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Local set-valued solutions of the Jensen and Pexider functional equations

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Abstract. Local solutions of the Jensen and Pexider functional equations for set-valued functions are given. The obtained results are applied to find a form of the locally Lipschitzian Nemytskii operator.

1. The Jensen functional equation

Z. Fifer in [2] (cf. also [5]) has proved that every set-valued Jensen function f defined in the interval $[0,\infty)$ with compact non-empty values in a normed space Y is of the form

$$f(x) = A(x) + B, \quad x \in [0,\infty), \tag{1}$$

where A is an additive set-valued function in $[0,\infty)$ with compact convex non-empty values in Y and B is a compact convex non-empty subset of Y. The main purpose of this paper is to give a local version of this result.

Example. The set-valued Jensen function given by the formula

$$f(x) = [0, 1-x] \quad \text{for} \quad x \in [0,1]$$

cannot be represented in the form (1).

Let $(Y, \|\cdot\|)$ be a normed space. We denote by $c(Y)$ the family of all compact non-empty subsets of Y and $cc(Y)$ the family of all convex sets from $c(Y)$. The symbol $\mathbb{R}$ stands for the set of all reals, and $\mathbb{N}$ for the set of positive integers.

Let $I = [0,a) \subset \mathbb{R}$ be an interval. A set-valued function $F : I \to 2^Y$ is said to be a Jensen function if

$$F\left(\frac{x+y}{2}\right) = \frac{1}{2}\left[F(x) + F(y)\right]$$

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for $x, y \in I$.

It is easily seen that the values of a Jensen function $F : I \to c(Y)$ belong to $cc(Y)$.

We shall apply the following

Lemma 1. (cf. [7]). *Let A, B and $C \neq 0$ be subsets of a topological Hausdorff vector space such that $A+C \subset B+C$. If B is convex and closed and C is bounded, then $A \subset B$.*

The Hausdorff metric in the set of all closed bounded and non-empty subsets of a normed space Y will be denoted by d. The following lemma collects the main properties of d:

Lemma 2. (cf. [7]).

(a) $d(A + C, B + C) = d(A, B)$;

(b) $d(\lambda A,\ \lambda B) = |\lambda| d(A, B)$;

(c) $d(A + C,\ B + D) \leq d(A, B) + d(C, D)$

for A, B, C, D from $cc(Y)$ and for any real number λ.

The main result of this paper is the following

Theorem 1. *If Y is a normed space and $F(0)$ is convex, then $F : I \to c(Y)$ is a Jensen function if and only if there exist sets $A, B \in cc(Y)$ and an additive function $a : \mathbb{R} \to Y$ such that*

$$F(x) + xB = F(0) + xA + a(x) \quad \text{for all} \quad x \in I.$$

Proof. The sufficiency is easily verifiable.

Necessity. Let $F : I \to c(Y)$ be a Jensen set-valued function. There exist an additive function $a : \mathbb{R} \to Y$ and a convex continuous (with respect to the Hausdorff metric d in $cc(Y)$) set-valued function $G : (0, a) \to cc(Y)$ such that

$$F(x) = a(x) + G(x) \quad \text{for all} \quad x \in (0, a)$$

(cf. K. Nikodem [4]). Put $G(0) := F(0)$. We notice that G is the Jensen function in I.

Let "$\approx$" denote Rådström's equivalence relation in $cc(Y)$ defined by

$$(A, B) \approx (C, D) \ \text{ if and only if } \ A + D = B + C$$

(cf. [7]). For any pair (A, B) denote by $[A, B]$ the equivalence class containing this pair. Define the addition of two equivalence classes by

$$[A, B] + [C, D] = [A + C, B + D]$$

and multiplication with a $\lambda \geq 0$ by

$$\lambda[A, B] = [\lambda A, \lambda B].$$

The metric δ on the space of all equivalence classes is given by

$$\delta([A,B],[C,D]) = d(A+D, B+C).$$

The formula

$$g(x) = [G(x), G(0)]$$

introduces a function of the type $g : I \to (cc(Y) \times cc(Y))/{\approx}$. We shall show that g is additive. Since

