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n-commuting maps on prime rings

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Abstract. We prove a result concerning additive n-commuting maps on prime rings and then apply it to n-commuting linear generalized differential polynomials.

1. Results

Throughout, unless specially stated, R always denotes a prime ring with center $\mathcal{Z}$. We let U be the maximal ring of right quotients of R and let Q stand for the symmetric Martindale quotient ring of R. The center C of U (and Q) is called the extended centroid of R. See [3] for its details. An additive map $d : R \to R$ is called a derivation if $(xy)^d = x^d y + xy^d$ for all $x, y \in R$. A map $f : R \to U$ is called *n-commuting* on a subset S of R, where n is a positive integer, if $[f(x), x^n] = 0$ for all $x \in S$. The map f is merely called *commuting* if it is 1-commuting. The study of these mappings was initiated by Posner's Theorem: The existence of a nonzero derivation commuting on R implies the commutativity of R [21, Theorem 2]. More related results have been obtained in [17]–[19], [4], [5], [13]–[6]. Also, see [11], [1], [2] for n-commuting maps. Applying [2, Theorem 1.1] and [1, Theorem 4.4] we have the result: *Let R be a prime*

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ring such that either $\operatorname{char} R = 0$ *or a prime* $p > n$, *or* $\deg(R) > n$. *Then every additive* n*-commuting map of* R *into* U *is commuting.* The goal of this paper is to prove a theorem related to the result above and then apply it to some applications on n-commuting linear differential polynomials. We now state the main result:

Theorem 1.1. *Let* R *be a prime ring with center* $\mathcal{Z}$, *its maximal ring of right quotients* U *and* n *a fixed positive integer. Suppose that* $f : R \to U$ *is an additive* n*-commuting map such that* f *is* $\mathcal{Z}$*-linear if* $\mathcal{Z} \neq 0$. *Then there exist* $\lambda \in C$ *and a map* $\mu : R \to C$ *such that* $f(x) = \lambda x + \mu(x)$ *for all* $x \in R$, *unless* $R \cong \mathrm{M}_2(\mathrm{GF}(2))$.

Here, GF(2) denotes the Galois field of two elements. The following gives a counterexample for the case $R = \mathrm{M}_2(\mathrm{GF}(2))$.

Example 1.2. Let $R = \mathrm{M}_2(\mathrm{GF}(2))$ and let $f : R \to R$ be defined by

$$f\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha+\gamma & 0 \\ 0 & \beta+\delta \end{pmatrix} \quad \text{for } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R.$$

Then f is a GF(2)-linear map. A direct computation proves that $[f(x), x^6] = 0$ for all $x \in R$. However, $\left[f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Hence, f is a 6-commuting linear map but it is not commuting.

We now apply Theorem 1.1 to n-commuting linear generalized differential polynomials. To state these results precisely, let us recall some notation. We denote by $\mathrm{Der}(U)$ the set of all derivations of U. For $d \in \mathrm{Der}(U)$ and $x \in U$, one usually writes x^d for $d(x)$. Also, if $\beta \in C$, define $x^{d\beta} = x^d\beta$. It follows that $\mathrm{Der}(U)$ forms a right C-module. Let $\mathbf{D}$ be the C-submodule of $\mathrm{Der}(U)$ defined by

$$\mathbf{D} = \{\delta \in \mathrm{Der}(U) \mid I^\delta \subseteq R \text{ for some nonzero ideal } I, \text{ depending on } \delta, \text{ of } R\}.$$

By a derivation word we mean an additive map Δ from U into itself assuming the form $\Delta = \delta_1\delta_2 \ldots \delta_t$, where $\delta_i \in \mathbf{D}$. If Δ is empty, we define $x^\Delta = x$ for $x \in U$. A linear generalized differential polynomial means a linear generalized polynomial with coefficients in U and with an indeterminate X

which are acted on by derivation words. Thus every linear generalized differential polynomial can be written in the form $\sum_i \sum_j a_{ij} X^{\Delta_i} b_{ij}$, where $a_{ij}, b_{ij} \in U$ and the Δ_i's are derivation words.

