Publ. Math. Debrecen
63/3 (2003), 495–510

A common characterization of euclidean and hyperbolic geometry by functional equations

By WALTER BENZ (Hamburg)

Dedicated to Professor Lajos Tamássy on the occasion of his 80th birthday, in friendship

Abstract. Non-standard translations are defined, which generalize euclidean as well as hyperbolic translations. If there exist appropriate distance functions invariant under such translations, Euclidean and Hyperbolic Geometry over arbitrary real inner product spaces of (finite or infinite) dimension ≥ 2 are characterized. The methods are based on the solution of special real Functional Equations.

1. Introduction

Let X be a *real inner product space* of (finite or infinite) dimension ≥ 2, $O(X)$ be its *orthogonal group* of all surjective linear orthogonal mappings $\omega : X \to X$, and e be a fixed element of X satisfying $e^2 = 1$. Observe $X = H \oplus \mathbb{R}e$ with $H := e^\perp$. If $x \in X$, there hence exist uniquely determined $h \in H$ and $x_0 \in \mathbb{R}$ with $x = h + x_0 e$. Suppose that φ is a monotonically increasing bijection of $\mathbb{R}$ with $\varphi(0) = 0$ and that ψ is a function from H

Mathematics Subject Classification: 39B22, 39B72, 51M05, 51M10.
Key words and phrases: translation equation, other functional equations, translation groups, separable translation groups, euclidean and hyperbolic geometry over real inner product spaces.

into the set $\mathbb{R}_{>0}$ of positive reals. There hence is exactly one $\tau \in \mathbb{R}$ with

$$x = h + x_0 e = h + \varphi(\tau)\psi(h)e.$$

For $t \in \mathbb{R}$ define the (*non-standard*) *translation*

$$T_t\big(h + \varphi(\tau)\psi(h)e\big) := h + \varphi(\tau + t)\psi(h)e, \tag{1}$$

which, obviously, is a bijection of X. Mainly the following result will be proved in this note.

Theorem 1. *Suppose that there exists a function*

$$d : X \times X \to \mathbb{R}_{\geq 0} := \mathbb{R}_{>0} \cup \{0\},$$

not identically 0, *satisfying*

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = d\big(\omega(x), \omega(y)\big)$,

(iii) $d(x, y) = d\big(T_t(x), T_t(y)\big)$,

(iv) $d(0, \beta e) = d\big(0, \alpha e) + d(\alpha e, \beta e)$

for all $x, y \in X$, $\omega \in O(X)$, $t, \alpha, \beta \in \mathbb{R}$ *with* $0 \leq \alpha \leq \beta$. *Then, up to isomorphism,* $\varphi(t) = t$, $\psi(h) = 1$ *and*

$$d(x, y) = \sqrt{(x - y)^2},$$

or $\varphi(t) = \sinh t$, $\psi(h) = \sqrt{1 + h^2}$ *and*

$$\cosh d(x, y) = \sqrt{1 + x^2}\sqrt{1 + y^2} - xy$$

hold true for all $x, y \in X$, $h \in H$, *and* $t \in \mathbb{R}$. *Hence,* (X, d) *is the Euclidean Metric Space with classical translations* (1), *or* (X, d) *is the Hyperbolic Metric Space in the form of the Weierstrass model with hyperbolic translations* (1).

Obviously, (i), (ii), (iii), (iv) are functional equations for the function d. Moreover, the proof of our theorem depends heavily on the solution of special real functional equations, which will be reduced to known ones.

2. Separable translation groups

Suppose that X is a (left) vector space over a (commutative or non-commutative) field F of (finite or infinite) dimension at least 2. Let

$$T : F \to \operatorname{Perm} X$$

be a mapping of F into the group of all permutations of X, and let $e \neq 0$ be a fixed element of X.

The mapping T is called a *translation group* ([3, p. 300]), of X with *axis* (or *direction*) e if and only if the following properties hold true.

(a) $T_{t+s} = T_t \cdot T_s$ *for all* $t, s \in F$,

(b) *For* $x, y \in X$ *satisfying* $y - x \in Fe$ *there exists exactly one* $t \in F$ *with* $T_t(x) = y$,

(c) $T_t(x) - x \in Fe$ *for all* $x \in X$ *and all* $t \in F$.

Here T_t designates the image of $t \in F$ under T, and $T_t(x)$ the image of $x \in X$ under the permutation T_t of X. Property (a) is the socalled *translation equation* (J. Aczél [1, pp. 245–253], Z. Moszner, J. Tabor [7]).

$\{T_t \mid t \in F\}$ is an abelian group under the multiplication in $\operatorname{Perm} X$, isomorphic to the additive group of the field F ([3, p. 304]).

