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A simple proof of a Paley–Wiener type theorem for the Chébli–Trimèche transform

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Abstract. We give a simple proof of a Paley–Wiener type theorem for the Chébli–Trimèche transform, characterising the Schwartz functions with compactly supported image.

1. Introduction

Let $\mathcal{F}$ denote the Chébli–Trimèche transform, let Δ denote the associated second order differential operator, $\rho \in \mathbb{R}$ a non-negative number to be defined later and $\mathcal{S}^p(\mathbb{R})^{\text{even}}$, $0 < p \leq 2$, the L^p-Schwartz spaces. Betancor, Betancor and Méndez showed in [2] that the limit

$$\lim_{n\to\infty} \left\|(\Delta - \rho^2)^n f\right\|_r^{1/2n}$$

exists for all $f \in \mathcal{S}^p(\mathbb{R})^{\text{even}}$, $0 < p < 2$, and all $1 \leq r \leq \infty$, and that it equals the radius of the support of $\mathcal{F}f$. This generalises [1, Theorem 1] for the classical Fourier transform.

In this paper, we use continuity of (the inverse of) $\mathcal{F}$ on the Schwartz spaces and the convolution structure associated with $\mathcal{F}$ to give a short and

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simple proof of the above Paley–Wiener type theorem (also including the $p = 2$ case).

2. The Chébli–Trimèche transform

The material in this section is from [3], [4] and [5]. Consider the following second order differential operator on $\mathbb{R}_+$:

$$\Delta := -\frac{1}{A(x)}\frac{d}{dx}\left(A(x)\frac{d}{dx}\right) = -\frac{d^2}{dx^2} - \frac{A'(x)}{A(x)}\frac{d}{dx},$$

where the (Chébli–Trimèche) function A is continuous on $[0,\infty)$, twice continuously differentiable on $(0,\infty)$ and satisfies the conditions:

(i) $A(0) = 0$ and $A(x) > 0$ for $x > 0$,

(ii) A is increasing and unbounded,

(iii) $\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x)$ on a neighbourhood of 0, where $\alpha > -\frac{1}{2}$ and B is an odd function on $\mathbb{R}$,

(iv) $\frac{A'(x)}{A(x)}$ is a decreasing smooth function on $(0,\infty)$ and hence $\rho := \frac{1}{2}\lim_{x\to\infty}\frac{A'(x)}{A(x)} \geq 0$ exists.

Assume furthermore that there exists a $\delta > 0$ such that, for all $x \in [x_o,\infty)$:

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\delta x}D(x) & \text{if } \rho > 0 \\ \dfrac{2\alpha+1}{x} + e^{-\delta x}D(x) & \text{if } \rho = 0 \end{cases}$$

where D is a smooth function whose derivatives of any order are bounded.

Let φ_λ, $\lambda \in \mathbb{C}$, denote the solutions of the differential equation:

$$\Delta\varphi_\lambda = (\lambda^2 + \rho^2)\varphi_\lambda, \qquad \varphi_\lambda(0) = 1, \qquad \varphi'_\lambda(0) = 0. \tag{1}$$

We notice that $|\varphi_\lambda(x)| \leq \varphi_0(x) \leq 1$, and also that $e^{-\rho x} \leq \varphi_0(x) \leq C(1 + |x|)e^{-\rho x}$, for all $\lambda, x \in \mathbb{R}$, where C is a positive constant, see [3, Lemma 3.4].

We define, for $f \in C_c^\infty(\mathbb{R})^{\text{even}}$, the Chébli–Trimèche transform $\mathcal{F}$ as:

$$\mathcal{F}f(\lambda) := \int_{\mathbb{R}_+} f(x)\varphi_\lambda(x)A(x)dx.$$

The (classical) Paley–Wiener theorem for the Chébli–Trimèche transform, see [4], states that $\mathcal{F}$ is a bijection from $C_c^\infty(\mathbb{R})^{\text{even}}$ onto $\mathcal{H}(\mathbb{C})^{\text{even}}$, the space of even entire rapidly decreasing functions of exponential type. More precisely; for $R > 0$, it maps $C_R^\infty(\mathbb{R})^{\text{even}} := \{f \in C_c^\infty(\mathbb{R})^{\text{even}} \mid \operatorname{supp} f \subset [-R,R]\}$, bijectively onto $\mathcal{H}_R(\mathbb{C})^{\text{even}}$, the space of even entire functions satisfying:

$$\sup_{\lambda\in\mathbb{C}} (1+|\lambda|)^k e^{-R|\Im\lambda|} g(\lambda) < \infty,$$

for all $k \in \mathbb{N}\cup\{0\}$. The inverse transform $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}g(x) = \int_{\mathbb{R}_+} g(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda, \qquad (x\in\mathbb{R}), \tag{2}$$

for $g \in \mathcal{H}(\mathbb{C})^{\text{even}}$, where $|c(\lambda)|^{-2}$ is continuous on $[0,\infty)$.