Theorem 1.3. *Let R be a noncommutative prime ring, $R \ncong \mathrm{M}_2(\mathrm{GF}(2))$, and n a fixed positive integer. Suppose that*

$$\big[\psi(x), x^n\big] = 0$$

for all $x \in R$, where $\psi(x)$ is a linear generalized differential polynomial. Then $\psi(x) = \lambda x + \mu(x)$ for all $x \in R$, where $\lambda \in C$ and $\mu : R \to C$.

Proof. Applying the identities (1)–(5) given in [8, p. 155], we can find finitely many distinct regular words $\Delta_0, \Delta_1, \cdots, \Delta_t$ with $\Delta_0 = \emptyset$ such that

$$\psi(x) = \sum_{i=0}^{t} \sum_{j} a_{ij} x^{\Delta_i} b_{ij} \tag{1.1}$$

for all $x \in R$, where $a_{ij}, b_{ij} \in U$. By assumption,

$$\left[\sum_{i=0}^{t} \sum_{j} a_{ij} x^{\Delta_i} b_{ij}, x^n\right] = 0 \tag{1.2}$$

for all $x \in R$. Applying Kharchenko's Theorem [9, Theorem 2] to (1.2) yields

$$\left[\sum_{i=1}^{t} \sum_{j} a_{ij} y_i b_{ij} + \sum_{j} a_{0j} x b_{0j}, x^n\right] = 0 \tag{1.3}$$

for all $y_i, x \in R$. For $i > 0$ we see that $[\sum_j a_{ij} y b_{ij}, x^n] = 0$ for all $x, y \in R$ and so for all $x, y \in U$ (see [3, Theorem 6.4.1] or [7, Theorem 2]). In view of [12, Theorem], we have $[\sum_j a_{ij} y b_{ij}, x] = 0$ for all $x, y \in R$. Thus $\sum_j a_{ij} y b_{ij} \in C$ for all $y \in U$. In particular, $\sum_{i=1}^{t} \sum_j a_{ij} x^{\Delta_i} b_{ij} \in C$ for all $x \in U$. Thus (1.3) is reduced to $[\sum_j a_{0j} x b_{0j}, x^n] = 0$ for all $x \in U$. By Theorem 1.1, there exist $\lambda \in C$ and $\eta : R \to C$ such that $\sum_j a_{0j} x b_{0j} = \lambda x + \eta(x)$ for all $x \in U$. We are now done by setting $\mu(x) = \eta(x) + \sum_{i=1}^{t} \sum_j a_{ij} x^{\Delta_i} b_{ij} \in C$ for all $x \in R$. This proves the theorem. $\square$

A special case of Theorem 1.3 is the following

Theorem 1.4. *Let R be a noncommutative prime ring, $R \not\cong \mathrm{M}_2(\mathrm{GF}(2))$, with a derivation δ, $n \geq 1$. Suppose that $[\psi(x), x^n] = 0$ for all $x \in R$, where $\psi(x) = \sum_{i=0}^{t} a_i x^{\delta^i}$ with $a_i \in R$. Then $a_0 \in \mathcal{Z}$ and $\psi(x) = a_0 x$ for all $x \in R$.*

We remark that Park and Jung studied the case: a derivation d on an $n!$-torsion-free semiprime ring R such that d^2 is n-commuting on R, where $n \geq 2$ [20, Theorem 3.1]. Applying the theory of orthogonal completion for semiprime rings (see [3]), [20, Theorem 3.1] can be reduced to the prime case and so can be solved as a special case of Theorem 1.4. To prove it we first quote Chang's Theorem [6, Theorem 3.2]:

Theorem 1.5 (Chang [6])**.** *Let R be a noncommutative prime ring with a derivation d. Suppose that $\sum_{i=1}^{n} a_i x^{d^i} \in \mathcal{Z}$, where $a_i \in R$. Then $\sum_{i=1}^{n} a_i x^{d^i} = 0$ for all $x \in R$.*

Before giving the proof of Theorem 1.4 we need the following generalization of Theorem 1.5

Theorem 1.6. *Let R be a noncommutative prime ring with a derivation d. Suppose that $\sum_{i=0}^{n} a_i x^{d^i} \in \mathcal{Z}$, where $a_i \in R$. Then $a_0 = 0$ and $\sum_{i=0}^{n} a_i x^{d^i} = 0$ for all $x \in R$.*

