An improved proof of Numata and Shibata's theorems on Finsler spaces of scalar curvature

By MAKOTO MATSUMOTO (Kyoto)

Abstract. Let F^n, $n \geq 3$, be a Finsler space of non-zero scalar curvature K. S. Numata proved in 1975 a theorem: If F^n is a Landsberg space, then it is a Riemannian space of constant curvature K. C. Shibata extended this theorem in 1978 under the condition that F^n has vanishing stretch curvature. But the notion of the stretch curvature given by L. Berwald in 1925 has little relation to metrical connections, and has been hidden from sight. We first clarifies the notion of this curvature, and then give a brief proof of Shibata's Theorem.

Introduction

The theorem called here SHIBATA's Theorem is "Theorem 4" of his paper [S]:

Theorem S. *Let F^n, $n \geq 3$, be a Finsler space of non-zero scalar curvature K. If F^n has the vanishing stretch curvature tensor, then F^n is a Riemannian space of constant curvature K.*

He added a remark: This is a generalization of "Theorem 2" of NU-MATA's paper [N]:

Theorem N1. *Let F^n, $n \geq 3$, be a Finsler space of scalar curvature K. If F^n is a Berwald space, then F^n is a Riemannian space of*

Mathematics Subject Classification: 53B40.
Key words and phrases: stretch curvature, Finsler space of scalar curvature.

constant curvature K or a locally Minkowski space, according as $K \neq 0$ or $K = 0$.

The main theorem of [N] is, however "Theorem 1":

Theorem N2. *Let F^n, $n \geq 3$, be a Finsler space of non-zero scalar curvature K. If F^n is a Landsberg space, then F^n is a Riamnnian space of constant curvature K.*

To prove this theorem, S. Numata showed first a lemma:

Lemma. *If a Finsler space F^n, $n \geq 3$, of non-zero scalar curvature is a Landsberg space, then F^n is a C-reducible space.*

Then, according to "Theorem 1" of [M1], the condition "Landsberg" of Theorem N2 is reduced to "Berwald", and hence Theorem N2 yields Theorem N1.

Numata's Theorem N2 will be well-known, because it was published in §30 of the monograph [M2] and the condition "Landsberg" is attractive. While the notion "stretch curvature" in Shibata's Theorem has little relation to metrical connections and has been hidden from sight ever since BERWALD gave the definition [B].

Recently we have an interesting paper [BF]. The authors gave a generalization of Numata's Theorem by generalizing the condition "Landsberg" to "generalized Landsberg" as follows:

Definition BF. A Finsler space F^n is called *genaralized Landsberg*, if the h-curvature tensors H and K of the Berwald connection BΓ and the Chern–Rund connection CRΓ coincide with each other.

The relation between H and K is written as

$$H_i{}^h{}_{jk} = K_i{}^h{}_{jk} + \tilde{A}_{(jk)}\{P^h{}_{ij/k} + P^h{}_{kr}P^r{}_{ij}\},$$

where $_{/k}$ is the h-covariant differentiation with respect to CRΓ or the Cartan connection CΓ and $P^h{}_{ij} = C_i{}^h{}_{j/0}$ ([M2], (18.16)). Hence "generalized Landsberg" is defined by $\tilde{A}_{(jk)}\{P^h{}_{ij/k} + P^h{}_{kr}P^r{}_{ij}\} = 0$, (cf. [M2], (25.2)), which is equivalent to

(a) $P_{hij/k} - P_{hik/j} = 0,$ (b) $P_{hkr}P^r{}_{ij} - P_{hjr}P^r{}_{ik} = 0.$

Indeed, it must be confessed that the stretch curvature tensor Σ_{hijk} is equal to $2(P_{hij/k} - P_{hik/j})$, as shown later on, and hence (a) asserts that *a generalized Landsberg space has the vanishing stretch curvature* Σ. Therefore their theorem is more restrictive than Shibata's, though more general than Numata's.

Furthermore their proof of Theorem 2 was done by means of the positive-definiteness: $g^{rs}C_{r/0}C_{s/0} = 0 \rightarrow C_{r/0} = 0$, though Numata and Shibata had not recoursed to such a restriction.

