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An explicit Boolean-valued model for non-standard arithmetic

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. We give an explicit construction of a Boolean-valued model for a certain version of the non standard arithmetic. The corresponding Boolean algebra is complete. The constructive character of our exposition enables us to give a constructive, algorithmic proof of the conservativity of the non-standard arithmetic over the usual Peano arithmetic.

1. Let L be an arithmetical language with the equality $=$, the constant 0 (zero) and functional symbols S (successor, unit addition), P (predecessor), $+$, $\cdot$ and (maybe) with some other additional functional symbols for arithmetical functions.

Let Ar be a formal axiomatic theory in L, an extension of Robinson's well-known induction-free arithmetic. Among the nonlogical axioms of Ar there are (the universal closures) of the following formulas:

(i) reflexivity, symmetry and transitivity of equality;

(ii) substitutional property with respect to functional symbols:

$$x = y \supset f(\dots x \dots) = f(\dots y \dots);$$

(iii) defining properties of distinguished functional symbols:

1) $Sx \neq 0$;	5) $x \cdot 0 = 0$;
2) $Sx = Sy \supset x = y$;	6) $x \cdot Sy = x \cdot y + x$;
3) $x + 0 = x$;	7) $P_0 = 0$;
4) $x + Sy = S(x + y)$;	8) $x = 0 \vee x = SPx$;

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(iv) some additional elementary properties which could be deduced with the help of formal induction:

9) $x + y = y + x$;
10) $x + (y + z) = (x + y) + z$;
11) $x \cdot y = y \cdot x$;
12) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
13) $x \cdot (y + z) = x \cdot y + x \cdot z$;

(v) some other closed formulas in L.

So the theory Ar might be very weak (without induction) or very strong (an extension of *Peano's arithmetic*, PA) and even inconsistent!

A *numeral* is a term of L of the form $SS \ldots S0$ with n occurrences of S, it will be denoted as $\overline{n}$ (or, abusingly, simply as n). The *order relation* is defined as usual:

$$x \leq y \rightleftharpoons \exists z\ (x + z = y),$$
$$x < y \rightleftharpoons Sx \leq y.$$

Our axioms allow us to prove (without induction, see [6], chapter 4) some elementary properties of this order, including the equivalence:

$$x \leq \overline{n} \equiv (x = \overline{0} \vee x = \overline{1} \vee \cdots \vee x = \overline{n})$$

for any number n.

Note that all our explicit nonlogical axioms (i) – (iv) are open formulas.

1.1. Now, L^c is an extension of L by one new constant c. Intuitively, c is a very large, "non-feasible" natural number. Let us consider the theory Ar^c in the language L^c which is an extension of Ar with an infinite number of axioms:

(vi) $\bar{n} < c$ for any numeral $\overline{n}$.

It is trivial that Ar^c is conservative over Ar. If $Ar^c \vdash A$ and A does not contain c, then only a finite number of axioms of the group (vi) occurs in the proof above:

$$n_1 < c, \ldots,\ n_k < c.$$

Let us put $n = \max\{n_i + 1 \mid i = 1 \ldots k\}$. Substituting c by $\overline{n}$ one gets a proof of $Ar \vdash A$.

1.2. The language LF is a further extension of L^c by a new predicate symbol: $F(x)$ – "x is a feasible number". The corresponding axiomatic theory ArF is an extension of Ar^c by the following nonlogical axioms:

(vii) 1) $F(0)$;
2) $\neg F(c)$;
3) $x = y \wedge F(x) \supset F(y)$;
4) $y \leq x \wedge F(x) \supset F(y)$;

(viii) closure properties of F with respect to functional symbols of $L : F(x_1) \wedge \cdots \wedge F(x_m) \supset F(f(x_1, \ldots, x_m))$, for instance,

$$F(x) \supset F(Sx);$$
$$F(x) \wedge F(y) \supset F(x+y) \quad \text{etc.}$$

(ix) a formal induction on *feasible* numbers:

$$A(0) \wedge \forall x(F(x) \wedge A(x) \supset A(Sx)) \supset \forall x(F(x) \supset A(x))$$

for any formula $A(x)$ in the language LF.

1.3. Let us reproduce now a short traditional reasoning concerning conservativity ArF over Ar. A similar reasoning due to T. Skolem, K. Gödel and L. Henkin might be found in (almost) any textbook on mathematical logic (see, for instance, S.C. Kleene [1], chapter VI, §53, theorem 38, or A.N. Kolmogoroff, A.G. Dragalin [2], chapter III, §2, sect. 3.).

Assume $ArF \vdash A$ but not $Ar \vdash A$, where A is a sentence (a closed formula) of the language L. Then not $Ar^c \vdash A$ (see 1.1.), so the theory $(Ar^c + \neg A)$ is consistent. By the completeness theorem there exists a model M for the theory Ar^c, such that $M \models \neg A$. It is easy to see that this model M can be extended to a model MF for the theory ArF. Namely, we put $MF \models F(a)$ for an object a of the model M iff there exists a natural m such that $M \models (a = \overline{m})$.

All additional axioms (vii)–(ix) of ArF are fulfilled in MF. So $MF \models \neg A$ and hence, not $ArF \vdash A$. Contradiction!

1.4. This elegant classic reasoning is, nevertheless, not quite satisfactory from the constructive point of view. After all, we do not get any concrete model for ArF and so we have no clear semantics for the arithmetic with feasible numbers.

The indirect reasoning does not give any direct connection between proofs in ArF and Ar. If one has a proof $ArF \vdash A$ for the sentence A (in L) how can one get a proof of $Ar \vdash A$? How can one estimate the calculability of the function which gives a proof of $Ar \vdash A$ when a proof of $ArF \vdash A$ is given? Is this function primitive recursive or, say, ε_0-recursive?