Proof. In view of Theorem 1.5, it is enough to show that $a_0 = 0$. Obviously we can assume that $d \neq 0$. We set $\phi(x) = \sum_{i=0}^{n} a_i x^{d^i}$ for $x \in U$, and note that $[\phi(x), y] = 0$ for all $x, y \in R$. According to [10, Theorem 2], $[\phi(x), y] = 0$ for all $x, y \in U$ and so $\phi(x) \in C$ for all $x \in U$. In particular, $a_0 = \phi(1) \in C$. Suppose that $a_0 \neq 0$. Replacing $\phi(x)$ with $a_0^{-1}\phi(x)$ we reduce the proof to the case when $a_0 = 1$. The aim is to derive a contradiction.

Given $x, y \in U$, it follows directly from Leibniz's rule that $\phi(yx) = \sum_{i=1}^{n} b_i x^{d^i} + \phi(y)x$ for some $b_i \in U$, depending on y. Therefore

$$\sum_{i=1}^{n} (b_i - \phi(y)a_i) x^{d^i} = \phi(yx) - \phi(y)\phi(x) \in C$$

for all $x, y \in U$. Theorem 1.5 now yields that $\phi(yx) = \phi(y)\phi(x)$ for all $x, y \in U$. Therefore $\phi : U \to C$ is a ring homomorphism. Next,

$\sum_{i=0}^{n} a_i x^{d^{i+1}} = \phi(x^d) \in C$ and so Theorem 1.5 yields that $x^d \in \ker(\phi)$. Since $d \neq 0$, $\ker(\phi) \neq 0$ as well. We see that $\ker(\phi)$ is a nonzero ideal of U and ϕ is a generalized differential polynomial identity on $\ker(\phi)$. Therefore [10, Theorem 2] implies that $\phi(x) = 0$ for all $x \in U$. In particular, $1 = \phi(1) = 0$, a contradiction. The proof is now complete. □

Proof of Theorem 1.4. In view of Theorem 1.3, $\psi(x) = \lambda x + \mu(x)$ for all $x \in R$, where $\lambda \in C$ and $\mu : R \to C$. That is, $\sum_{i=0}^{t} a_i x^{\delta^i} - \lambda x \in C$ for all $x \in R$ and so for all $x \in U$ [10, Theorem 2]. In view of Theorem 1.6, $\sum_{i=0}^{t} a_i x^{\delta^i} - \lambda x = 0$ for all $x \in U$. In particular, we set $x = 1$ to get $a_0 = \lambda \in \mathcal{Z}$, and hence $\sum_{i=1}^{t} a_i x^{\delta^i} = 0$ for all $x \in U$. Thus $\psi(x) = a_0 x$ for all $x \in R$. This proves the theorem. □

2. Proof of Theorem 1.1

We begin with the following special case.

Lemma 2.1. *Theorem 1.1 holds if $R = \mathrm{M}_m(C)$, the $m \times m$ matrix ring over a field C, unless $m = 2$ and $C = \mathrm{GF}(2)$.*

Proof. For $n = 1$ we are done by Brešar's Theorem [4, Theorem A]. Therefore, we always assume $n > 1$. Let $\{e_{ij} \mid 1 \leq i,\ j \leq m\}$ be the set of usual matrix units of R. The aim is to prove that there exists $\lambda \in C$ such that $f(e_{ij}) - \lambda e_{ij} \in C$ for all $1 \leq i,\ j \leq m$. Indeed, we then have $f(x) - \lambda x \in C$ for all $x \in R$ as f is C-linear. Hence, the lemma is proved by setting $\mu(x) = f(x) - \lambda x \in C$ for $x \in R$.