The main purpose of the present paper is to explane the notion of the stretch curvature and to show an improved proof of Shibata's Theorem from the standpoint of recent developments of Finsler Geometry. (The author's recent paper [M3] is not published yet.)

1. The stretch curvature tensor

We consider a Finsler space $F^n = (M, L(x, y))$, equipped with a Finsler connection $F\Gamma = (N^i{}_j, F_j{}^i{}_k, U_j{}^i{}_k)$. Let us denote the fundamental tensor by $g_{ij}(x, y)$ and the h- and v-covariant differentiations with respect to $F\Gamma$ by (; , :) respectively.

Take an infinitesimal circuit of M which consists of four points $P(x)$, $Q(x + d_1x)$, $R(x + d_1x + d_2(x + d_1x))$, $S(x + d_2x)$ and P, and a vector field $v(v^i)$ which is defined along the circuit and transformed parallel with respect to a parallel supporting element y. Thus we have

$$dv^i + v^r F_r{}^i{}_j dx^j = 0, \quad dy^j + N^j{}_i dx^i = 0.$$

For a function $f(x, y)$, we have

$$df = (\partial_i f)dx^i + (\dot\partial_j f)dy^j = (\delta_i f)dx^i,$$

where $\delta_i := \partial_i - (\dot\partial_j)N^j{}_i$.

We treat of the fundamental tensor $g_{hi}(x, y)$:

$$dg_{hi} = (\delta_j g_{hi})dx^j = (g_{hi;\,j} + g_{ri}F_h{}^r{}_j + g_{hr}F_i{}^r{}_j)dx^j.$$

Then, for the covariant components $v_i = g_{ij}v^j$ of v we have

$$dv_i = (dg_{ij})v^j + g_{ij}dv^j = (g_{ij;\,h} + g_{rj}F_i{}^r{}_h)v^j dx^h.$$

We are concerned with the length V of $v^i : V^2 = v^i v_i$.

$$d_1 V^2 = (d_1 v^i) v_i + v^i(d_1 v_i) = (-v^r F_r{}^i{}_j d_1 x^j) v_i$$
$$+ v^i(g_{ij;\,k} + g_{rj} F_i{}^r{}_k) v^j d_1 x^k = g_{ij;\,k} v^i v^j d_1 x^k,$$

$$d_2 d_1 V^2 = d_2(g_{ij;\,k} v^i v^j d_1 x^k)$$
$$= \delta_h(g_{ij;\,k} v^i v^j) d_2 x^h d_1 x^k + g_{ij;\,k} v^i v^j d_2 d_1 x^k$$
$$= \left[\delta_h(g_{ij;\,k}) v^i v^j + g_{ij;\,k}\{(-v^r F_r{}^i{}_h) v^j\right.$$
$$\left. + v^i(-v^r F_r{}^j{}_h)\}\right] d_2 x^h d_1 x^k + g_{ij;\,k} v^i v^j d_2 d_1 x^k$$
$$= \{\delta_h(g_{ij;\,k}) - g_{rj;\,k} F_i{}^r{}_h - g_{ir;\,k} F_j{}^r{}_h\} v^i v^j d_2 x^h d_1 x^k$$
$$+ g_{ij;\,k} v^i v^j d_2 d_1 x^k$$
$$= (g_{ij;\,k;\,h} + g_{ij;\,r} F_k{}^r{}_h) v^i v^j d_2 x^h d_1 x^k + g_{ij;\,k} v^i v^j d_2 d_1 x^k.$$

Therefore, putting

$$(d_2 d_1 - d_1 d_2) V^2 = -1/2 \Sigma_{ijhk} v^i v^j (d_1 x^h d_2 x^k - d_2 x^h d_1 x^k),$$

then we have

$$\Sigma_{ijhk} = g_{ij;\,k;\,h} - g_{ij;\,h;\,k} - g_{ij;\,r} T_h{}^r{}_k, \tag{1.1}$$

where T is the $(h)h$-torsion tensor of FΓ ([M2], §10). The tensor field $\Sigma = \big(\Sigma_{ijhk}(x, y)\big)$ is called the stretch curvature tensor of FΓ. As shown by (1.1), if we are concerned with a h-metrical connection FΓ, then we have $g_{ij;\,k} = 0$, and hence its stretch curvature tensor Σ vanishes identically.