$$2g\left(\frac{x}{2}\right) = \left[2G\left(\frac{x}{2}\right),\ 2G(0)\right] =$$

$$= [G(x) + G(0), G(0) + G(0)] = [G(x), G(0)] = g(x)$$

for all $x \in I$, thus we get

$$g(x+y) = 2g\left(\frac{x+y}{2}\right) = 2\left[G\left(\frac{x+y}{2}\right), G(0)\right] = 2\left[\frac{G(x)+G(y)}{2}, G(0)\right] =$$

$$= [G(x) + G(y), 2G(0)] = [G(x), G(0)] + [G(y), G(0)] = g(x) + g(y)$$

for all $x, y \in I$ such that $x + y \in I$. Moreover

$$\delta(g(x), g(y)) = \delta([G(x), G(0)],\ [G(y), G(0)]) =$$
$$= d(G(x) + G(0),\ G(0) + G(y)) = d(G(x), G(y))$$

for $x, y \in I$. Therefore for every $x \in (0, a)$ we have

$$\lim_{y \to x} \delta(g(x), g(y)) = \lim_{y \to x} d(G(x), G(y)) = 0.$$

Thus g is continuous in $(0, a)$ and it is an additive function. Consequently there is an equivalence pair $[A, B]$ for which

$$g(x) = x[A, B], \quad x \in I.$$

This equality can be rewritten in the following form

$$[G(x), G(0)] = [xA, xB], \quad x \in I,$$

whence by the definition of the relation "$\approx$" we have

$$G(x) + xB = G(0) + xA, \quad x \in I.$$

Adding $a(x)$ to both sides of the above equality we obtain

$$F(x) + xB = F(0) + xA + a(x), \quad x \in I$$

and the proof is complete.

Remark 1. Professor K. Nikodem pointed out that the above theorem can be extended to set-valued functions defined in arbitrary intervals $[a, b)$.

Remark 2. Theorem 1 generalizes Fifer's result (see Theorem 1 in [2]). In fact, let Y be a Banach space and let $I = [0, \infty)$. Then

$$F(x) + xB = F(0) + xA + a(x),$$

where $A, B \in cc(Y)$ and $a : \mathbb{R} \to Y$ is an additive function, hence

$$F(2^n) + 2^n B = F(0) + 2^n A + 2^n a(1), \quad n \in \mathbb{N}.$$

Thus

$$\frac{1}{2^n} F\left(2^n\right) + B = \frac{1}{2^n} F(0) + A + a(1), \quad n \in \mathbb{N}. \tag{2}$$

Now we observe that the sequence of sets with terms $\frac{1}{2^n} F(2^n)$, $\quad n \in \mathbb{N}$ fulfils the Cauchy condition. Indeed,

$$\begin{aligned} d\left(\frac{1}{2^n} F(2^n),\ \frac{1}{2^m} F(2^m)\right) &= d\left(\frac{1}{2^n} F(2^n) + B,\ \frac{1}{2^m} F(2^m) + B\right) = \\ &= d\left(\frac{1}{2^n} F(0) + A + a(1),\ \frac{1}{2^m} F(0) + A + a(1)\right) = \\ &= d\left(\frac{1}{2^n} F(0),\ \frac{1}{2^m} F(0)\right) = \left|\frac{1}{2^n} - \frac{1}{2^m}\right| \|F(0)\| \end{aligned}$$

for $n, m \in \mathbb{N}$, where $\|A\| = \sup\{\|x\| :\ x \in A\}$. Consequently this sequence converges to a set $C \in cc(Y)$ (see [1]).
By (2) we get

$$C + B = A + a(1)$$

and

$$F(x) + xB = F(0) + x[C + B - a(1)] + a(x).$$

Using Rådström's Lemma 1 we have

$$F(x) = F(0) + a(x) + x(C - a(1)).$$

2. The Pexider functional equation

In the paper [6] K. Nikodem characterized set-valued solutions of the Pexider functional equation

$$F(x + y) = G(x) + H(y) \tag{3}$$

with three unknown functions F, G and H. In this section we shall establish a form of a local set-valued solution of equation (3).