One of the aims of this article is just to give a direct proof of the conservativity result by the *explicit* construction of a model MF for the theory ArF such that

$$MF \models A \quad \text{iff} \quad ArF \vdash A.$$

This is impossible if MF is a usual two-valued model but, nevertheless, we reach this result by the substitution of usual classical models by *Boolean–valued* ones and using *constructive* methods of [3] – [5].

This way we get a completely finitistic notion of truth. Some standard metamathematical investigation of the applied method allows further to give a constructive estimated connection between proofs in ArF and Ar.

2. Some general algebraic constructions

The construction of our Boolean-valued model is quite general and applicable to an arbitrary first-order axiomatic theory so it is worth-while to make our exposition as general as possible.

2.1. We begin with some general algebraic considerations. Most of the proofs can be found in [3], [4] but we give here full definitions and a full list of all essential facts we shall need later on.

Elementary information about Boolean and Heyting algebras can be found, for example, in [7] (note that Heyting algebras are called *pseudo–Boolean algebras* in [7]).

Let us remind shortly, that a *Heyting algebra* is a structure of type

$$\langle H, \sqsubseteq, \mathbf{0}, \mathbf{1}, \wedge, \vee, \Rightarrow, \neg \rangle,$$

where $\sqsubseteq$ is a (partial) order on H (a so-called *basic* order), $\mathbf{0}$ is the least element of H (zero), $\mathbf{1}$ is the largest element of H (unit),

$\wedge$, $\vee$ are the usual lattice operations of a *distributive* lattice;

$\Rightarrow$ is a lattice implication, i.e. $a \Rightarrow b$ is the maximal c such that $a \wedge c \sqsubseteq b$;

$\neg$ is correspondingly the lattice negation, $\neg a = (a \Rightarrow \mathbf{0})$.

A *complete* Heyting algebra contains a meet and a join, $\bigwedge \mathcal{Q}$ and $\bigvee \mathcal{Q}$ for any subset $\mathcal{Q} \subseteq H$. We consider a complete Heyting algebra as an abstract structure:

$$\left\langle H, \sqsubseteq, \mathbf{0}, \mathbf{1}, \wedge, \vee, \Rightarrow, \neg, \bigwedge, \bigvee \right\rangle$$

where the two latter operations are defined on the family of all subsets of H.

A Heyting algebra is said to be a *Boolean algebra* if some additional equality takes place; namely , for any $p \in H$

$$\neg\neg p = p.$$

Note, that all operations in a Heyting algebra are definable uniquely by the basic relation $\sqsubseteq$, so we shall denote sometimes a Heyting algebra shortly as $(H, \sqsubseteq)$.

Our definitions do not exclude the situation when $\mathbf{0} = \mathbf{1}$ in a given Heyting algebra. If $\mathbf{0} = \mathbf{1}$ then H is a one-element set and the algebra H is called a degenerate one. The question whether a given H is degenerate is rather often quite complicate and non-elementary in our constructions.

2.2. Let $(T, \leq)$ be an arbitrary (partially) ordered set; this means, of course, that the familiar properties: a) $p \leq p$; b) $p \leq q$, $q \leq r \Longrightarrow p \leq r$; c) $p \leq q$, $q \leq p \Longrightarrow p = q$ take place for any p, q, $r \in T$.

A set $x \subseteq T$ is said to be *(order) open* (on T) if: $p \in x$, $q \leq p \Longrightarrow q \in x$, for any $p, q \in T$. (Cf. [4], sect. 2.2., note that here we use the more convenient *dual notations*, i.e. we write here $p \leq q$ for $q \leq p$ in [3], [4]) The family of all order open subsets of T we denote as $\mathcal{O}$.

The *minimal order open extending* could be manufactured from an arbitrary subset $x \subseteq T$. Namely:

$$i(x) = \{p \in T \mid (\exists q \in x)(p \leq q)\}.$$

2.2.1. Fact.

(i) $x \subseteq i(x), \quad i(x) \in \mathcal{O}$;
(ii) $x \subseteq a, \quad a \in \mathcal{O} \Longrightarrow i(x) \subseteq a$;
(iii) $x \in \mathcal{O} \Longleftrightarrow i(x) = x$.

If $x, y \subseteq T$, we define an *open implication* of x and y as follows:

$$(x \supset^{\circ} y) = \{p \in T \mid (\forall q \leq p)\,(q \in x \Rightarrow q \in y)\}\,.$$

It is evident that always $(x \supset^{\circ} y) \in \mathcal{O}$. The most important property of $\mathcal{O}$ is the following fact (see [4], sect. 2.2.):

2.2.2. Fact. *The structure* $(\mathcal{O}, \subseteq)$ *is a complete Heyting algebra. Operations in this algebra are calculated in the following way:*

$$\mathbf{1} = T;\ \mathbf{0} = \emptyset;\ a \mathbin{\wedge\!\!\!\wedge} b = a \cap b;\ a \mathbin{\vee\!\!\!\vee} b = a \cup b;$$
$$(a \mapsto b) = (a \supset^{\circ} b); \quad \neg a = (a \supset^{\circ} \emptyset).$$

If $\mathcal{Q} \subseteq \mathcal{O}$ *then* $\bigwedge\!\!\!\bigwedge \mathcal{Q} = \bigcap \mathcal{Q} = \{p \in T \mid (\forall a \in \mathcal{Q})\,(p \in a)\}\,,$

$$\bigvee\!\!\!\bigvee \mathcal{Q} = \bigcup \mathcal{Q} = \{p \in T \mid (\exists a \in \mathcal{Q})(p \in a)\}.$$

2.3. We need a slightly more complicated construction in the sequel.

A *completion relation* on $(T, \leq)$ is by definition a relation $J(d, p)$, where $d \subseteq T$ and $p \in T$, such that

$$(\forall q \in T)(J(d, p), q \in d \Longrightarrow q \leq p).$$

We shall say that d is a *set of premises* for p according to J, if $J(d, p)$ takes place.