We observe the standard four connections ([M2], §17, §18):

$$B\Gamma = (G^i{}_j, G_j{}^i{}_k, 0) \qquad\qquad C\Gamma = (G^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k),$$

$$CR\Gamma = (G^i{}_j, F_j{}^i{}_k, 0), \qquad\qquad H\Gamma = (G^i{}_j, G_j{}^i{}_k, C_j{}^i{}_k).$$

The *Cartan connection* CΓ is h- and v-metrical and, throughout the following, we denote by $(_/, |)$ the h- and v-covariant differentiations with respect to CΓ.

The *Berwald connection* BΓ is not h-metrical $(g_{ij;\,k} = -2C_{ijk/0})$ and not v-metrical $(g_{ij:k} = \dot{\partial}_k g_{ij} = 2C_{ijk})$. In the following the symbols $(;, :)$ are mainly used for BΓ.

The *Chern–Rund connection* CRΓ has the same $F_j{}^i{}_k$ with CΓ, while the *Hashiguchi connection* HΓ has the same $G_j{}^i{}_k$ with BΓ.

Definition. The *stretch curvature tensor* Σ of a Finsler space $F^n = (M, L(x, y))$ is the stretch curvature tensor of the Berwald connection BΓ.

We have one of the Ricci identities of BΓ: For a $(1, 1)$-tensor field K we have

$$K^h{}_{i;\,j;\,k} - K^h{}_{i;\,k;\,j} = K^r{}_i H_r{}^i{}_{jk} - K^h{}_r H_i{}^r{}_{jk} - K^h{}_{i:r} R^r{}_{jk},$$

where $H_i{}^h{}_{jk}$ is the h-curvature tensor and $R^i{}_{jk} = (y^r H_r{}^i{}_{jk})$ the $(v)h$-torsion tensor. It is noted that the $(h)h$-torsion $G_j{}^i{}_k - G_k{}^i{}_j$ of BΓ vanishes identically. Thus, for the fundamental tensor g_{hi}

$$g_{hi;\,j;\,k} - g_{hi;\,k;\,j} = -g_{ri} H_h{}^r{}_{jk} - g_{hr} H_i{}^r{}_{jk} - 2C_{hir} R^r{}_{jk}.$$

Consequently (1.1) shows

Proposition 1.1. *The stretch curvature tensor Σ of a Finsler space F^n is given by*

$$\Sigma_{hijk} = H_{hijk} + H_{ihjk} + 2C_{hir} R^r{}_{jk},$$

in the Berwald connection BΓ.

On the other hand, if we are concerned with HΓ, [M2], (14.16) gives its h-curvature tensor ${}^*R_h{}^i{}_{jk} = H_h{}^i{}_{jk} + C_h{}^i{}_r R^r{}_{jk}$. Thus

$$\Sigma_{hijk} = {}^*R_{hijk} + {}^*R_{ihjk}, \quad \text{in HΓ.} \tag{1.2}$$

In the following it is important to write the stretch curvature tensor Σ in terms of the Cartan connection CΓ. We have [M2], (17.22), (18.2) and (18.16):

$$P_{ijk} = C_{ijk/0}, \quad R_{hijk} = K_{hijk} + C_{hir} R^r{}_{jk},$$

$$K_{hijk} = H_{hijk} - \tilde{A}_{(jk)}\{P_{hij/k} + P_{hjr} P^r{}_{ik}\}.$$

The symbol $\tilde{A}_{(jk)}$ is used to interchange indices j, k and subtract. It follows from the well-known $R_{hijk} = -R_{ihjk}$ that we get

$$\Sigma_{hijk} = 2(P_{hij/k} - P_{hik/j}), \quad \text{in CΓ.} \tag{1.3}$$

The well-known relation $G_j{}^i{}_k = F_j{}^i{}_k + P_j{}^i{}_k$ leads to

$$\Sigma_{hijk} = 2(P_{hij;\,k} - P_{hik;\,j}), \quad \text{in B}\Gamma. \tag{1.4}$$