Theorem 2. *If Y is a normed space, then set-valued functions F, G, $H : I \to cc(Y)$ fulfil equation (3) for $x, y \in I$ such that $x + y \in I$ if and only if there exist sets $A, B, K, L \in cc(Y)$ and an additive function $a : \mathbb{R} \to Y$ such that*

$$\begin{aligned} F(x) + xB &= K + L + xA + a(x), \\ G(x) + xB &= K + xA + a(x), \\ H(x) + xB &= L + xA + a(x) \end{aligned}$$

for $x \in I$.

Proof. The sufficiency is easily seen. To prove necessity take $x, y \in I$. We have

$$F\left(\frac{x+y}{2}\right) = G\left(\frac{x}{2}\right) + H\left(\frac{y}{2}\right) \tag{4}$$

and

$$F\left(\frac{x+y}{2}\right) = G\left(\frac{y}{2}\right) + H\left(\frac{x}{2}\right). \tag{5}$$

Hence

$$2F\left(\frac{x+y}{2}\right) = G\left(\frac{x}{2}\right) + G\left(\frac{y}{2}\right) + H\left(\frac{x}{2}\right) + H\left(\frac{y}{2}\right). \tag{6}$$

Putting $y = x$ in (4) and $x = y$ in (5) we get

$$F(x) + F(y) = G\left(\frac{x}{2}\right) + G\left(\frac{y}{2}\right) + H\left(\frac{x}{2}\right) + H\left(\frac{y}{2}\right). \tag{7}$$

The comparison of equalities (6) and (7) gives

$$F\left(\frac{x+y}{2}\right) = \frac{1}{2}\left[F(x) + F(y)\right]$$

for $x, y \in I$. In virtue of Theorem 1 there are sets $A, B \in cc(Y)$ and an additive function $a : \mathbb{R} \to Y$ such that

$$F(x) + xB = F(0) + xA + a(x), \quad x \in I.$$

Putting $x+y$ instead of x in the above equality and applying equation (3) we obtain

$$G(x) + H(y) + (x+y)B = G(0) + H(0) + xA + yA + a(x) + a(y)$$

for all $x, y \in I$ for which $x + y \in I$. Hence for $y = 0$ we have

$$G(x) + xB = G(0) + xA + a(x), \quad x \in I.$$

Similarly

$$H(y) + yB = H(0) + yA + a(y), \quad y \in I.$$

Putting $K := G(0)$, $L := H(0)$ and applying the equality $F(0) = G(0) + H(0)$ we get the assertion of the theorem.

3. Application

Let $(X, |\cdot|)$, $(Y, |\cdot|)$ be normed spaces and let $U \subset X$ be a convex set with zero. Denote by $\mathrm{lip}(U, I)$ the set of all functions $\phi : U \to I$ such that

$$\sup_{x \neq \overline{x}} \frac{|\phi(x) - \phi(\overline{x})|}{|x - \overline{x}|} < \infty,$$

where supremum is taken over all $x, \overline{x} \in U$. In the set $\mathrm{lip}(U, I)$ we introduce the metric defined by the formula

$$D(\phi_1, \phi_2) := |\phi_1(0) - \phi_2(0)| + \sup_{x \neq \overline{x}} \frac{|\phi_1(x) - \phi_2(x) - \phi_1(\overline{x}) + \phi_2(\overline{x})|}{|x - \overline{x}|}.$$

Let $\mathrm{Lip}(U, Y)$ denote the set

$$\left\{ \phi : U \to cc(Y) : \sup_{x \neq \overline{x}} \frac{d(\phi(x), \phi(\overline{x}))}{|x - \overline{x}|} < \infty \right\}.$$

In this set the metric may be defined by

$$\rho(\phi_1, \phi_2) := d\left(\phi_1(0), \phi_2(0)\right) + \sup_{x \neq \overline{x}} \frac{d(\phi_1(x) + \phi_2(\overline{x}), \phi_1(\overline{x}) + \phi_2(x))}{|x - \overline{x}|}.$$

Every set-valued function $h : U \times \mathbb{R} \to cc(Y)$ generates the Nemytskii operator $\mathcal{N}$

$$\mathcal{N}(\phi)(x) := h(x, \phi(x)), \quad x \in U \tag{8}$$

mapping the space of all functions $\phi : U \to \mathbb{R}$ with values in the space of all functions $\phi : U \to cc(Y)$.