In [3] we proved the following

Theorem 2. *Let* $e \neq 0$ *be an element of* X *and* H *be a maximal subspace of* X *with*

$$H \oplus Fe = X$$

and let $\varrho : H \times F \to F$ *satisfy*

(α) *For all* $h \in H$ *and* $\xi \in F$ *there exists exactly one* $t = t(h, \xi)$ *in* F *with* $\varrho(h, t) = \xi$.

Then for all $h \in H$ *and all* $t, \tau \in F$

$$T_t\big(h + \varrho(h, \tau)e\big) := h + \varrho(h, \tau + t)e \tag{2}$$

defines a translation group of X *with axis* e. *There are no other translation groups of* X *with direction* e.

In addition to (α) it is possible to assume ([3, p. 304]) $\varrho(h,0) = 0$ for all $h \in H$, without loss of generality. So, in fact, we will add $\varrho(h,0) = 0$ to (α). The function ϱ will be called the *kernel* of the translation group T.

From now on let X be a real inner product space of (finite or infinite) dimension at least 2. We will call a translation group

$$T : \mathbb{R} \to \operatorname{Perm} X$$

with kernel ϱ and axis $e, e^2 = 1$, *separable* if and only if the following property holds true, where $H = e^{\perp} := \{h \in X \mid he = 0\}$.

(β) *$\varrho(h,\xi) = \varphi(\xi)\psi(h)$ for all $\xi \in \mathbb{R}$ and $h \in H$ with functions $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : H \to \mathbb{R}_{>0}$ satisfying $\varphi(0) = 0$ and $\varphi(t_1) \leq \varphi(t_2)$ for all reals $t_1 \leq t_2$.*

Property (α) implies, by (β), that φ is a monotonically increasing bijection of $\mathbb{R}$. Without loss of generality we may assume $\psi(0) = 1$, because otherwise we would work with

$$\varphi_1(\xi) = \varphi(\xi)\psi(0) \quad \text{and} \quad \psi_1(h) = \frac{\psi(h)}{\psi(0)}.$$

Important examples of separable translation groups are given by

$$\varphi(\xi) = \xi, \qquad \psi(h) = 1 \text{ (euclidean case)},$$

$$\varphi(\xi) = \sinh \xi, \qquad \psi(h) = \sqrt{1 + h^2} \text{ (hyperbolic case, [3, p. 305])}$$

for all $\xi \in \mathbb{R}$ and $h \in H$.

3. Consequences for the distance function *d*

For the sections to come let T be a separable translation group of X with kernel $\varrho = \varphi \cdot \psi$, $\psi(0) = 1$, and axis e, and let

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

be a function, not identically 0, satisfying properties (i), (ii), (iii), (iv) of Theorem 1.

In view of property (ii) and of $d(x,y) \geq 0$ for all $x,y \in X$ we obtain, by Theorem 2 ([4, p. 20]), that there exists a function $f : K \to \mathbb{R}_{\geq 0}$ with

$$K := \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1, \xi_2 \in \mathbb{R}_{\geq 0} \quad \text{and } \xi_3^2 \leq \xi_1 \xi_2\}$$

and

$$d(x,y) = f(x^2, y^2, xy) \tag{3}$$

for all $x, y \in X$.

Given reals $0 \leq \alpha \leq \beta$ we get $0 \leq \varphi(\alpha) \leq \varphi(\beta)$, by $\varphi(0) = 0$ and $\varphi(t_1) \leq \varphi(t_2)$ for $t_1 \leq t_2$. Hence, in view of (iv) with $\varphi(\beta)$ instead of β and $\varphi(\alpha)$ instead of α,

$$d\big(0, \varphi(\beta)e\big) = d\big(0, \varphi(\alpha)e\big) + d\big(\varphi(\alpha)e, \varphi(\beta)e\big). \tag{4}$$

Observe, by (1),

$$T_{-\alpha}\big(0 + \varphi(\beta)\psi(0)e\big) = 0 + \varphi(\beta - \alpha)\psi(0)e,$$

i.e. $T_{-\alpha}\big(\varphi(\beta)e\big) = \varphi(\beta - \alpha)e$, i.e., by (iii),

$$d\big(\varphi(\alpha)e, \varphi(\beta)e\big) = d\big(0, \varphi(\beta - \alpha)e\big).$$

Hence, in view of (3), (4),

$$f\big(0, \varphi^2(\beta), 0\big) = f\big(0, \varphi^2(\alpha), 0\big) + f\big(0, \varphi^2(\beta - \alpha), 0\big),$$

which implies, for reals $0 \leq \xi \leq \eta$ with $\alpha := \xi$, $\beta := \xi + \eta$,

$$f\big(0, \varphi^2(\xi + \eta), 0\big) = f\big(0, \varphi^2(\xi), 0\big) + f\big(0, \varphi^2(\eta), 0\big).$$

There hence exists a real constant $k \geq 0$ with

$$f\big(0, \varphi^2(\xi), 0\big) = k \cdot \xi \tag{5}$$

for all $\xi \geq 0$.