In order to derive an interesting expression of Σ we need the identity

$$y_r G_i{}^r{}_{jk} = -2P_{ijk}, \quad y_r := g_{ri}y^i.$$

The proof is as follows: One of the Ricci identities of BΓ shows

$$y_{i;\,j:k} - y_{i:k;\,j} = -y_r G_i{}^r{}_{jk}$$
$$= (g_{ir}y^r)_{;\,j:k} - (g_{ir}y^r)_{:k;\,j} = -g_{ik;\,j} = 2P_{ikj}.$$

The Bianchi identities (18.21) and (18.22) of [M2] lead to

$$y_r(H_h{}^r{}_{kj:i} + G_h{}^r{}_{ji;\,k} - G_h{}^r{}_{ik;\,j})$$
$$= y_r R^r{}_{kj:h:i} - 2(P_{hij;\,k} - P_{hik;\,j}) = 0.$$

Thus (1.4) yields

$$\Sigma_{hijk} = -y_r \dot{\partial}_h \dot{\partial}_i R^r{}_{jk}. \tag{1.5}$$

Remark. The four connections we considered have the common non-linear connection $(G^i{}_j)$ and hence the common $(v)h$-torsion tensor

$$R^i{}_{jk} = \tilde{A}_{(jk)}\{\partial_k G^i{}_j - (\dot{\partial}_r G^i{}_j)G^r{}_k\}.$$

Thus the formula (1.5) is common to those connections.

Proposition 1.2. *The stretch curvature tensor Σ of a Finsler space F^n is written in the forms (1.2) in HΓ, (1.3) in CΓ, (1.4) in BΓ and (1.5).*

We pay attention to a two-dimensional Finsler space F^2 with the Berwald frame $(1, m)$ ([M2], §28; [AIM], §3.5). Then we have

$$P_{hij} = I_{,\,1}m_h m_i m_j, \quad P_{hij/k} = (I_{,\,1,\,1}\ell_k + I_{,\,1,\,2}m_k)m_h m_i m_j,$$

where I is the main scalar. Thus (1.3) leads to the stretch curvature tensor Σ of the form

$$\Sigma_{hijk} = -2I_{,\,1,\,1}m_h m_i G_{jk}, \quad G_{jk} := \ell_j m_k - \ell_k m_j. \tag{1.6}$$

We have the Bianchi identity of the form

$$\varepsilon R;_2 + RI + I,_{1,1} = 0,$$

where R is the h-scalar curvature: $R_{hijk} = \varepsilon R G_{hi} G_{jk}$. Hence we have another form

$$\Sigma_{hijk} = 2(\varepsilon R;_2 + RI)m_h m_i G_{jk}. \tag{1.6'}$$

The latter shows that $\Sigma = 0$ is equivalent to $R;_2 + RI\varepsilon = 0$, that is,

$$\partial \log(\varepsilon R)/\partial\theta + I = 0, \tag{1.7}$$

where θ is the Landsberg angle ([M2], (28.6)).

Next we consider $g = \det(g_{ij})$. From $L\dot\partial_i g = 2\varepsilon g I m_i$ we get $L\dot\partial_i(R^2 g) = 2Rg\varepsilon (R;_2 + RI)m_i$ and hence

$$L\dot\partial_i(R\sqrt{|g|}) = -\varepsilon\sqrt{|g|}I,_{1,1}m_i.$$

Therefore we have

Theorem 1.1. (1) *The main scalar I of a Finsler surface F^2 having zero stretch curvature is given from the h-scalar curvature R by (1.7).*

(2) *$R\sqrt{|g|}$ of F^2 depends on a position alone, if and only if F^2 has the zero stretch curvature.*

Therefore the total curvature $\iint R\sqrt{|g|}dx^1 dx^2$ can be defined only in a Finsler surface with $\Sigma = 0$.