In the paper [8] it has been proved that the Nemytskii operator $\mathcal{N}$ mapping the space $\mathrm{lip}(U, C)$ into $\mathrm{Lip}(U, Z)$, where C is a convex cone with zero in Y and Z is a normed space, and globally Lipschitzian must be of the form

$$\mathcal{N}(\phi)(x) = A(x, \phi(x)) + B(x),$$

where $A := U \times C \to cc(Z)$, $A(x, \cdot)$ is an additive set-valued function and $B \in \mathrm{Lip}(U, Z)$. In this part of the paper we are going to give the following analogue of Matkowski's theorem (cf. [3]) for set-valued functions:

Theorem 3. *Let $(X, |\cdot|)$, $(Y, |\cdot|)$ be normed spaces and $U \subset X$ be a convex set such that $0 \in U$. Assume that $h : U \times I \to cc(Y)$ and the Nemytskii operator $\mathcal{N}$ generated by h satisfies the two conditions*

(i) $\mathcal{N} : \mathrm{lip}(U, I) \to \mathrm{Lip}(U, Y)$;

(ii) *there is $c \geq 0$ such that*

$$\rho\left[\mathcal{N}(\phi_1), \mathcal{N}(\phi_2)\right] \leq c D(\phi_1, \phi_2), \quad \phi_1, \phi_2 \in \mathrm{lip}(U, I),$$

then there exist set-valued functions $A, B : U \to cc(Y)$ for which

$$h(x,y) + yB(x) = yA(x) + h(x,0), \quad x \in U, \quad y \in I.$$

Moreover there exists a constant $l \geq 0$ such that

$$d[A(x_1) + B(x_2),\ A(x_2) + B(x_1)] \leq l|x_1 - x_2|, \quad x_1, x_2 \in U. \tag{9}$$

Proof. Fix $y \in [0,a)$. The constant function $\phi(x) = y, \quad x \in U$ belongs to $\mathrm{lip}(U,I)$. It follows by (i) that

$$h(\cdot, y) \in \mathrm{Lip}(U,Y), \quad y \in I.$$

In particular the function h is continuous with respect to the first variable for every fixed y belongig to I.

Using the definition of ρ and (ii) we have

$$\frac{d[h(t,\phi_1(t)) + h(\bar{t},\phi_2(\bar{t})),\ h(\bar{t},\phi_1(\bar{t})) + h(t,\phi_2(t))]}{|t-\bar{t}|} \leq cD(\phi_1,\phi_2) \tag{10}$$

for all ϕ_1, $\phi_2 \in \mathrm{Lip}(U,Y)$, $t, \bar{t} \in U$, $t \neq \bar{t}$.

Let us take $x \in U$, $x \neq 0$ and $\overline{x} \in U$ such that $|\overline{x}| < |x|$. Fix $y_1, y_2, \overline{y}_1, \overline{y}_2 \in I$ and define

$$\phi_i(t) = \begin{cases} \overline{y}_i, & |t| \leq |\overline{x}| \\ \frac{y_i - \overline{y}_i}{|x| - |\overline{x}|}(|t| - |\overline{x}|) + \overline{y}_i, & |\overline{x}| \leq |t| \leq |x| \\ y_i, & |t| \geq |x| \end{cases} \tag{11}$$

for $t \in U$ and $i = 1, 2$. It is evident that $\phi_i \in \mathrm{lip}(U,I)$. Moreover

$$D(\phi_1,\phi_2) = |\overline{y}_1 - \overline{y}_2| + \frac{|y_1 - y_2 - \overline{y}_1 + \overline{y}_2|}{|x| - |\overline{x}|}.$$

Putting in (10) ϕ_1 and ϕ_2 as given by (11), $t = x$, $\bar{t} = \overline{x}$ we get

$$\begin{aligned} &\frac{d[h(x,y_1) + h(\overline{x},\overline{y}_2),\ h(\overline{x},\overline{y}_1) + h(x,y_2)]}{|x - \overline{x}|} \leq \\ &\leq c\left[|\overline{y}_1 - \overline{y}_2| + \frac{|y_1 - y_2 - \overline{y}_1 + \overline{y}_2|}{|x| - |\overline{x}|}\right]. \end{aligned} \tag{12}$$