For $m \geq 3$ we claim that

$$[f(u), e] = 0 \text{ if } u^2 = eu = ue = 0 \quad \text{and} \quad e = e^2 \quad \text{for } e, u \in R. \tag{2.1}$$

Indeed, $(e + u)^n = e$ since $n > 1$. Thus, by assumption, $0 = [f(e + u), (e + u)^n] = [f(e) + f(u), e] = [f(u), e]$, as desired. We claim that there exist $\lambda_{ij} \in C$ such that

$$f(e_{ij}) - \lambda_{ij} e_{ij} \in C \quad \text{and so} \quad [f(e_{ij}), e_{ij}] = 0 \tag{2.2}$$

for $i \neq j$. Let $1 \leq p \leq m$ be distinct from i, j. Note that ${e_{pp}}^2 = e_{pp}$ and ${e_{ij}}^2 = 0 = e_{pp} e_{ij} = e_{ij} e_{pp}$. Thus, by (2.1), $[f(e_{ij}), e_{pp}] = 0$ follows. Write

$f(e_{ij}) = \sum_{s,t} \alpha_{st} e_{st}$, where $\alpha_{st} \in C$. A direct computation proves that

$$f(e_{ij}) = \sum_{s=1}^{m} \alpha_{ss} e_{ss} + \alpha_{ij} e_{ij} + \alpha_{ji} e_{ji}. \tag{2.3}$$

Since the idempotent $e_{ip}+e_{pp}$ satisfies $e_{ij}(e_{ip}+e_{pp}) = 0 = (e_{ip}+e_{pp})e_{ij}$, by (2.1) we have $[f(e_{ij}), e_{ip}+e_{pp}] = 0$ and so $[f(e_{ij}), e_{ip}] = 0$. Applying (2.3) we obtain that $\alpha_{ii}e_{ip} + \alpha_{ji}e_{jp} = \alpha_{pp}e_{ip}$. Hence, $\alpha_{ji} = 0$ and $\alpha_{ii} = \alpha_{pp}$. On the other hand, the idempotent $e_{pj} + e_{pp}$ satisfies $e_{ij}(e_{pj} + e_{pp}) = (e_{pj} + e_{pp})e_{ij} = 0$. By (2.1) again, $[f(e_{ij}), e_{pj} + e_{pp}] = 0$ and so $[f(e_{ij}), e_{pj}] = 0$. Applying (2.3) and $\alpha_{ji} = 0$ we obtain that $\alpha_{pp}e_{pj} = \alpha_{jj}e_{pj}$ and so $\alpha_{pp} = \alpha_{jj}$. This implies that $f(e_{ij}) - \alpha_{ij}e_{ij} \in C$. Set $\lambda_{ij} = \alpha_{ij} \in C$. In particular, $[f(e_{ij}), e_{ij}] = 0$. This proves (2.2).

Next, we write $f(e_{ii}) = \sum_{s,t} \beta_{ist} e_{st}$, where $\beta_{ist} \in C$. By assumption, we have $[f(e_{ii}), e_{ii}] = 0$. This implies $f(e_{ii})e_{ii} = e_{ii}f(e_{ii})$ and so $e_{ii}f(e_{ii})e_{pp} = 0$ for all $p \neq i$. Hence $\beta_{iip} = 0$. Using the fact that $e_{ii} + e_{ij}$ is an idempotent where $j \neq i$, we have that

$$0 = [f(e_{ii} + e_{ij}), e_{ii} + e_{ij}] = [f(e_{ii}), e_{ij}] + [f(e_{ij}), e_{ii}].$$

Note that $f(e_{ij}) - \lambda_{ij}e_{ij} \in C$. This implies that

$$[f(e_{ii}), e_{ij}] + [\lambda_{ij}e_{ij}, e_{ii}] = 0. \tag{2.4}$$

Right-multiplying by e_{pp} where $p \neq j$, we see that $\beta_{ijp} = 0$ and so $f(e_{ii})$ is diagonal, that is, $\beta_{ist} = 0$ for $s \neq t$ and so $f(e_{ii}) = \sum_{t=1}^{m} \beta_{itt}e_{tt}$. Making use of (2.4), we get $[\sum_{t=1}^{m} \beta_{itt}e_{tt}, e_{ij}] + [\lambda_{ij}e_{ij}, e_{ii}] = 0$ and so $\beta_{iii} = \beta_{ijj} + \lambda_{ij}$. Let $1 \leq k \leq m$ be such that $k \neq i, j$. By assumption,