A comletion structure $(T, \leq, J)$ allows to define the family $\mathcal{E}$ of *complete subsets* of T. Namely, an $x \subseteq T$ is said to be *complete* (symbolically, $x \in \mathcal{E}$) iff

$$(\forall d \subseteq T)(\forall p \in T)(J(d, p), d \subseteq x \Longrightarrow p \in x).$$

Then we define an *operation of completion* in the following way:

$$\boldsymbol{\mathcal{D}} x = \bigcap \{b \in \mathcal{E} \mid x \subseteq b\}$$

for any $x \subseteq T$.

2.3.1. Fact. *The operation $\boldsymbol{\mathcal{D}}$ acts as a closure operation. Namely,*

(i) $x \subseteq \boldsymbol{\mathcal{D}}x$; $\boldsymbol{\mathcal{D}}x \in \mathcal{E}$; .

(ii) $x \subseteq b,\ b \in \mathcal{E} \Longrightarrow \boldsymbol{\mathcal{D}}x \subseteq b$;

(iii) $\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}}x = \boldsymbol{\mathcal{D}}x$;

(iv) $x \in \mathcal{E} \iff \boldsymbol{\mathcal{D}}x = x$;

(v) $x \subseteq y \Longrightarrow \boldsymbol{\mathcal{D}}x \subseteq \boldsymbol{\mathcal{D}}y$.

2.3.2. Fact.

$$a \in \mathcal{O}, x \subseteq T \Longrightarrow a \cap \boldsymbol{\mathcal{D}}x \subseteq \boldsymbol{\mathcal{D}}(a \cap x).$$

Proof. Let us consider the set

$$c = \{p \in T \mid p \in a \Longrightarrow p \in \boldsymbol{\mathcal{D}}(a \cap x)\}$$

and check subsequently

$$c \in \mathcal{E},\ x \subseteq c,\ \boldsymbol{\mathcal{D}}x \subseteq c,\ a \cap \boldsymbol{\mathcal{D}}x \subseteq \boldsymbol{\mathcal{D}}(a \cap x). \qquad \square$$

A completion structure $(T, \leq, J)$ is said to be *ordered* if the following condition holds:

$$p \leq q,\ J(d, q) \Longrightarrow \exists e(J(e, p), e \subseteq i(d)).$$

If a completion structure is ordered then some further useful properties take place (cf. [4], sect. 2.4.2.) .

2.3.3. Fact.

(i) $a \in \mathcal{O} \Longrightarrow \boldsymbol{\mathcal{D}}a \in \mathcal{O}$;

(ii) $a \in \mathcal{O},\ b \in \mathcal{E} \Longrightarrow (a \supset^\circ b) \in \mathcal{E} \cap \mathcal{O}$;

(iii) $a, b \in \mathcal{O} \Longrightarrow \boldsymbol{\mathcal{D}}a \cap \boldsymbol{\mathcal{D}}b = \boldsymbol{\mathcal{D}}(a \cap b)$;

(iv) $a \in \mathcal{O},\ b \in \mathcal{E} \Longrightarrow (a \supset^\circ b) = (\boldsymbol{\mathcal{D}}a \supset^\circ b)$.

These properties provide the following important fact (see [4], sect. 2.5.).

2.3.4. Fact. *Let $(T, \leq, J)$ be an ordered completion structure. Then the structure $(\mathcal{E} \cap \mathcal{O}, \subseteq)$ is a complete Heyting algebra. Operations in this algebra are calculated in the following way:*

$$\mathbf{1} = T;\ \mathbf{0} = \boldsymbol{\mathcal{D}}\emptyset;\ a \wedge b = a \cap b;\ a \vee b = \boldsymbol{\mathcal{D}}(a \cup b);$$
$$(a \Rightarrow b) = (a \supset^\circ b);\ \neg a = (a \supset^\circ \boldsymbol{\mathcal{D}}\emptyset).$$

If $\mathcal{Q} \subseteq \mathcal{E} \cap \mathcal{O}$ then $\bigwedge \mathcal{Q} = \bigcap \mathcal{Q}$, $\bigvee \mathcal{Q} = \boldsymbol{\mathcal{D}}(\bigcup \mathcal{Q})$.

Now we deal with a standard *double-negation* construction manufacturing a *Boolean* algebra from a given Heyting one. In our situation this construction looks as follows.

Let $(T, \leq, J)$ be an ordered completion structure and $v \in \mathcal{E} \cap \mathcal{O}$. Let us consider the family of *stable* subsets of T:

$$\mathcal{N} = \{a \in \mathcal{O} \mid a = ((a \supset^\circ v) \supset^\circ v)\}.$$

2.3.5. Fact.

(i) $v \in \mathcal{N}$;

(ii) $a \in \mathcal{O},\ b \in \mathcal{N} \Longrightarrow (a \supset^\circ b) \in \mathcal{N}$;

(iii) $a \in \mathcal{N} \Longrightarrow v \subseteq a$;

(iv) $a \in \mathcal{O} \Longrightarrow a \subseteq ((a \supset^\circ v) \supset^\circ v)$;

(v) $a \in \mathcal{N} \Longrightarrow a \in \mathcal{E} \cap \mathcal{O}$.

2.3.6. Fact. (Cf. [3], sect. 1.9.). *The structure $(\mathcal{N}, \subseteq)$ is a complete Boolean algebra. In this algebra*

$$\mathbf{1} = T; \quad \mathbf{0} = v; \quad a \mathbin{\wedge\!\!\!\wedge} b = a \cap b;$$
$$a \mathbin{\vee\!\!\!\vee} b = ((a \supset^\circ v) \cap (b \supset^\circ v)) \supset^\circ v;$$
$$(a \Rrightarrow b) = (a \supset^\circ b); \quad \neg\!\!\neg a = (a \supset^\circ v);$$

If $\mathcal{Q} \subseteq \mathcal{N}$, then

$$\bigwedge\!\!\!\!\bigwedge \mathcal{Q} = \bigcap \mathcal{Q},\ \bigvee\!\!\!\!\bigvee \mathcal{Q} = (\bigcap\{(a \supset^\circ v) \mid a \in \mathcal{Q}\}) \supset^\circ v.$$

Note that $\mathcal{N}$ is a subset of $\mathcal{E} \cap \mathcal{O}$ but not a subalgebra of $\mathcal{E} \cap \mathcal{O}$.