Obviously $\frac{|x-\overline{x}|}{|x|-|\overline{x}|} \geq 1$. Now for $\overline{x} = \lambda x$, where $0 < \lambda < 1$, we have

$$\frac{|x - \overline{x}|}{|x| - |\overline{x}|} = \frac{|x - \lambda x|}{|x| - \lambda|\overline{x}|} = 1,$$

whence $\liminf\limits_{\overline{x}\to x} \frac{|x-\overline{x}|}{|x|-|\overline{x}|} = 1$. Taking the lim inf as $\overline{x} \to x$ in (12) and using the continuity of $h(\cdot, y)$ we obtain

$$d\left(h(x,y_1)+h(x,\overline{y}_2),\ h(x,\overline{y}_1)+h(x,y_2)\right) \le c|y_1-y_2-\overline{y}_1+\overline{y}_2| \tag{13}$$

for all $x \neq 0$, $x \in U$ and $y_1, y_2, \overline{y}_1, \overline{y}_2 \in I$. The inequality (13) holds also for $x = 0$ on account of the continuity of $h(\cdot, y)$. Putting in (13) $y_1 = \overline{y}_2 = \frac{y+w}{2}$, $y_2 = y$, $\overline{y}_1 = w$, where $y, w \in I$, we get

$$d\left(2h\left(x, \frac{y+w}{2}\right), h(x,y)+h(x,w)\right) = 0,$$

whence

$$h\left(x, \frac{y+w}{2}\right) = \frac{1}{2}\left[h(x,y)+h(x,w)\right] \tag{14}$$

for all $x \in U$, $y, w \in I$. In virtue of Theorem 1 there exist two set-valued functions $A : U \times [0, +\infty) \to cc(Y)$, $B : U \to cc(Y)$ such that

$$h(x,y) + yB(x) = h(x,0) + A(x,y), \quad x \in U, \quad y \in I \tag{15}$$

and $A(x, \cdot)$ is additive. Putting $\overline{y}_1 = \overline{y}_2 = 0$ in (13) we have

$$\begin{aligned} d(h(x,y_1), h(x,y_2)) = d(h(x,y_1)+h(x,0),\ h(x,y_2)+h(x,0)) \le \\ \le c|y_1-y_2| \end{aligned} \tag{16}$$

for all $x \in U$ and $y_1, y_2 \in [0, a)$. The inequality (16) implies the continuity of $h(x, \cdot)$, $x \in U$ and by (15) the continuity of $A(x, \cdot)$. Thus there exists $A(x) \in cc(Y)$ for which $A(x, y) = yA(x)$, $x \in U$, $y \in [0, +\infty)$. The first part of our theorem is proved. Now for $x_1, x_2 \in U$ and $y \in I$ we get

$$\begin{aligned} & d(yB(x_1)+yA(x_2),\ yB(x_2)+yA(x_1)) = \\ &= d(yB(x_1)+h(x_1,y)+yA(x_2)+h(x_2,y),\ yB(x_2)+h(x_2,y)+ \\ &\qquad +yA(x_1)+h(x_1,y)) = \\ &= d(yA(x_1)+h(x_1,0)+yA(x_2)+h(x_2,y),\ yA(x_2)+h(x_2,0)+ \\ &\qquad +yA(x_1)+h(x_1,y)) = \\ &= d(h(x_1,0)+h(x_2,y),\ h(x_2,0)+h(x_1,y)) \le \\ &\le d(h(x_1,0),\ h(x_2,0)) + d(h(x_2,y),\ h(x_1,y)). \end{aligned}$$

Since $h(\cdot, y) \in \mathrm{Lip}(U, Y)$ for $y \in I$ we can find a constant $l(y) \ge 0$ such that

$$d(B(x_1)+A(x_2), B(x_2)+A(x_1)) \le \frac{l(y)}{y}|x_1-x_2|.$$

Taking $l := \inf\left\{\frac{l(y)}{y} : y \in I\right\}$ we get the inequality (9) which ends the proof.

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