$$\begin{aligned} 0 &= \left[f(e_{ii} + e_{kj} + e_{ji}),\ (e_{ii} + e_{kj} + e_{ji})^n\right] \\ &= \left[\sum_{t=1}^{m} \beta_{itt}e_{tt} + \lambda_{kj}e_{kj} + \lambda_{ji}e_{ji},\ e_{ii} + e_{ki} + e_{ji}\right] \\ &= (\beta_{ikk} - \beta_{iii} + \lambda_{kj})e_{ki} + (\beta_{ijj} - \beta_{iii} + \lambda_{ji})e_{ji}, \end{aligned}$$

since $n > 1$. This implies that $(\lambda_{kj} - \lambda_{ik})e_{ki} + (\lambda_{ji} - \lambda_{ij})e_{ji} = 0$, since $\beta_{iii} - \beta_{ikk} = \lambda_{ik}$. That is, $\lambda_{ji} = \lambda_{ij}$ and $\lambda_{kj} = \lambda_{ik} = \lambda_{ki}$. So $\beta_{iii} =$

$\beta_{ijj} + \lambda_{ij} = \beta_{ikk} + \lambda_{ik}$. But $\lambda_{ik} = \lambda_{jk} = \lambda_{kj} = \lambda_{ij}$, this implies that $\beta_{ijj} = \beta_{ikk}$ and so

$$f(e_{ii}) - \lambda_{ij}e_{ii} = \sum_{s=1}^{m} \beta_{iss}e_{ss} - \lambda_{ij}e_{ii} = \beta_{ijj}\sum_{s=1}^{m} e_{ss} \in C.$$

We let $\lambda = \lambda_{ij} \in C$. Then $f(e_{st}) - \lambda e_{st} \in C$ for $1 \le s, t \le m$.

We assume next that $m = 2$. By assumption, we have $[f(e_{11}), e_{11}] = 0$, implying that $f(e_{11}) = \alpha e_{11} + \beta e_{22}$ for some $\alpha, \beta \in C$. Setting $\lambda_{11} = \alpha - \beta$ we have $f(e_{11}) - \lambda_{11}e_{11} \in C$. Analogously, $f(e_{22}) - \lambda_{22}e_{22} \in C$ for some $\lambda_{22} \in C$. As $|C| > 2$, there exists $\alpha \in C$ with $\alpha \ne 0, 1$. Note that $e_{11} + e_{12}$ and $e_{11} + \alpha e_{12}$ are two idempotents. Thus $\big[f(e_{11} + e_{12}), e_{11} + e_{12}\big] = 0$ and $\big[f(e_{11} + \alpha e_{12}), e_{11} + \alpha e_{12}\big] = 0$. Since f is C-linear and $\alpha \ne 0, 1$, this implies $[f(e_{12}), e_{12}] = 0$. So $f(e_{12}) - \lambda_{12}e_{12} \in C$ for some $\lambda_{12} \in C$. Analogously, $f(e_{21}) - \lambda_{21}e_{21} \in C$ for some $\lambda_{21} \in C$. On the other hand, $0 = \big[f(e_{11} + e_{12}), e_{11} + e_{12}\big] = [f(e_{11}), e_{12}] + [f(e_{12}), e_{11}] = [\lambda_{11}e_{11}, e_{12}] + [\lambda_{12}e_{12}, e_{11}] = (\lambda_{11} - \lambda_{12})e_{12}$, implying that $\lambda_{11} = \lambda_{12}$. It follows from an analogous argument that $\lambda_{12} = \lambda_{22}$ and $\lambda_{11} = \lambda_{21}$. Set $\lambda = \lambda_{11}$. We see that $f(e_{ij}) - \lambda e_{ij} \in C$ for $i, j = 1, 2$. This proves the lemma. □

Lemma 2.2. *Let R be a prime* PI*-ring with center $\mathcal{Z}$. Then every $\mathcal{Z}$-linear map from R into RC is defined by a linear generalized polynomial with coefficients in RC.*