2.4. Let us consider the relation $p \in \boldsymbol{\mathcal{D}}x$ as a two argument predicate of arguments $p \in T$ and $x \subseteq T$. According to the definition

$$p \in \boldsymbol{\mathcal{D}}x \iff (\forall b \subseteq T) \quad (x \subseteq b,\ b \in \mathcal{E} \Longrightarrow p \in b),$$

so we use in this definition an *analytical quantifier* $(\forall b \subseteq T)$ on subsets of T. From a general constructive point of view it would be much more plausible to avoid this nonelementary quantifier. This is the only place where we use a subset quantifier for the definitions of a set. All other set definition of ours are *strongly predicative:* they use only quantifiers limited by elements of earlier predicted sets. We show that in an important case of the completion structure with *finite premises* this analytical universal quantifier can be substituted by the much more elementary existential quantifier on some finite objects. In fact, in this article we use completion structures with finite premises exclusively. This circumstance is essential in our metamathematical observations on the connection between proofs in ArF and in Ar.

We say that a set X is *finite* if there exists a natural number m and a function $f : \{0, 1, \ldots, m-1\} \to X$ such that: $\mathrm{dom} f = \{0, \ldots, m-1\}$, $\mathrm{rng} f = X$, and f is a *bijection.* The latter means

$$(\forall i, j \in \mathrm{dom} f)(i = j \iff f(i) = f(j)).$$

If this definition seems unnecessary for somebody, please, remember that, from the constructive point of view, there are a few natural *nonequivalent* definitions of finiteness and we are working in the frame of constructive metamathematics. For example, for any finite X we have a constructive disjuction

$$X = \emptyset \vee \exists x (x \in X)$$

and this statement is by no means true for an arbitrary set.

We say that a completion relation J on $(T, \leq)$ is a relation *with finite premises* if it follows from $J(d, p)$ that d is a finite set.

2.4.1. Now we consider a well known primitive recursive enumeration ω^* of finite sequences (corteges) of natural numbers by natural numbers. So any natural number p could be considered as a cortege

$$p = \langle n_0, \ldots, n_{k-1} \rangle,$$

where $k = \partial p$ is the *length* of the cortege p and $n_i = [p]_i$ for $i = 0, \ldots, k-1$ is the i–th member of p. We suppose that ∂p and $[p]_i$ are primitive recursive functions of p (and i) as well as $\langle n_0, \ldots, n_{k-1} \rangle$ for any fixed k. Furthermore, there can be defined a primitive recursive concatenation operation $p * q$, such that if $p = \langle u_0, \ldots, u_{k-1} \rangle$ and $q = \langle v_0, \ldots, v_{\ell-1} \rangle$, then

$$p * q = \langle u_0, \ldots, u_{k-1}, v_0, \ldots, v_{\ell-1} \rangle.$$

The *main* relation $\sqsubseteq$ on the cortege system ω^* is defined by the statement

$$q_1 \sqsubseteq q_2 \iff \exists p \ (q_2 * p = q_1),$$

so a prolonged cortege is *less* in this notation. It is the tree–like partial ordering with the empty cortege $\langle \ \rangle$ as the *largest* element. We suppose that $\sqsubseteq$ is a primitive recursive relation on ω.

The set of all corteges containing as members only 0 and 1 (the set of all *binary* corteges) we denote by $\{0, 1\}^*$. It is a primitive recursive subset of ω^*.

A subset $a \subseteq \omega^*$ is called *homogeneous* if

$$p, q \in a, \quad p \sqsubseteq r \sqsubseteq q \Longrightarrow r \in a.$$

A set a is, by definition, *standard* if it is homogeneous and, moreover,

$$p * \langle m+1 \rangle \in a \Longrightarrow p * \langle m \rangle \in a$$

for any natural p and m.

For example, $\{0, 1\}^*$ is standard. Further, the set of all binary corteges containing at least one unit (i.e. not pure zero corteges) is also standard.

A $p \in a$ is said to be a *leaf* of a if p is *minimal* in a

i. e. $$(\forall q \in a)(q \sqsubseteq p \Longrightarrow q = p).$$

An element $p \in a$ is said to be a *root* of a if p is the *largest* element of a, i.e. $(\forall q \in a)(q \sqsubseteq p)$.

2.4.2. Let us consider now an arbitrary completion structure $(T, \leq, J)$.

A *finite inference* over $(T, \leq, J)$ is, by definition, a couple of functions (π, τ), such that

(i) $\mathrm{dom}\pi$ is a finite standard subset of ω^* with a root $\langle\ \rangle$ (the empty cortege) and $\mathrm{rng}\pi \subseteq T$;

(ii) if m is not a leaf of $\mathrm{dom}\pi$ then

$$J(\{\pi(m * \langle n\rangle) \mid m * \langle n\rangle \in \mathrm{dom}\pi\},\ \pi(m));$$

(iii) $\mathrm{dom}\tau$ is the set of leaves of $\mathrm{dom}\pi$ and $\mathrm{rng}\tau \subseteq \{0, 1\}$;

(iv) if $\tau(m) = 0$ then $J(\emptyset, \pi(m))$.

An inference (π, τ) is an inference *for* $p \in T$, if $\pi(\langle\ \rangle) = p$. Let $x \subseteq T$ be a subset of T. We say that (π, τ) is an inference *from* x if $\tau(m) = 1 \Longrightarrow \pi(m) \in x$.

Intuitively, a finite inference for p from x is some sort of finite proof for the fact $p \in \boldsymbol{\mathcal{D}}x$.

Now we are ready to fomulate the main fact of this section.