The Chébli–Trimèche transform reduces to the Hankel transform when $A(x) = x^{2\alpha+1}$, $\alpha > -\frac{1}{2}$; and to the Jacobi transform when $A(x) = \cosh^{2\alpha+1}(x)\sinh^{2\beta+1}(x)$, $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha \neq -\frac{1}{2}$.

3. Schwartz spaces

The material is this section is taken from [3]. Let $L^p(\mathbb{R}, A(x)dx)^{\text{even}}$, $1 \leq p \leq \infty$, denote the space of even measurable functions f such that

$$\|f\|_p := \left(\int_{\mathbb{R}_+} |f(x)|^p A(x)dx\right)^{1/p} < \infty,\ 1 \leq p < \infty, \qquad \text{and}$$

$$\|f\|_\infty := \operatorname{ess}\sup_{x\in\mathbb{R}} |f(x)| < \infty.$$

Let $0 < p \leq 2$. The L^p-Schwartz space $\mathcal{S}^p(\mathbb{R})^{\text{even}}$ is defined as the space of all functions $f \in C^\infty(\mathbb{R})^{\text{even}}$ such that:

$$\mu_{k,l}^p(f) := \sup_{x\in\mathbb{R}} e^{\frac{2}{p}\rho|x|}(1+|x|)^k \left|\frac{d^l}{dx^l}f(x)\right| < \infty,$$

for all $k, l \in \mathbb{N}\cup\{0\}$. We equip $\mathcal{S}^p(\mathbb{R})^{\text{even}}$ with the topology defined by the seminorms $\mu_{k,l}^p$. We note that our definition is equivalent to the definition of generalised Schwartz spaces in [3] by the estimates on $\varphi_\lambda(x)$. We also note that

$$A(x) \leq C(1+x)^\beta e^{2\rho x}, \qquad x \geq 0, \tag{3}$$

where C and β are positive constants, see [3, (3.5)].

Let $0 \leq \varepsilon < \infty$. Let $\mathbb{R}_\varepsilon := \{\lambda \in \mathbb{C} : |\Im\lambda| \leq \varepsilon\rho\}$, and let $\mathbb{R}_\varepsilon^o$ denote the interiour of $\mathbb{R}_\varepsilon$ when $\varepsilon\rho > 0$. We define the extended Schwartz space $\mathcal{S}(\mathbb{R}_\varepsilon)^{\text{even}}$ as the space of all even analytic functions g in $\mathbb{R}_\varepsilon^o$, such that all the derivatives of g extend continuously to $\mathbb{R}_\varepsilon$, and such that

$$\tau_{k,l}^\varepsilon(g) = \sup_{\lambda\in\mathbb{R}_\varepsilon} (1+|\lambda|)^k \left| \frac{d^l}{d\lambda^l} g(\lambda) \right| < \infty,$$

for all $k, l \in \mathbb{N} \cup \{0\}$. We equip $\mathcal{S}(\mathbb{R}_\varepsilon)^{\text{even}}$ with the topology defined by the seminorms $\tau_{k,l}^\varepsilon$.

Theorem 3.1. *Let $0 < p \leq 2$ and $\varepsilon = \frac{2}{p} - 1$. The Chébli–Trimèche transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{S}^p(\mathbb{R})^{\text{even}}$ onto $\mathcal{S}(\mathbb{R}_\varepsilon)^{\text{even}}$. The inverse transform $\mathcal{F}^{-1}$ is again given by* (2).

Proof. See [3, Theorem 4.27]. □

Remark 3.2. Let $\rho = 0$. Then $\mathcal{S}^p(\mathbb{R})^{\text{even}}$ and $\mathcal{S}(\mathbb{R}_\varepsilon)^{\text{even}}$ are identical to the classical Schwartz space $\mathcal{S}(\mathbb{R})^{\text{even}}$ of even functions. Let now $\rho > 0$. Then $\mathcal{S}^p(\mathbb{R})^{\text{even}} \subset L^q(\mathbb{R}, A(x)dx)^{\text{even}}$ for all $0 < p \leq q \leq \infty$, but $\mathcal{S}^p(\mathbb{R})^{\text{even}} \not\subset L^q(\mathbb{R}, A(x)dx)^{\text{even}}$ for $0 < q < p \leq 2$.