Proof. By Posner's Theorem for prime PI-rings, RC is a finite-dimensional central simple C-algebra. Moreover, $\mathcal{Z} \ne 0$ [22, Theorem 2.10] and C is the quotient field of $\mathcal{Z}$. Suppose that $f : R \to RC$ is a $\mathcal{Z}$-linear map. Then it is obvious that f is uniquely extended to a C-linear map from RC into RC. Note that $RC \otimes_C RC^o \cong \mathrm{End}_C(RC)$ via a canonical map ϕ, defined by $\phi(\sum_i a_i \otimes b_i{}^o)(x) = \sum_i a_i x b_i$ for $x \in RC$, where RC^o denotes the ring opposite to RC. Thus there exist $a_i, b_i \in RC$ such that $f = \phi(\sum_i a_i \otimes b_i{}^o)$. That is, $f(x) = \sum_i a_i x b_i$ for all $x \in R$, proving the lemma. □

Lemma 2.3. *If $xa - bx \in C$ for all $x \in R$, where $a, b \in U$, then either R is commutative or $a = b \in C$.*

Proof. Suppose that R is not commutative. Choose a dense right ideal ρ of R such that $b\rho \subseteq R$. Let $y \in \rho$. Then $by \in R$ and so $(by)a -$

$b(by) \in C$. That is, $b(ya - by) \in C$. Since $ya - by \in C$, either $b \in C$ or $ya = by$. If $b \in C$, then $R(a - b) \subseteq C$, implying that $a = b$ since R is not commutative. Suppose next that $ya = by$ for all $y \in \rho$. In view of [7, Theorem 2], $ya = by$ for all $y \in U$. In particular, set $y = 1$. Then $a = b$ follows. So $[a, R] \subseteq C$, implying $a \in C$ again. This proves the lemma. □

We are now ready to the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that $R \not\cong \mathrm{M}_2(\mathrm{GF}(2))$. By assumption, we have $[f(x), x^n] = 0$ for all $x \in R$. Suppose first that R is not a PI-ring. Then $\deg(R) = \infty$ in the sense of [1]. In view of [1, Theorem 4.4], there exist $a, b \in U$ and maps $\mu, \nu : R \to C$ such that $f(x) = xa + \mu_1(x) = bx + \nu_2(x)$ for all $x \in R$. Thus $xa - bx \in C$ for all $x \in R$. It follows from Lemma 2.3 that either R is commutative or $a = b \in C$. Since R is not a PI-ring, R is not commutative. So $a = b \in C$. We are done in this case by setting $\lambda = a \in C$.

Suppose next that R is a PI-ring. Then $\mathcal{Z} \neq 0$ [22, Theorem 2.10]. By assumption, f is a $\mathcal{Z}$-linear map. In view of Lemma 2.2, there exist finitely many $a_i, b_i \in RC$ such that $f(x) = \sum_i a_i x b_i$ for all $x \in R$. By assumption, we see that

$$\left[\sum_i a_i x b_i, x^n\right] = 0 \tag{2.5}$$

for all $x \in R$ and hence for all $x \in RC$ ([3, Theorem 6.4.1] or [7, Theorem 2]). Define F to be the algebraic closure of C if C is infinite. Otherwise, let $F = C$. Then (2.5) holds for all $x \in RC \otimes_C F$. Note that $x \in RC \otimes_C F \cong \mathrm{M}_m(F)$ for some $m \geq 1$. Define $g : RC \otimes_C F \to RC \otimes_C F$ by $g(x) = \sum_i a_i x b_i$ for all $x \in RC \otimes_C F$. Then, by Lemma 2.1, there exist $c \in F$ and $\nu : RC \otimes_C F \to F$ such that $g(x) = cx + \nu(x)$ for all $x \in RC \otimes_C F$. Choose a basis $\{\beta_1, \beta_2, \ldots\}$ of F over C with $\beta_1 = 1$. Write $c = \lambda\beta_1 + \sum_{j=2}^{s} \lambda_j \beta_j$ for some $s \geq 1$ and $\lambda, \lambda_j \in C$. Set $\mu(x) = g(x) - \lambda x$ for $x \in RC$. Then $\mu(x) \in C$ for $x \in RC$. Note that $f(x) = g(x)$ for all $x \in R$. Thus we see that $f(x) = \lambda x + \mu(x)$ for all $x \in R$, proving the theorem. □

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