2. Finsler spaces of scalar curvature

Let $F^n = (M, L)$, $n \geq 3$, be a Finsler space of dimension n, equipped with the Berwald connection BΓ, and denote by $(; , :)$ the h- and v-covariant differentiations in BΓ, where $:_i = \partial/\partial y^i$. The $(v)h$-torsion tensor $R^i{}_{jk}$ and the h-curvature tensor $H_h{}^i{}_{jk}$ of BΓ satisfy the equations

$$H_h{}^i{}_{jk} = R^i{}_{jk}:_h, \quad R^i{}_{jk} = (1/3)(R^i{}_{0k}:_j - R^i{}_{0j}:_k).$$

It is noted that $R_{i0k} = g_{ir}R^r{}_{0k}$ is a symmetric tensor. Then we get

(a) $H_{hijk} = R_{ijk}:_h - 2C_i{}^r{}_h R_{rjk},$

(b) $R_{ijk} = (1/3)\tilde{A}_{(jk)}\{R_{i0k}:_j - 2C_i{}^r{}_j R_{r0k}\}.$

$$\tag{2.1}$$

It is well-known that F^n is called *of scalar curvature* K, if and only if

$$R_{i0k} = L^2 K h_{ik}, \tag{2.2}$$

holds, where $h_{ik} = g_{ik} - \ell_i \ell_k$ is the angular metric tensor.

It follows from (b) of (2.1) that

$$R_{ijk} = h_{ik} K_j - h_{ij} K_k, \quad K_j := L(K\ell_j + LK_{:j}/3). \tag{2.3}$$

$K_j(x, y)$ is $(1)p$-homogeneous in y and $K_0 = L^2 K$. Next it follows from (a) of (2.1) that

$$H_{hijk} = \tilde{A}_{(jk)}\{h_{ik} K_{j:h} + (h_{ih}\ell_j + h_{jh}\ell_i)K_k/L\}. \tag{2.4}$$

These three equations are equivalent to each other.

Since any Finsler surface satisfies (2.2), we are concerned in the following with $n(\geq 3)$-dimensional Finsler spaces of scalar curvature. A Riemannian space of scalar curvature K is necessarily of constant curvature K.

We treat of the stretch curvature tensor Σ of a Finsler space F^n of scalar curvature K. (2.4) gives

$$
\begin{aligned}
H_{hijk} + H_{ihjk} = \tilde{A}_{(jk)}\{ & h_{ik}(K_{j:h} - K_j\ell_h/L) \\
& + h_{hk}(K_{j:i} - K_j\ell_i/L) + 2h_{ih}\ell_j K_k/L\}.
\end{aligned}
$$

Then, putting

$$
\begin{aligned}
M_{ij} &= K_{j:i} - K_j\ell_i/L - Kh_{ij} \\
&= L^2 K_{:i:j}/3 + L(K_{:i}\ell_j + K_{:j}\ell_i/3), \tag{2.5}
\end{aligned}
$$

we get

$$H_{hijk} + H_{ihjk} = \tilde{A}_{(jk)}\{h_{ik} M_{hj} + h_{hk} M_{ij} + h_{hi} M_{kj}\}.$$

On the other hand, (18.16) and (17.22) of [M2] give

$$H_{hijk} + H_{ihjk} = K_{hijk} + K_{ihjk} + 2\tilde{A}_{(jk)}\{P_{hij/k}\},$$

and hence (18.11) of [M2] and (1.3) lead to

$$= -2C_h{}^r{}_i R_{rjk} + \Sigma_{hijk}.$$

Consequently (2.3) yields the form of Σ as

$$\Sigma_{hijk} = \tilde{A}_{(jk)}\{h_{ik} M_{hj} + h_{hk} M_{ij} + h_{hi} M_{kj} - 2C_{hij} K_k\}. \tag{2.6}$$

Proposition 2.1. *The stretch curvature tensor of a Finsler space F^n of scalar curvature K is given by (2.6) where M_{ij} is defined by (2.5).*

Now (1.3) and (2.5) give

$$\Sigma_{hij0} = 2P_{hij/0},$$

$$M_{i0} = (2L^2/3)K_{:i}, \quad M_{0j} = 0.$$

Then (2.6) yields

$$P_{hij/0} + (L^2/3)\$_{(hij)}\{h_{ij}K_{:h}\} + L^2 K C_{hij} = 0,$$

where and in the following the symbol $\$_{(hij)}$ is used to permute indices h, i, j and sum.