Lemma 1. *If $a \neq b$ are elements of X, there exist $\omega_1, \omega_2 \in O(X)$ and $\lambda, t \in \mathbb{R}$ with $\lambda > 0$ and*

$$\omega_1 T_t \omega_2(a) = 0, \ \omega_1 T_t \omega_2(b) = \lambda e.$$

Proof. There exists $\omega_2 \in O(X)$ with $\omega_2(a) = \mu e$ for a suitable $\mu \in \mathbb{R}$ (see step 1 of the proof of Theorem 1 in [4, p. 19]). Now take, by (b) of Section 2, $t \in \mathbb{R}$ with $T_t(\mu e) = 0$. Hence $T_t \omega_2(a) = 0$. Finally take $\omega_1 \in O(X)$ with $\omega_1(c) = \lambda e$, $\lambda \geq 0$, where $c := T_t \omega_2(b)$. Hence $g(a) = 0$ and $g(b) = \lambda e$ with $g := \omega_1 T_t \omega_2$. Since $a \neq b$, we obtain $\lambda > 0$. □

Since the function d is not identically 0, there exist $p, q \in X$ with $d(p,q) > 0$. If $p = q$, take, on account of the proof of Lemma 1, $T_t\omega_2$ with $T_t\omega_2(p) = 0$. Hence, by (ii), (iii),

$$0 < d(p,q) = d\big(T_t\omega_2(p), T_t\omega_2(p)\big) = d(0,0),$$

contradicting (iv), here $\alpha = \beta = 0$. Hence $p \neq q$. In view of Lemma 1 and (ii), (iii), we obtain

$$0 < d(p,q) = d(0, \lambda e)$$

with a real $\lambda > 0$. Define $\xi \in \mathbb{R}$ by $\lambda = \varphi(\xi)$. Since $0 < \lambda$ implies $0 = \varphi(0) < \varphi(\xi)$, we get $0 < \xi$. Now (3), (5) lead to

$$0 < d\big(0, \varphi(\xi)e\big) = f\big(0, \varphi^2(\xi), 0\big) = k \cdot \xi.$$

The constant $k \geq 0$ in (5) must hence be positive.

A consequence of our considerations before is

Proposition 1. *$d(x,y) = 0$ if and only if $x = y$ holds true for all $x, y \in X$.*

Proof. We will prove that $d(x,y) > 0$ is equivalent with $x \neq y$. But we already realized that $d(p,p) > 0$ leads to a contradiction. If $x \neq y$, we already got, for a real $\xi > 0$,

$$d(x,y) = d\big(0, \varphi(\xi)e\big) > 0.$$

Assume $t < 0$ and define $\varphi(\tau) := -\varphi(t)$. Hence $\tau > 0$. Observe, by (i),

$$d\big(0, \varphi(-t)e\big) = d\Big(T_t(0), T_t\big(\varphi(-t)e\big)\Big) = d\big(\varphi(t)e, 0\big) = d\big(0, \varphi(t)e\big),$$

i.e. $f\big(0, \varphi^2(-t), 0\big) = f\big(0, \varphi^2(t), 0\big) = f\big(0, \varphi^2(\tau), 0\big)$. Hence, by (5),

$$k \cdot (-t) = k \cdot \tau,$$

i.e. $\varphi(-t) = -\varphi(t)$ for all $t < 0$. Thus

$$\varphi(-t) = -\varphi(t) \text{ for all reals } t. \tag{6}$$

This implies, by (5),

$$f\big(0, \varphi^2(\xi), 0\big) = k \cdot |\xi| \text{ for all reals } \xi. \tag{7}$$

For $t \geq 0$ put $t = \varphi(\xi)$. Hence $\xi \geq 0$. Now (7) implies

$$\varphi\left(\frac{f\big(0, \varphi^2(\xi), 0\big)}{k}\right) = \varphi(\xi),$$

i.e.

$$\varphi\left(\frac{f(0, t^2, 0)}{k}\right) = t \quad \text{for all } t \geq 0. \tag{8}$$

Hence

$$\varphi\left(\frac{d(0, y)}{k}\right) = \sqrt{y^2} =: \|y\| \quad \text{for all } y \in X. \tag{9}$$

Let now x, y be elements of X with $x \neq 0$, and let $j \in H$ be given with $j^2 = 1$. Hence, by (3) and $j \in e^\perp$,

$$d(x, y) = f(x^2, y^2, xy) = d\left(\|x\|e, \frac{1}{\|x\|}\left[(xy)e + \sqrt{x^2y^2 - (xy)^2}j\right]\right). \tag{10}$$