Remark 3.3. Let $\rho > 0$ and let $0 \neq f \in \mathcal{S}^p(\mathbb{R})^{\text{even}}$, with $0 < p < 2$. Then $\mathcal{F}f$ is analytic in $\mathbb{R}_\varepsilon^o$, with $\varepsilon = \frac{2}{p} - 1$, and hence cannot have compact support.

Continuity of $\mathcal{F}^{-1}$ implies in particular that, for any $m \in \mathbb{N}$ and $g \in C_c^\infty(\mathbb{R})^{\text{even}}$:

$$\sup_{x\in\mathbb{R}_+} e^{\frac{2}{p}\rho x}(1+x)^m |\mathcal{F}^{-1}g(x)| \leq C \sup_{\lambda\in\mathbb{R}_+} \sum_{1\leq k,l\leq M} (1+\lambda)^k \left| \frac{d^l}{d\lambda^l} g(\lambda)) \right|, \tag{4}$$

for a positive constant C and a positive integer M.

We finally see that, for all $f \in \mathcal{S}^p(\mathbb{R})$, $0 < p \leq 2$, and all $n \in \mathbb{N} \cup \{0\}$,

$$\mathcal{F}(\Delta - \rho^2)^n f(\lambda) = \int_{\mathbb{R}_+} (\Delta - \rho^2)^n f(x) \varphi_\lambda(x) A(x) dx$$

$$= \int_{\mathbb{R}_+} f(x)(\Delta - \rho^2)^n \varphi_\lambda(x) A(x) dx$$
$$= \int_{\mathbb{R}_+} f(x)\lambda^{2n}\varphi_\lambda(x)A(x)dx = \lambda^{2n}\mathcal{F}f(\lambda), \qquad (\lambda \in \mathbb{R}),$$

using symmetry of Δ and (1). This yields the following identities:

$$(\Delta - \rho^2)^n f = \mathcal{F}^{-1}\mathcal{F}(\Delta - \rho^2)^n f = \mathcal{F}^{-1}(\lambda^{2n}\mathcal{F}f), \tag{5}$$

for all $n \in \mathbb{N} \cup \{0\}$.

4. A Paley–Wiener type theorem

This section contains our short and simple proof of [2, Proposition 2.2]. Let $f \in \mathcal{S}^p(\mathbb{R})$, then the radius of the support of $\mathcal{F}f$ is given by: $R_f := \sup\{|\lambda| : \lambda \in \operatorname{supp} \mathcal{F}f\}$.

Theorem 4.1. *Let $0 < p \leq 2$, $1 \leq r \leq \infty$, if $\rho = 0$ and $0 < p \leq r \leq 2$, if $\rho > 0$. Then*

$$\lim_{n\to\infty} \left\|(\Delta - \rho^2)^n f\right\|_r^{1/2n} = R_f,$$

for any $f \in \mathcal{S}^p(\mathbb{R})^{\text{even}}$.

Proof. Let $f \in \mathcal{S}^p(\mathbb{R})^{\text{even}}$ and assume that $\mathcal{F}f$ has compact support. Write $(\Delta - \rho^2)^n f(x) = (1+x)^{-2}(1+x)^2(\Delta - \rho^2)^n f(x)$, for $x \in \mathbb{R}_+$. The estimate (3) on $A(x)$, the identities (5) and continuity of $\mathcal{F}^{-1}$ (4) yield:

$$\begin{aligned}
&\left\|(\Delta - \rho^2)^n f\right\|_r \\
&\quad \leq C \left(\int_{\mathbb{R}_+} \left|(1+x)^{-2}(1+x)^2(\Delta - \rho^2)^n f(x)\right|^r e^{2\rho x}(1+x)^\beta dx \right)^{1/r} \\
&\quad \leq C \sup_{x\in\mathbb{R}_+} e^{\frac{2}{r}\rho x}(1+x)^{\frac{\beta}{r}+2}|(\Delta - \rho^2)^n f(x)| \\
&\quad \leq C \sup_{x\in\mathbb{R}_+} e^{\frac{2}{p}\rho x}(1+x)^m|\mathcal{F}^{-1}(\lambda^{2n}\mathcal{F}f)(x)| \\
&\quad \leq C \sup_{\lambda\in\mathbb{R}_+} \sum_{1\leq k,l\leq M} (1+\lambda)^k \left|\frac{d^l}{d\lambda^l}(\lambda^{2n}\mathcal{F}f(\lambda))\right|,
\end{aligned}$$

for positive constants C and integers m, M, independent of n. Leibniz's rule yields

$$\frac{d^l}{d\lambda^l}(\lambda^{2n}\mathcal{F}f(\lambda)) = \sum_{j=0}^{l}\binom{l}{j}\frac{(2n)!}{(2n-j)!}\lambda^{2n-j}\frac{d^{l-j}}{d\lambda^{l-j}}\mathcal{F}f(\lambda).$$