2.4.3. Fact. *If $(T, \leq, J)$ is a completion structure with finite premises, then $p \in \boldsymbol{\mathcal{D}}x$ iff there exists a finite inference (π, τ) from x for p.*

Proof. Let us consider a set c such that $p \in c$ iff there exists a finite inference (π, τ) from x for p. Firstly, we check $x \subseteq c$ and $c \in \mathcal{E}$, so $\boldsymbol{\mathcal{D}}x \subseteq c$ (2.3.1.(ii)). Conversely, if $x \subseteq b$, $b \in \mathcal{E}$, then $c \subseteq b$. Indeed, the implication $(\forall p \in c)(p \in b)$ could be proven by arithmetical induction on the number of $\mathrm{dom}\pi$ in the finite inference (π, τ) for $p \in c$. $\square$

2.4.4. In the proof above

$$c = \bigcap\{b \in \mathcal{E} \mid x \subseteq b\} = \boldsymbol{\mathcal{D}}x,$$

so one can *define* $\boldsymbol{\mathcal{D}}x$ as c.

The situation is particularly simple if T is a set of natural numbers with recursive $\leq$. In this case a finite inference (π, τ) is a completely finite object which can be coded by a natural number, so the predicate $p \in \boldsymbol{\mathcal{D}}x$ turns out to be a $\sum_1^0$ arithmetical predicate (the original definition gives only a $\prod_1^1$ *analytical* estimation for this predicate).

3. Let us construct now some concrete standard $T \subseteq \omega^*$. We construct T subsequently by induction on the length ∂p for elements $p \in \omega^*$. Simultaneously we construct a function $h(p)$ with $\mathrm{dom}h = T$. For $p \in T$, the corresponding $h(p)$ will be a *queue of marked formulas*, i.e. finite (not empty) sequence $\varphi_1 \ldots \varphi_m$ $(m > 0)$ of marked formulas. A *marked formula* is a construction of the kind ℓA (read "A is on the left" and it could be thought of as "A is presumably true") or rA (read "A is on the right side" and in this case it could be thought of as "A is presumably false"). Here

A is an arbitrary formula of our language (in our situation it is a formula of L^c, but, as we have said already, our construction is applicable to an arbitrary explicit first order theory).

A queue of marked formulas is a close analogon of a *sequent* in a usual theory of cut-elimination. For example a queue

$$\ell A\ rD\ \ell B\ rA\ \ell C$$

represents a sequent $A, B, C \to D, A$ in a usual notation.

3.1. Let us arrange into a simple sequence

$$\text{Form}_0, \text{Form}_1, \ldots, \text{Form}_n, \ldots$$

all formulas of our language (i.e. of the language L^c). Similarly,

$$Ax_0, Ax_1, \ldots, Ax_n, \ldots$$

is the sequence of all axioms of our theory (i.e. Ar^c) and

$$\text{Term}_0, \text{Term}_1, \ldots, \text{Term}_n, \ldots$$

is the sequence of all terms.

3.1.1. As a basis (the first step) of the construction T and $h(p)$ we form the part T_1 of T; $T_1 \subseteq \{0,1\}^*$: $T_1 = \{1, 01, 001, 0001, \ldots\}$, and put $h(0^n 1) = r\text{Form}_n$; it is a one-element queue of marked formulas.

3.1.2. Let us suppose now that we have constructed already the part T_k of the set T and the function h on the set T_k on some step k.

A point $p \in T_k$ is *inconsistent*, if $h(p)$ contains ℓA and rA for the same formula A, or $h(p)$ contains $\ell\bot$ (where $\bot$ is the logical constant "false"). All inconsistent $p \in T_k$ are leaves of T_k. In the next step we shall prolong all *consistent* leaves of T_k, so inconsistent leaves of T_k will be minimal in Theither.

3.1.3. Let us represent the number k of the current step as $k = 3m+i$ with $i = 0, 1, 2$.

(i) If $i = 0$, we prolong every *consistent* point $p \in T_k$ by two points $p * \langle 0\rangle$, $p * \langle 1\rangle$ and put

$$h(p * \langle 0\rangle) = (h(p), \ell\text{Form}_m),$$
$$h(p * \langle 1\rangle) = (h(p), r\text{Form}_m),$$

so we prolong the current queue by the formula Form_m with the two possible marks ℓ and r. The number m of this formula is defined by the number k of the current step.

(ii) If $i = 1$ we prolong every consistent leaf $p \in T_k$ by one cortege $p * \langle 0\rangle$ with $h(p * \langle 0\rangle) = (h(p), \ell Ax_m)$.

(iii) Finally, if $i = 2$ then the prolongation of the consistent $p \in T_k$ depends on the *first* element of the queue $h(p)$. Let $h(p) = (\varphi, \mathcal{Q})$ where φ is a marked formula, the first member of $h(p)$.

1) φ is an atomic marked formula. Then we use a one prologation and put φ at the *end* of the queue:

$$h(p * \langle 0\rangle) = (\mathcal{Q}, \varphi).$$

2) $\varphi = r(A \wedge B)$. We use two prolongations:

$$h(p * \langle 0\rangle) = (\mathcal{Q}, rA), \quad h(p * \langle 1\rangle) = (\mathcal{Q}, rB).$$

3) $\varphi = \ell(A \wedge B)$, use one prolongation

$$h(p * \langle 0\rangle) = (\mathcal{Q}, \ell A, \ell B).$$

4) $\varphi = r(A \vee B)$;

$$h(p * \langle 0\rangle) = (\mathcal{Q}, rA, rB).$$

5) $\varphi = \ell(A \vee B)$;

$$h(p * \langle 0\rangle) = (\mathcal{Q}, \ell A), \quad h(p * \langle 1\rangle) = (\mathcal{Q}, \ell B).$$

6) $\varphi = r(A \supset B)$;

$$h(p * \langle 0\rangle) = (\mathcal{Q}, \ell A, rB).$$

7) $\varphi = \ell(A \supset B)$;

$$h(p * \langle 0\rangle) = (\mathcal{Q}, \ell B), \quad h(p * \langle 1\rangle) = (\mathcal{Q}, rA).$$

8) $\varphi = r\neg A$;

$$h(p * \langle 0\rangle) = (\mathcal{Q}, \ell A).$$

9) $\varphi = \ell\neg A$;

$$h(p * \langle 0\rangle) = (\mathcal{Q}, rA).$$

10) $\varphi = r\forall x A(x)$; we use a one prolongation of P. Let y be a the first variable which does not occur in $h(p)$. We put $h(p * \langle 0\rangle) = (\mathcal{Q}, rA(y))$.