Observe $(xy)^2 \leq x^2y^2$ because of the inequality of Cauchy–Schwarz. Put

$$\|x\| =: \varphi(\xi). \tag{11}$$

Hence $\xi > 0$ because of $x \neq 0$, and, moreover, $T_{-\xi}(\|x\|e) = 0$. Define $\lambda x^2 := x^2y^2 - (xy)^2$, and $\eta \in \mathbb{R}$ by

$$\frac{xy}{\|x\|} = \varphi(\eta) \cdot \psi\big(\sqrt{\lambda}j\big). \tag{12}$$

Hence $T_{-\xi}\big(\sqrt{\lambda}j + \varphi(\eta)\psi(\sqrt{\lambda}j)e\big) = \sqrt{\lambda}j + \varphi(\eta - \xi)\psi(\sqrt{\lambda}j)e$.

Applying (iii), we obtain, by transforming the elements of X of the right hand side of (10) under $T_{-\xi}$,

$$d(x, y) = d\big(0, \sqrt{\lambda}j + \varphi(\eta - \xi)\psi(\sqrt{\lambda}j)e\big).$$

Hence, by (9),

$$\varphi^2\left(\frac{d(x,y)}{k}\right) = \lambda + \varphi^2(\eta-\xi)\psi^2(\sqrt{\lambda}j).$$

Applying this formula for $xy = 0$, we get, with $\eta = 0$ from (12), and $\lambda = y^2$,

$$\psi^2(\|y\|j) = \left(\varphi^2\left(\frac{d(x,y)}{k}\right) - y^2\right)\cdot\frac{1}{x^2},$$

by observing $\varphi^2(-\xi) = \varphi^2(\xi) = x^2$, in view of (6). The right-hand side of this equation does not depend on the chosen j of H satisfying $j^2 = 1$. So we get

$$\psi^2(\alpha j) = \psi^2(\alpha j')$$

for all real $\alpha \geq 0$ and all $j, j' \in H$ with $j^2 = 1 = j'^2$. Since ψ-values are positive, we hence get

$$\psi(h) = \psi(h')$$

for all $h, h' \in H$ with $h^2 = (h')^2$. So we may define

$$\psi_0(\eta) := \psi(\sqrt{\eta}j)$$

for $\eta \geq 0$, where $j \in H$ is chosen arbitrarily with $j^2 = 1$. Hence we obtain

$$\varphi^2\left(\frac{d(x,y)}{k}\right) = \lambda + \varphi^2(\eta-\xi)\psi_0^2(\lambda) \tag{13}$$

for all $x, y \in X$ with $x \neq 0$ and $\lambda x^2 = x^2y^2 - (xy)^2$.

Take an arbitrary element $h \neq 0$ of H. Because of (i) we get $d(e,h) = d(h,e)$. Hence, by (13), we obtain

$$h^2 + \varphi^2(\eta-\xi)\psi_0^2(h^2) = 1 + \varphi^2(\eta'-\xi')\psi_0^2(1) \tag{14}$$

with (compare (11) and (12))

$$1 = \varphi(\xi),\ \|h\| = \varphi(\xi'),\ 0 = \varphi(\eta)\psi_0(h^2), 0 = \varphi(\eta')\psi_0(1),$$

i.e. $\eta = 0 = \eta'$. Thus, by (14), (6),

$$h^2 + \psi_0^2(h^2) = 1 + h^2\psi_0^2(1),$$

i.e. $\psi_0^2(h^2) = 1 + h^2\big(\psi_0^2(1) - 1\big)$. If $\psi_0^2(1)$ were < 1, then for sufficiently large h^2, $\psi_0^2(h^2)$ would become negative. So we get with $\psi_0(h^2) \geq 1$ for all $h \in H$

$$\psi_0(h^2) = \sqrt{1 + \delta h^2} \quad \text{with } \delta := \psi_0^2(1) - 1 \geq 0. \tag{15}$$

From (15) we get with $\psi(h) = \psi_0(h^2)$ for $h \in H$,

$$\psi(h) = \sqrt{1 + \delta h^2} \tag{16}$$

with a constant $\delta \geq 0$. □

4. Motions, the triangle inequality, other directions

A surjective mapping $f : X \to X$ is called a *distance preserving transformation* or a *motion* of X (or of (X, d)) if and only if

$$d(x, y) = d\big(f(x), f(y)\big)$$

holds true for all $x, y \in X$. Distance preserving transformations must also be injective on account of Proposition 1. We hence will speak of the group $\mathcal{G}$ of distance preserving transformations of X, or also of the group of *motions* of X.