Multiplying by g^{hj}, the above gives

$$(n+1)L^2 K_{:i} + 3(L^2 K C_i + P_{i/0}) = 0. \tag{2.7}$$

Consequently the above is rewritten in the form

$$P_{hij/0} + L^2 K C_{hij} = \$_{(hij)}\{h_{hi}(P_{j/0} + L^2 K C_j)\}/(n+1). \tag{2.8}$$

Proposition 2.2. (1) *In a Finsler space of scalar curvature K the equation (2.8) are satisfied.* (2) *If a Finsler space of non-zero scalar curvature K satisfies $P_{hij/0} = 0$, then the space is C-reducible.*

The notion of *C*-reducibility was defined in the paper [M1]: A Finsler space $F^n = (M, L)$, $n \geq 3$, is called *C-reducible*, if the *C*-tensor is of the form

$$(n+1)C_{hij} = h_{hi}C_j + h_{ij}C_h + h_{jh}C_i. \tag{2.9}$$

It was proved by [MH] that the fundamental function $L(x, y)$ of a *C*-reducible space is given only by a quadratic form

$$R(x)L^2 + 2\{R_i(x)y^i\}L + R_{ij}(x)y^i y^j = 0,$$

where three R's are functions of position alone. According as $R(x)$ vanishes or not, the metric L is called *Kropina or Randers*.

For a C-reducible Finsler space the equation (2.9) leads to

$$
\begin{aligned}
&\text{(a)} \quad (n+1)P_{hij} = h_{hi}P_j + h_{ij}P_h + h_{jh}P_i, \\
&\text{(b)} \quad (n+1)P_{hij/0} = h_{hi}P_{j/0} + h_{ij}P_{h/0} + h_{jh}P_{i/0}.
\end{aligned}
\tag{2.10}
$$

Hence we get (2.8).

Let us recall the T-tensor

$$
T_{hijk} = LC_{hij}|_k + \ell_h C_{ijk} + \ell_i C_{jkh} + \ell_j C_{khi} + \ell_k C_{hij},
$$

where $(|)$ is the v-covariant differentiation with respect to the Cartan connection CΓ ([M2], (28.20)).

It is remarked that the T-tensor is completely symmetric because $C_{hij}|_k$ is completely symmetric. For a C-reducible space we shall observe this fact. First, we have from (2.9)

$$
(n+1)C_{hij}|_k = \$_{(hij)}\{h_{hi}C_j|_k - h_{hk}(C_i\ell_j + C_j\ell_i)/L\}.
$$

If we consider the contracted T-tensor

$$
T_{jk} = g^{hi}T_{hijk} = LC_j|_k + C_j\ell_k + C_k\ell_j,
$$

the identity $C_{hij}|_k - C_{hik}|_j = 0$ is written in the form

$$
h_{ij}T_{hk} + h_{jh}T_{ik} - h_{ik}T_{hj} - h_{hk}T_{ij} = 0,
$$

which gives

$$
T_{ij} = Th_{ij}/(n-1), \quad T := g^{ij}T_{ij} = LC^r|_r.
\tag{2.11}
$$

Therefore we have

$$
\begin{aligned}
C_{hij}|_k = {}&\{T/L(n^2-1)\}\$_{(hij)}\{h_{hk}h_{ij}\} \\
&- (\ell_h C_{ijk} + \ell_i C_{hjk} + \ell_j C_{hik} + \ell_k C_{hij})/L,
\end{aligned}
$$

and hence

$$
T_{hijk} = \{T/(n^2-1)\}\$_{(hij)}\{h_{hi}h_{jk}\}.
\tag{2.12}
$$

It is obvious that (2.12) implies (2.11) ([M2], Proposition 30.2).

Proposition 2.3. *The T-tensor of a C-reducible Finsler space is written in the form* (2.12).

