convexities of Orlicz and Orlicz–Bochner spaces, *Comment. Math. Univ. Carolin.* **29**, 1 (1988), 13–29.

[3] W. J. Davis, D. J. H. Darling and N. Tomczak-Jaegermann, The complex convexity of quasi-normed linear spaces, *J. Funct. Anal.* **55** (1984), 110–150.

[4] S. J. Dilworth, Complex convexity and the geometry of Banach spaces, *Math. Proc. Camb. Philos. Soc.* **99** (1986), 495–506.

[5] J. Globevnik, On complex strict and uniform convexity, *Proc. Amer. Math. Soc.* **47** (1975), 175–178.

[6] H. Hudzik, B. Wang and T. Wang, On the convexity characteristic of Orlicz spaces, *Math. Japonica* **37**, 4 (1992), 691–699.

[7] H. Hudzik and M. Mastyło, Strongly extreme points in Köthe–Bochner spaces, *Rocky Mountain Journal of Math.* **23**, 3 (1993), 899–909.

[8] A. Kamińska, Some convexity properties of Musielak–Orlicz spaces of Bochner type, *Rend. Circ. Math. Palermo, (2), suppl.* **10** (1985), 63–73.

[9] G. Leśniak, Complex convexity and finitely additive vector measure, *Proc. Amer. Math. Soc.* **302** (1988), 867–873.

[10] M. Pavlovic, On the complex uniform convexity of Quasi-normed spaces, *Math. Bolkanica* **5** (1991), 92–98.

[11] E. Thorp and R. Whitley, The strong maximum modules theorem for analytic functions into a Banach space, *Proc. Amer. Math. Soc.* **18** (1967), 640–646.

[12] T. Wang and Y. Teng, Complex locally uniform rotundity of Musielak–Orlicz spaces, *Science in China* **30** (1) (2000), 24–31.

[13] W. Wei and P. Liu, PL uniform rotundity and the characterization of martingale in complex spaces, *Chinese Sci. Bull.* **2** (1997), 234–217.

[14] C. Wu and S. Sun, Norm calculation and complex convexity of Musielak–Orlicz sequence spaces, *Chinese Ann. Math.* **12A** (1991 suppl.), 98–102.

[15] C. Wu and S. Sun, On the complex convexity of Musielak–Orlicz sequence spaces, *Comment. Math. Univ. Carolin.* **30** (1989), 397–408.

[16] C. Wu and H. Sun, On the complex extreme point and complex strictly rotundity of Musielak–Orlicz spaces, *J. Systyms Sci. Math.* **7** (1987), 7–13.

[17] C. Wu and H. Sun, On the complex uniformly rotundity of Musielak–Orlicz spaces, *J. Northeast Math.* **4** (1988), 389–396.

CUIXIA HAO
DEPARTMENT OF MATHEMATICAL SCIENCES
HEILONGJIANG UNIVERSITY
HARBIN, 150080
P.R. CHINA
and
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF ALBERTA
EDMONTON, T6G, 2G1
CANADA

E-mail: haocuixia@yahoo.com

LIFANG LIU
DEPARTMENT OF MATHEMATICS
XIAMEN UNIVERSITY
XIAMEN, 361005
P.R. CHINA

E-mail: lifang.liu@263.net

(Received March 14, 2003; revised October 14, 2003)