Proposition 2. *The group of motions of* (X, d) *is given by*

$$\mathcal{G} = \{\alpha T_t \beta \mid \alpha, \beta \in O(X),\ t \in \mathbb{R}\}.$$

Proof. By (ii), (iii), $\alpha T_t \beta$ must be a motion. Let now γ be an arbitrary motion and put $\gamma(0) =: a$. Then there exists $\omega \in O(X)$ with $\omega(a) = \|a\|e$. According to property (b) of a separable translation group there exists $s \in \mathbb{R}$ with

$$T_s\big(\omega(a)\big) = T_s(\|a\|e) = 0.$$

Hence $\lambda(0) = 0$ with $\lambda := T_s \omega \gamma$. Since, in view of (9),

$$z^2 = \varphi^2\left(\frac{d(0, z)}{k}\right) = \varphi^2\left(\frac{d\big(0, \lambda(z)\big)}{k}\right) = \big(\lambda(z)\big)^2$$

for all $z \in X$, we get $vw = \lambda(v)\lambda(w)$ for all $v, w \in X$. This implies $\lambda \in O(X)$, i.e. $\gamma = \omega^{-1} T_{-s} \lambda \in \mathcal{G}$. □

From Proposition 2 follows

Proposition 3. *The stabilizer of $\mathcal{G}$ in 0 is $O(X)$.*

Proof. If $\alpha T_t \beta(0) = 0$, then $\alpha^{-1}(0) = T_t \beta(0)$, i.e. $0 = T_t(0)$, i.e. $t = 0$. □

Proposition 4. *For all $x, y, z \in X$,*

$$d(x,y) \leq d(x,z) + d(z,y) \tag{17}$$

holds true.

Proof. Instead of (17) we prove

$$d(0, \lambda e) \leq d(0,p) + d(\lambda e, p) \tag{18}$$

for all real $\lambda > 0$ and all $p \in X$: if x, y, z are arbitrary elements of X, then, excluding the trivial case $x = y$, we take, by Lemma 1, $g \in \mathcal{G}$ with $g(x) = 0$ and $g(y) = \lambda e$, $\lambda > 0$, and we obtain from (25) for $p := g(z)$,

$$d\big(g^{-1}(0), g^{-1}(\lambda e)\big) \leq d\big(g^{-1}(0), g^{-1}(p)\big) + d\big(g^{-1}(\lambda e), g^{-1}(p)\big),$$

i.e. (17), by (i), (ii), (iii). Put

$$p := h + \varphi(\tau)\sqrt{1 + \delta h^2}e, \ h \in H.$$

Then, by (9), (11)–(13),

$$\varphi\left(\frac{d(0,\lambda e)}{k}\right) = \lambda, \varphi\left(\frac{d(0,p)}{k}\right) = \sqrt{h^2 + \varphi^2(\tau)(1 + \delta h^2)}$$

and

$$\varphi\left(\frac{d(\lambda e, p)}{k}\right) = h^2 + \varphi^2(\eta - \xi)(1 + \delta h^2)$$

with $\varphi(\xi) = \lambda$ and $\eta = \tau$. For $\alpha \in \mathbb{R}$ put

$$\varphi\big(F(\alpha)\big) := \sqrt{h^2 + \varphi^2(\alpha)(1 + \delta h^2)},$$

and observe $F(-\alpha) = F(\alpha)$, moreover, for $0 \leq \alpha < \beta$,

$$0 \leq F(\alpha) < F(\beta). \tag{19}$$

Obviously, $\varphi(\alpha) \leq \sqrt{h^2 + \varphi^2(\alpha)(1 + \delta h^2)}$, i.e. $\alpha \leq F(\alpha)$. In order to prove (18), we must show

$$\varphi^{-1}(\lambda) \leq F(\tau) + F\big(\varphi^{-1}(\lambda) - \tau\big), \tag{20}$$

by noticing $F(\eta - \xi) = F(\xi - \eta) = F\big(\varphi^{-1}(\lambda) - \tau\big)$.

Case 1. $0 \leq \varphi(\tau) \leq \lambda$.
Hence $0 \leq \tau \leq \varphi^{-1}(\lambda)$, and thus

$$\varphi^{-1}(\lambda) = \tau + \big(\varphi^{-1}(\lambda) - \tau\big) \leq F(\tau) + F\big(\varphi^{-1}(\lambda) - \tau\big),$$

i.e. (20).

Case 2. $\varphi(\tau) < 0$.
Hence $\tau < 0$, i.e. $\varphi^{-1}(\lambda) - \tau > \varphi^{-1}(\lambda)$, and thus, by (19),

$$\varphi^{-1}(\lambda) \leq F\big(\varphi^{-1}(\lambda)\big) < F\big(\varphi^{-1}(\lambda) - \tau\big),$$

i.e. (20).