3. Shibata's theorem

We consider a Finsler space F^n, $n \geq 3$, of non-zero scalar curvature K and vanishing stretch curvature Σ. For F^n we have $P_{hij/0} = 0$ from (1.3) and hence it is C-reducible from Proposition 2.2. Further (2.7) gives

$$(n+1)K_{:i} + 3KC_i = 0, \tag{3.1}$$

and hence (2.6) leads to

$$
\begin{aligned}
&\text{(a)} \quad \tilde{A}_{(jk)}\{h_{ik}N_{hj} + h_{hk}N_{ij} + h_{hi}N_{kj}\} = 0, \\
&\text{(b)} \quad N_{ij} := M_{ij} + 2C_iK_j/(n+1).
\end{aligned}
\tag{3.2}
$$

Substituting from (2.5) and (3.1), N_{ij} can be written in the form

$$3N_{ij} = L^2K_{:i:j} + L(K_{:i}\ell_j + K_{:j}\ell_i) - (2L^2/3K)K_{:i}K_{:j},$$

which is symmetric and $N_{i0} = 0$. Multiplying by g^{hj}, (3.2) (a) yields

$$N_{ik} = Nh_{ik}/(n-1), \quad N := g^{hj}N_{hj}. \tag{3.3}$$

On the other hand, (3.1) gives

$$K_{:i:j} = -3(C_iK_{:j} + KC_{:j})/(n+1). \tag{3.4}$$

(2.11) is written in the form

$$
\begin{aligned}
LC_{i:j} = {} &\{LC^2/(n+1) + T/(n-1)\}h_{ij} \\
&+ \{2L/(n+1)\}C_iC_j - \ell_iC_j - \ell_jC_i,
\end{aligned}
$$

where $C^2 = g^{ij}C_iC_j$. Consequently (3.4) can be rewritten as

$$
\begin{aligned}
(n+1)K_{:i:j} = {} &-(3K/L)\{LC^2/(n+1) + T/(n-1)\}h_{ij} \\
&+ (3K/(n+1))C_iC_j + (3K/L)(\ell_iC_j + \ell_jC_i).
\end{aligned}
$$

Thus, according to (3.1), etc., (3.3) is written in the form

$$\Lambda h_{ij} = KC_iC_j,$$

with some scalar Λ. If Λ does not vanish, then the rank$(h_{ij}) = n-1$ is less than two, contradict to $n \geq 3$. Thus we must have $\Lambda = 0$, and hence $C_i = 0$, which shows $C_{ijk} = 0$ from the C-reducibility. Therefore the space F^n must be Riemannian and we proved Theorem S.

References

[AIM] P. ANTONELLI, R. S. INGARDEN and M. MATSUMOTO, The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, *Kluwer Acad. Publ.*, *Dordrecht*, 1993.

[B]   L. BERWALD, Über Parallelübertraung in Räumen mit allgemeiner Maßbestimmung, *Jahresber. der Deutschen Math.-Ver.* **34** (1925), 213–220.

[BF]   A. BEJANCU and H.-R. FARRAN, Generalized Landsberg manifolds of scalar curvature, *Bull. Korean Math. Soc.* **37** (2000), 543–550.

[M1]   M. MATSUMOTO, On C-reducible Finsler spaces, *Tensor, N. S.* **24** (1972), 29–37.

[M2]   M. MATSUMOTO, Foundations of Finsler Geometry and Special Finsler Spaces, *Kaiseisha Press*, 1986.

[M3]   M. MATSUMOTO, On the stretch curvature of a Finsler space and certain open problem, *J. Nat. Acad. Math. India, Gorakhpur*, (unpublished).

[MH]   M. MATSUMOTO and HÔJÔ, A conclusive theorem on C-reducible Finsler spaces, *Tensor, N. S.* **32** (1978), 225–230.

[N]   S. NUMATA, On Landsberg spaces of scalar curvature, *J. Korean Math. Soc.* **12** (1975), 97–100.

[S]   C. SHIBATA, On the curvature tensor R_{hijk} of Finsler spaces of scalar curvature, *Tensor, N. S.* **32** (1978), 311–317.

MAKOTO MATSUMOTO
15 ZENBU-CHO, SHIMOGAMO
SAKYO-KU
KYOTO, 606-0815
JAPAN