11) $\varphi = \ell\forall x A(x)$;

$$h(p * \langle 0\rangle) = (\mathcal{Q}, \ell A(\mathrm{Term}_0), \ldots, \ell A(\mathrm{Term}_{m-1}), \ell\forall x A(x)),$$

so we use a one prolongation and put at the end of the current queue the substitution of the first m terms. Note that the original marked formula $\ell\forall x A(x)$ still *remains* at the end of the resulting queue for the further using.

12) $\varphi = r\exists x A(x)$; this case is similar to 11):

$$h(p * \langle 0\rangle) = (\mathcal{Q}, rA(\mathrm{Term}_0), \ldots, rA(\mathrm{Term}_{m-1}), r\exists x A(X)).$$

13) $\varphi = \ell\exists x A(x)$; we put $h(p * \langle 0\rangle) = (\mathcal{Q}, \ell A(y))$ for some new variable y not occurring in $h(p)$.

3.1.4. Now we define T as $\bigcup_k T_k$ and the function h on T as it follows from 3.1.1–3.1.3. In fact, T is a primitive recursive set, h is a primitive recursive function. Note $T \subseteq \{0, 1\}^*$, T is a *standard* subset of ω^*.

3.2. Let us notice now some simple facts concerning h. First we define a formula $h(p)^f$ for any sequence $h(p)$ in the following way (note that this definition excellently harmonizes with the usual interpretation of a sequent). Namely, let us put $\ell^f = \neg$ and r^f is an empty expression. Let

$$h(p) = (\varepsilon_1 A_1, \dots, \varepsilon_k A_k)$$

where ε_i is either ℓ or r. We define

$$h(p)^f = \forall \left(\varepsilon_1^f A_1 \vee \varepsilon_2^f A_2 \vee \cdots \vee \varepsilon_k^f A_k \right).$$

Here the quantifier $\forall(\dots)$ means the closure by universal quantifiers, so $h(p)^f$ is a *closed* formula.

3.2.1. Fact.
(i) *If $p \in T$ is inconsistent then $h(p)^f$ is deducible (in the classical predicate logic).*

(ii) *If $p \in T$, $p * \langle 0 \rangle \in T$, $p * \langle 1 \rangle \notin T$ then $h(p)^f$ is deducible from $h(p * \langle 0 \rangle)^f$ in our theory (in our case it is the theory Ar^c).*

(iii) *If $p \in T$, $p * \langle 0 \rangle \in T$, $p * \langle 1 \rangle \in T$ then $h(p)^f$ is deducible in our theory from $h(p * \langle 0 \rangle)^f$ and $h(p * \langle 1 \rangle)^f$.*

Proof. The proof is provided by a straightforward observation of the construction of 3.1. Note the point 3.1.3. (ii) where axioms of our theory are used. □

3.2.2. Fact. *If p is inconsistent, then p is a leaf of T.*

Proof. Cf. 3.1.2. □

3.2.3. Fact. *For every closed formula A there exists $p \in T$, such that $h(p)^f = A$.*

Proof. See 3.1.1. □

3.3. We define the completion structure on the set T. The fact $J(d, p)$ depends on the $h(p)$, namely
(i) if p is not a leaf in T then $J(d, p)$ iff

$$d = T \cap \{p * \langle 0 \rangle,\ p * \langle 1 \rangle\};$$

(ii) if p is a leaf in T (i.e. p is inconsistent) then $J(d, p)$ iff $d = \emptyset$.

It is evident, J is a relation with finite premises. Moreover, the structure $(T, \sqsubseteq, J)$ (with the main relation $\sqsubseteq$, 2.4.1.) is an *ordered* completion structure, the set T is homogeneous.

So we have the complete Heyting algebra $(\mathcal{E} \cap \mathcal{O}, \subseteq)$ (cf. 2.3.4.). Let us define the set $v \subseteq T$ as follows:

$$v = \boldsymbol{\mathcal{D}}\{p \in T \mid p \text{ is inconsistent}\}.$$

3.3.1. Fact.
(i) *$v \in \mathcal{E} \cap \mathcal{O}$;*
(ii) *If $p \in v$ then $h(p)^f$ is deducible in our therory.*

PROOF. Cf. 2.3.3., 3.2.2., 3.2.1. □

So, accordind to 2.3.6. we have the complete Boolean algebra $(\mathcal{N}, \subseteq)$.

3.4. Now we construct the *formula distribution* for (J, v). It is the couple of functions (L, R), such that for any formula A of our language (in L^c for our situation), $L(A)$ and $R(A)$ are subsets of T. Intuitively, (L, R) could be viewed as a some "intermediate product" for the model. $L(A)$ is the place where A is "certainly true" and, correspondigly, $R(A)$ is the place where A is "certainly false". The set v represent the zero of the algebra, the place where *all* statement are true (and false). We extend further this distribution to the real Boolean-valued model, so we need to check some preliminary conditions on this distribution (cf. 3.4.2. below) making possible the subsequent extension. For example, it has to be $L(A) \cap R(A) \subseteq v$ (the place where A is "certainly" true and false is the zero palce).