Case 3. $\lambda < \varphi(\tau)$.
Because of $\varphi^{-1}(\lambda) < \tau \leq F(\tau)$, we obtain (20). □

Now (i) and Propositions 1, 4 imply that (X, d) is a metric space.

If $v \in X$ satisfies $v^2 = 1$, there exists $\omega' \in O(X)$ with $\omega'(v) = \mu e$ for a suitable $\mu \in \mathbb{R}$. Hence $v \cdot v = \omega'(v)\omega'(v)$ implies $\mu^2 = 1$. For $\mu = 1$ put $\omega := \omega'$, otherwise $\omega := -\omega'$. So $\omega(v) = e$ with $\omega \in O(X)$. Observe $v^\perp = \omega(H)$. If

$$x = h + \varphi(\tau)\sqrt{1 + \delta h^2} e,$$

we get $\omega(x) = \omega(h) + \varphi(\tau)\sqrt{1 + \delta[\omega(h)]^2}\omega(e)$, and thus

$$\omega T_t \omega^{-1}\big(\omega(x)\big) = \omega(h) + \varphi(\tau + t)\sqrt{1 + \delta[\omega(h)]^2}\omega(e).$$

So $\{\omega T_t \omega^{-1} \mid t \in \mathbb{R}\}$ is a separable translation group in the direction of v with kernel

$$\varrho(h', \xi) = \varphi(\xi)\sqrt{1 + \delta(h')^2}$$

for $\xi \in \mathbb{R}$ and $h' \in v^\perp$. The arbitrary motion $\alpha T_t \beta$ can be written as

$$\alpha T_t \alpha^{-1}(\alpha\beta) = (\alpha\beta) \cdot \beta^{-1} T_t \beta,$$

where $\alpha T_t \alpha^{-1}, \beta^{-1} T_t \beta$ are translations with axis $\alpha^{-1}(e)$, $\beta(e)$, respectively, and where $\alpha\beta$ is in $O(X)$.

5. A functional equation for φ

If $0 \neq h \in H$ and $t \in \mathbb{R}$, then, by (i), (iii),

$$d(0,h) = d(h,0) = d\big(T_t(h), T_t(0)\big)$$

holds true, i.e., by (9), (11)–(13), (16), we get the functional equation for φ,

$$h^2 = \frac{h^2\varphi^2(t)}{\varphi^2(\xi)} + \varphi^2(\eta-\xi)\left(1 + \frac{\delta h^2\varphi^2(t)}{\varphi^2(\xi)}\right), \tag{21}$$

where $\xi > 0$ and η are given by

$$\varphi^2(\xi) = h^2 + \varphi^2(t)(1+\delta h^2), \tag{22}$$

$$\varphi^2(t)\sqrt{1+\delta h^2} = \varphi(\xi)\varphi(\eta)\sqrt{1 + \frac{\delta h^2\varphi^2(t)}{\varphi^2(\xi)}}. \tag{23}$$

Take a fixed $j \in H$ with $j^2 = 1$, and take an arbitrary real number $\mu > 0$. Put $h = \sqrt{\mu}j$. Then $0 \neq h \in H$. Defining $\xi > 0$ by, see (22),

$$\varphi^2(\xi) = \mu + \varphi^2(t)(1+\delta\mu), \tag{24}$$

and η by, see (23),

$$\varphi^2(t)\sqrt{1+\delta\mu} = \varphi(\eta)\sqrt{\varphi^2(\xi) + \delta\mu\varphi^2(t)}, \tag{25}$$

we obtain, see (21),

$$\mu \cdot \big(\varphi^2(\xi) - \varphi^2(t)\big) = \varphi^2(\xi-\eta)\big(\varphi^2(\xi) + \delta\mu\varphi^2(t)\big), \tag{26}$$

by observing (6).

We pose the following question: Given arbitrarily real numbers $\xi > \eta \geq 0$, is it possible to find real numbers $\mu > 0$ and t such that (24), (25) hold true? The answer is yes. $\xi > \eta \geq 0$ implies $\varphi(\xi) > \varphi(\eta) \geq 0$, i.e.

$$\mu := \varphi^2(\xi) - \varphi(\xi)\varphi(\eta)\sqrt{\frac{1+\delta\varphi^2(\xi)}{1+\delta\varphi^2(\eta)}} > 0. \tag{27}$$

(27) yields $\varphi^2(\xi) - \mu \geq 0$. Take a $t \in \mathbb{R}$ with

$$\varphi^2(t) := \frac{\varphi^2(\xi) - \mu}{1 + \delta\mu}. \tag{28}$$

Obviously, (28) implies (24), and (27),

$$\mu^2 - 2\varphi^2(\xi)\mu = \varphi^2(\xi)\frac{\varphi^2(\eta) - \varphi^2(\xi)}{1 + \delta\varphi^2(\eta)}, \tag{29}$$

i.e. (25), if we square both sides of (25) by observing $\varphi(\eta') \geq 0$, and replace there $\varphi^2(t)$ by (28).