Using the estimates $\sum_{j=0}^{l}\binom{l}{j}\frac{(2n)!}{(2n-j)!} \leq MM!(2n)^M$ and $R_f^{2n-j} \leq (1+1/R_f)^M R_f^{2n}$, we get:

$$\left\|(\Delta-\rho^2)^n f\right\|_r \leq C(2n)^M MM!(1+R_f+1/R_f)^M R_f^{2n},$$

for a positive constant C, independent of n. We thus see that

$$\limsup_{n\to\infty}\left\|(\Delta-\rho^2)^n f\right\|_r^{1/2n} \leq R_f \lim_{n\to\infty} n^{M/2n} = R_f.$$

There is a generalised convolution structure $*$ associated with the Chébli–Trimèche transform, see [5] and [6] for details. We need the following properties, for $f_1, f_2 \in \mathcal{S}^p(\mathbb{R})^{\text{even}}$:

(a) $f_1 * f_2 \in \mathcal{S}^2(\mathbb{R})^{\text{even}}$.

(b) $\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1)\mathcal{F}(f_2)$.

(c) $\|f_1 * f_2\|_\infty \leq \|f_1\|_r \|f_2\|_{r'}$, if $f_1 \in L^r(\mathbb{R}, A(x)dx)^{\text{even}}$, $f_2 \in L^{r'}(\mathbb{R}, A(x)dx)^{\text{even}}$, with $1/r + 1/r' = 1$.

Consider a general function $0 \neq f \in \mathcal{S}^p(\mathbb{R})^{\text{even}}$. Let $R > 0$ such that $R < R_f \leq \infty$. Then

$$\begin{aligned}
\left\|(\Delta-\rho^2)^n f\right\|_r \left\|\bar{f}\right\|_{r'} &\geq \left\|(\Delta-\rho^2)^n f * \bar{f}\right\|_\infty \\
&= \left\|\mathcal{F}^{-1}\mathcal{F}((\Delta-\rho^2)^n f * \bar{f})\right\|_\infty = \left\|\mathcal{F}^{-1}(\lambda^{2n}|\mathcal{F}f|^2)\right\|_\infty \\
&= \sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}_+}\lambda^{2n}|\mathcal{F}f(\lambda)|^2\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda\right| \\
&= \int_{\mathbb{R}_+}\lambda^{2n}|\mathcal{F}f(\lambda)|^2|c(\lambda)|^{-2}d\lambda \geq \int_R^\infty \lambda^{2n}|\mathcal{F}f(\lambda)|^2|c(\lambda)|^{-2}d\lambda \\
&\geq R^{2n}\int_R^\infty |\mathcal{F}f(\lambda)|^2|c(\lambda)|^{-2}d\lambda,
\end{aligned}$$

since $|\varphi_\lambda(x)| \le \varphi_\lambda(0) = 1$, and thus,

$$\liminf_{n\to\infty} \left\|(\Delta - \rho^2)^n f\right\|_r^{1/2n} = R_f,$$

considering $R = R_f - \epsilon$, for any $\epsilon > 0$, if $R_f < \infty$, and $R \to \infty$ otherwise. □

References

[1] H. H. Bang, A property of infinitely differentiable functions, *Proc. Amer. Math. Soc.* **108** (1990), 73–76.

[2] J. J. Betancor, J. D. Betancor and J. M. R. Méndez, Paley–Wiener type theorems for Chébli-Trimèche transforms, *Publ. Math. Debrecen.* **60** (2002), 347–358.

[3] W. R. Bloom and Z. Xu, Fourier transforms of Schwartz functions on Chébli–Trimèche hypergroups, *Monatsh. Math.* **125** (1998), 89–109.

[4] H. Chébli, Sur un théorème de Paley–Wiener associé à la décomposition spectrale d'un opérateur de Sturm–Liouville sur $]0,\infty[$, *J. Funct. Anal.* **17** (1974), 447–461.

[5] K. Trimèche, Transformation intégrale de Weyl et théorème de Paley–Wiener associés à un opérateur différentiel singulier sur $(0,\infty)$, *J. Math. Pures Appl.* (9) **60** (1981), 51–98.

[6] K. Trimèche, Inversion of the Lions transmutation operators using generalized wavelets, *Appl. Comput. Harmon. Anal.* **4** (1997), 97–112.

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