Definition. $p \in L(A)$ iff either
(i) p is inconsistent or
(ii) p is consistent and there exists $q \in T$, $p \sqsubseteq q$, such that ℓA occurs in the sequence $h(q)$.
Correpondingly, $p \in R(A)$ iff either
(i) p is inconsistent or
(ii) p is consistent and there exists $q \in T$, $p \sqsubseteq q$, such that rA occurs in the sequence $h(q)$.

Note that the set $\{q \in T \mid p \sqsubseteq q\}$ is finite. As is evident from the definition, we have the

3.4.1. Fact. *$L(A)$, $R(A) \in \mathcal{O}$ for any A.*

The following fact means that our distribution is a *systematic* one and provides the subsequent extending to the Boolean-valued model.

3.4.2. Fact.

1) $L(\bot) \subseteq v, \quad L(A) \cap R(A) \subseteq v;$

2) $L(A \wedge B) \subseteq \boldsymbol{\mathcal{D}}(L(A) \cap L(B));$

3) $R(A \wedge B) \subseteq \boldsymbol{\mathcal{D}}(R(A) \cup R(B));$

4) $L(A \vee B) \subseteq \boldsymbol{\mathcal{D}}(L(A) \cup L(B))$;

5) $R(A \vee B) \subseteq \boldsymbol{\mathcal{D}}(R(A) \cap R(B))$;

6) $L(A \supset B) \subseteq \boldsymbol{\mathcal{D}}(R(A) \cup L(B))$;

7) $R(A \supset B) \subseteq \boldsymbol{\mathcal{D}}(L(A) \cap R(B))$;

8) $L(\neg A) \subseteq \boldsymbol{\mathcal{D}}(R(A))$;

9) $R(\neg A) \subseteq \boldsymbol{\mathcal{D}}(L(A))$;

10) $L(\forall x A(x)) \subseteq \boldsymbol{\mathcal{D}}(L(A(t)))$
for any term t of our langauge;

11) $R(\forall x A(x)) \subseteq \boldsymbol{\mathcal{D}}(\bigcup_{y \in \mathrm{Var}} R(A(y)))$,
where Var *is the set of all variables;*

12) $L(\exists x A(x)) \subseteq \boldsymbol{\mathcal{D}}(\bigcup_{y \in \mathrm{Var}} L(A(y)))$;

13) $R(\exists x A(x)) \subseteq \boldsymbol{\mathcal{D}}(R(A(t)))$.

Proof. 1) If $p \in L(A) \cap R(A)$ then p is inconsistent, hence (3.3.) $p \in v$. Cases 2)–13) are consequences of the definition 3.1.3. (iii). For instance, let us consider 3). Let us suppose $p \in R(A \wedge B)$. If p is inconsistent, then $p \in R(A)$ (and $p \in R(B)$), so $p \in R(A) \cup R(B)$. If p is consistent then $r(A \wedge B)$ occurs in $h(q)$, $p \sqsubseteq q$; q is consistent and is not a leaf (3.1.2., 3.1.3.), note that $h(q)$ contains a nonatomic marked formula, namely $r(A \wedge B)$. So q should be prolonged. At each step of the prolongation according to 3.1.3. (iii) we *delete* the first member of the current h (and maybe put some new formulas at the end of the queue). So, at some step n of this process we get a subtree T_n such that $h(m)$ begins with $r(A \wedge B)$ for any m, $m \sqsubseteq p$, and m is a leaf of T_n. At the next step $(n+1)$ any such m will be prolonged in a such a way that $h(m * \langle 0 \rangle)$ contains rA and $h(m * \langle 1 \rangle)$ contains rB. So, for every leaf s of T_{m+1}, $s \sqsubseteq p$, and $h(s)$ contains either rA or rB. Hence $p \in \boldsymbol{\mathcal{D}}(R(A) \cup R(B))$. □

The following fact means that our formula distribution is a *formula complete* one.

3.4.3. Fact. *For every formula A of our language we have*

$$\boldsymbol{\mathcal{D}}(R(A) \cup L(A)) = T.$$

Proof. Cf. 3.1.2. (i) □

3.4.4. Fact. *If A is an axiom of our theory then*

$$\boldsymbol{\mathcal{D}}(L(A)) = T.$$

PROOF. Cf. 3.1.2. (ii) □

3.5. We use our systematic formula distribution in the more convenient form of a *semivaluation*. Namely, for any formula we define

$$|A|^- = (L(A) \supset^\circ v) \supset^\circ v;$$
$$|A|^+ = (R(A) \supset^\circ v).$$

The main convenience of using semivaluations is that $|A|^-$, $|A|^+$ are the members of the main algebra $(\mathcal{N}, \subseteq)$ (2.3.5., 3.4.1., 3.3.1.). As a consequence of 3.4.2. we have the following fact, that precisely means that $|A|^-$, $|A|^+$ give a *semivaluation* in the sense of Takahashi.

3.5.1. Fact.

1) $|\bot|^- \subseteq v$; $|A|^+ \subseteq |A|^+$;
2) $|A \wedge B|^- \subseteq |A|^- \mathbin{\wedge\!\!\!\wedge} |B|^- \subseteq |A|^+ \mathbin{\wedge\!\!\!\wedge} |B|^+ \subseteq |A \wedge B|^+$;
3) $|A| \vee B|^- \subseteq |A|^- \mathbin{\vee\!\!\!\vee} |B|^- \subseteq |A|^+ \mathbin{\vee\!\!\!\vee} |B|^+ \subseteq |A \vee B|^+$;
4) $|A \supset B|^- \subseteq |A|^+ \Rrightarrow |B|^- \subseteq |A|^- \Rrightarrow |B|^+ \subseteq |A \supset B|^+$;
5) $|\neg A|^- \subseteq \neg\!\!\neg |A|^+ \subseteq \neg\!\!\neg |A|^- \subseteq |\neg A|^+$;
6) $|\forall x A(x)|^- \subseteq \bigwedge\!\!\!\bigwedge_{t \in Tm} |A(t)|^- \subseteq \bigwedge\!\!\!\bigwedge_{y \in \mathrm{Var}} |A(y)|^+ \subseteq |\forall x A(x)|^+$;
7) $|\exists x A(x)|^- \subseteq \bigvee\!\!\!\bigvee_{y \in \mathrm{Var}} |A(y)|^- \subseteq \bigvee\!\!\!\bigvee_{t \in Tm} |A(t)|^+ \subseteq |\exists x A(x)|^+$.