Hence (26) holds true for arbitrarily given $\xi > \eta \geq 0$, if we define μ by (27), and $\varphi^2(t)$ by (28). Replacing now these values in (26), we obtain with $\alpha := \varphi^2(\xi),\ \beta := \varphi^2(\eta)$,

$$\mu\left(\alpha - \frac{\alpha - \mu}{1 + \delta\mu}\right) = \varphi^2(\xi - \eta)\left(\alpha + \delta\mu \cdot \frac{\alpha - \mu}{1 + \delta\mu}\right),$$

i.e. $\mu^2(1 + \delta\alpha) = \varphi^2(\xi - \eta)(\alpha + \delta[2\alpha\mu - \mu^2])$, i.e. by (29), $\mu^2(1 + \delta\beta) = \alpha\varphi^2(\xi - \eta)$, i.e., by (27),

$$\left(\sqrt{\alpha(1 + \delta\beta)} - \sqrt{\beta(1 + \delta\alpha)}\,\right)^2 = \varphi^2(\xi - \eta). \tag{30}$$

Since $\varphi(\xi) > \varphi(\eta) \geq 0$, we get $\alpha > \beta$, i.e. $\alpha(1 + \delta\beta) > \beta(1 + \delta\alpha)$. Hence, by (30) and $\xi > \eta \geq 0$,

$$\varphi(\xi - \eta) = \varphi(\xi)\sqrt{1 + \delta\varphi^2(\eta)} - \varphi(\eta)\sqrt{1 + \delta\varphi^2(\xi)} \tag{31}$$

holds true for all $\xi > \eta \geq 0$. Since $\varphi(0) = 0$, (31) also holds true for $\xi = \eta \geq 0$.

By observing $\delta \geq 0$ (see (15)), we will distinguish two cases, namely $\delta = 0$, $\delta > 0$, respectively.

For $\delta = 0$ we get the well-known functional equation
(Aczél–Dhombres [2])

$$\varphi(\xi - \eta) = \varphi(\xi) - \varphi(\eta) \tag{32}$$

for all $\xi \geq \eta \geq 0$, from (31). Given real numbers $t \geq 0$ and $s \geq 0$, put $\eta := s$ and $\xi := t + s$. Hence

$$\varphi(t + s) = \varphi(t) + \varphi(s).$$

Since $\varphi(t) \geq 0$ for $t \geq 0$, there exists a constant $l \geq 0$ with $\varphi(t) = lt$. Because of $1 > 0$, i.e. of $\varphi(1) > 0$ we get $l > 0$. Hence, in view of (6), we obtain for all $t \in \mathbb{R}$,

$$\varphi(t) = lt. \tag{33}$$

For $\delta > 0$ write $f(t) := \sqrt{\delta} \cdot \varphi(t)$ for $t \geq 0$, and (31) leads to the well-known functional equation

$$f(\xi - \eta) = f(\xi)\sqrt{1 + f^2(\eta)} - f(\eta)\sqrt{1 + f^2(\xi)} \tag{34}$$

(Aczél–Dhombres [2], in the form of two unknown functions; see in this context also Z. Daróczy [6]). Since φ is a monotonically increasing bijection of $\mathbb{R}$, satisfying (6), f must be a monotonically increasing bijection of $\mathbb{R}_{\geq 0}$. So put

$$f(\xi) =: \sinh g(\xi), \quad \xi \geq 0,$$

and g must be a monotonically increasing bijection of $\mathbb{R}_{\geq 0}$ as well. (34) implies

$$\sinh g(\xi - \eta) = \sinh\big(g(\xi) - g(\eta)\big)$$

for all $\xi \geq \eta \geq 0$. Hence $g(\xi - \eta) = g(\xi) - g(\eta)$ and we get again

$$g(\xi) = l\xi$$

for all $\xi \geq 0$ with a constant $l > 0$. Thus

$$\varphi(t) = \frac{1}{\sqrt{\delta}} \sinh(lt) \tag{35}$$

for all $t \geq 0$. This implies, in view of (6), that (35) holds true for all $t \in \mathbb{R}$ with a constant $l > 0$.

6. The case $\delta = 0$ as euclidean geometry

With (33) we obtain, by (9), (13), (11), (12), for all $x, y \in X$,

$$d(x, y) = \operatorname{eucl}\big(\sigma(x), \sigma(y)\big) \tag{36}$$

where we put $\sigma(x) := \frac{k}{l} \cdot x$ for $x \in X$ and

$$\text{eucl}(x, y) := \sqrt{(x - y)^2}.$$

A geometry in the sense of Felix Klein is a set $S \neq \emptyset$ together with a subgroup $\mathcal{G}$ of the group of permutations of S. Two geometries $(S, \mathcal{G}), (S', \mathcal{G}')$ are called isomorphic if, and only if, there exist bijections

$$\sigma : S \to S' \quad \text{and} \quad \tau : \mathcal{G} \to \mathcal{G}'$$

with $\tau(g_1 g_2) = \tau(g_1)\tau(g_2)$ and $\sigma\big(g(s)\big) = \tau(g)\sigma(s)$ for all $x \in S$ and $g_1, g_2, g \in \mathcal{G}$ (see [5]).

Define $S = X = S'$ and let $\mathcal{G}$, $\mathcal{G}'$ be the group of motions of (X, d), (X, eucl), respectively. If $g \in \mathcal{G}$, then (36) implies

$$\text{eucl}\Big(\sigma g \sigma^{-1}\big(\sigma(x)\big), \sigma g \sigma^{-1}\big(\sigma((y))\big)\Big) = \text{eucl}\ \big(\sigma(x), \sigma(y)\big),$$

in view of

$$d\big(g(x), \varrho(y)\big) = d(x, y).$$

Hence $\tau(g) := \sigma g \sigma^{-1}$ leads to an isomorphism $\tau : \mathcal{G} \to \mathcal{G}'$. Moreover, $\sigma : X \to X$ is a bijection satisfying for $x \in X$ and $g \in \mathcal{G}$,

$$\tau(g)\sigma(x) = \sigma g \sigma^{-1}\big(\sigma(x)\big) = \sigma g(x).$$

Hence $(X, \mathcal{G}) \simeq (X, \mathcal{G}')$.

7. The case $\delta > 0$ as hyperbolic geometry

With (35) we obtain, by (9), (13), (11), (12), for all $x, y \in X$,

$$d(x, y) = \frac{k}{l}\text{hyp}(\sqrt{\delta}x, \sqrt{\delta}y), \tag{37}$$

where we put $\text{hyp}(x, y) \geq 0$ and

$$\cosh \text{hyp}(x, y) := \sqrt{1 + x^2}\sqrt{1 + y^2} - xy$$

(see [4]). Define $\sigma(x) := \sqrt{\delta} \cdot x$, $S = X = S'$, and let $\mathcal{G}$, $\mathcal{G}'$ be the group of motions of (X, d), (X, hyp), respectively. If $g \in \mathcal{G}$, then (37) implies

$$d(x, y) = d\big(g(x), g(y)\big) = \frac{k}{l} \mathrm{hyp}\big(\sigma g(x), \sigma g(y)\big),$$

i.e. $\mathrm{hyp}\big(\sigma(x), \sigma(y)\big) = \mathrm{hyp}\Big(\sigma g \sigma^{-1}\big(\sigma(x)\big), \sigma g \sigma^{-1}\big(\sigma(y)\big)\Big)$. Hence $\sigma g \sigma^{-1} \in \mathcal{G}'$. If $g' \in \mathcal{G}'$, a similar argument leads to $\sigma^{-1} g' \sigma \in \mathcal{G}$. We thus get an isomorphism $\tau : \mathcal{G} \to \mathcal{G}'$ with $\tau(g) = \sigma g \sigma^{-1}$. We also observe that the bijection $\sigma : X \to X$ satisfies

$$\tau(g)\sigma(x) = \sigma g \sigma^{-1}\big(\sigma(x)\big) = \sigma g(x).$$

Hence $(X, \mathcal{G}) \simeq (X, \mathcal{G}')$.

References

[1] J. Aczél, Lectures on functional equations and their applications, *Academic Press, New York, London*, 1966.

[2] J. Aczél and J. Dhombres, Functional Equations in Several Variables, *Cambridge University Press, Cambridge, New York, Sydney*, 1989.

[3] W. Benz, Translation equation and some new geometries, *Publ. Math. Debrecen* **52** (1998), 299–308.

[4] W. Benz, Hyperbolic distances in Hilbert spaces, *Aequationes Math.* **58** (1999), 16–30.

[5] W. Benz, Real Geometries, *BI–Wissenschaftsverlag, Leipzig, Wien, Zürich*, 1994.

[6] Z. Daróczy, Elementare Lösung einer mehrere unbekannte Funktionen enthaltenden Funktionalgleichung, *Publ. Math. Debrecen* **8** (1961), 160–168.

[7] Z. Moszner and J. Tabor, L'équation de translation sur une structure avec zéro, *Ann. Polon. Math.* **31** (1975), 255–264.

WALTER BENZ
UNIVERSITY OF HAMBURG
DEPARTMENT OF MATHEMATICS
BUNDESSTR. 55
D-20146 HAMBURG
GERMANY

(Received February 17, 2003)