where all operations are in the algebra $(\mathcal{N}, \subseteq)$*;* Tm *is the set of all terms of our theory and* Var *is the set of all its variables.*

PROOF. This fact is a straightforward consequence of 3.4.2. Let us consider, for instance, $|A|^- \subseteq |A|^+$. According to 3.4.2.1 $L(A) \cap R(A) \subseteq v$, so in the algebra $(\mathcal{O}, \subseteq)$ we have $R(A) \subseteq (L(A) \supset^\circ v)$. On the other hand, in $\mathcal{O}$:

$$(L(A) \supset^\circ v) \cap ((L(A) \supset^\circ v) \supset^\circ v) \subseteq v,$$

hence $R(A) \cap ((L(A) \supset^\circ v) \supset^\circ v) \subseteq v$ and acting in $\mathcal{O}$:

$$((L(A) \supset^\circ v) \supset^\circ v) \subseteq (R(A) \supset^\circ v). \qquad \Box$$

3.5.2. Fact. *For every formula A we have*

$$|A|^- = |A|^+.$$

PROOF. In view of 3.5.1. we have $|A|^- \subseteq |A|^+$. Let us use now the property 3.4.3. As $\boldsymbol{\mathcal{D}}(L(A) \cup R(A)) = T$, we have in the algebra $\mathcal{N}$:

$\boldsymbol{\mathcal{D}}(L(A) \cup R(A)) \supseteq^\circ v) = v$, and hence by (2.3.3.(iv)): $(L(A) \cup T(A) \supseteq^\circ v) = v$. Now, acting in $\mathcal{O}$:

$$(L(A) \supset^\circ v) \cap (R(A) \supset^\circ v) = v$$

and, further,

$$(R(A) \supset^\circ v) \subseteq (L(A) \supset^\circ v) \supset^\circ v,$$

i.e. $|A|^+ \subseteq |A|^-$. □

3.5.3. Fact. *If A is an axiom of our theory, then $|A|^- = T$.*

Proof. If A is an axiom, then $\boldsymbol{\mathcal{D}}(L(A)) = T$ (3.4.4.), so in the algebra $(\mathcal{N}, \subseteq)$: $(\boldsymbol{\mathcal{D}}(L(A)) \supset^\circ v) = v$ and by (2.3.3.(iv)): $(L(A) \supset^\circ v) = v$ and acting in $\mathcal{O} : T \subseteq (L(A) \supset^\circ v) \supset^\circ v$, i.e. $|A|^- = T$. □

3.6. Now we are ready to define our designed model M for our theory (i.e. for Ar^c).

The domain $\mathcal{D}$ of our theory will be the set of figures $[t]$, where t is a term of the theory.

As the Boolean algebra of truth values we use the algebra $(\mathcal{N}, \subseteq)$. Further, for a functional symbol f of our theory we define its values in M as follows:

$$\| f\left([t_1], \ldots, [t_m]\right) \| = \left[f\left(t_1, \ldots, t_m\right)\right].$$

And, finally, for a predicate symbol P of our theory we define:

$$\|P([t_1], \ldots, [t_m])\| = |P(t_1, \ldots, t_m)|^+.$$

The definition of the model M is finished.

We remind that the truth values of evaluated formulas are calculated in M according to the operations in the algebra $(\mathcal{N}, \subseteq)$. For example, $\|A \supset B\| = \|A\| \Rrightarrow \|B\|$, $\|A \vee B\| = \|A\| \Cup \|B\|$, $\|\forall x A(x)\| = \bigwedge_{a \in \mathcal{D}} \|A(a)\|$ etc.

3.6.1. Fact. (the substitution property) *For any formula $A(x, y)$ and terms $t(x, y)$ and $r(x)$ such that all parameters in them, except those explicitely mentioned, are evaluated in M, we have*

$$\|t(a, r(a))\| = \|t(a, \|r(a)\|)\|,$$
$$\|A(a, r(a))\| = \|A(a, \|r(a)\|)\|.$$

Proof. By straightforward induction on the construction of A and t. □

The fundamental fact concerning our model M can be expressed by the following

3.6.2. Fact. *Let $A(x_1, \ldots, x_n)$ be an arbitrary formula with the parameters $x_1, \ldots, x_n$ and let $t_1, \ldots, t_n$ be arbitrary terms. Then* $|A(t_1, \ldots, t_n)|^- \subseteq \|A([t_1], \ldots, [t_n])\| \subseteq |A(t_1, \ldots, t_n)|^+$.

PROOF. The proof is by induction on the construction of A using 3.5.1. Let us denote $A([t_1]), \ldots, [t_n])$ by A' and $A(t_1, \ldots, t_n)$ by A^*. Now let us consider, as an example, the case when A is an implication $B \supset C$. By the inductive supposition we have $|B^*|^- \subseteq \|B'\| \subseteq |B^*|^+$ and $|C^*|^- \subseteq \|C'\| \subseteq |C^*|^+$. Hence in view of 3.5.1.4):

$$|(B \supset C)^*|^- = |B^* \supset C^*|^- \subseteq |B^*|^+ \Rrightarrow |C^*|^- \subseteq \|B'\| \Rrightarrow \|C'\| \\ = \|(B \supset C)'\| \subseteq |B^*|^- \Rrightarrow |C^*|^+ \subseteq |(B \supset C)^*|^+. \qquad \square$$

3.6.3. Theorem. *Let $A(x_1, \ldots, x_n)$ be an arbitrary formula with the parameters $x_1, \ldots, x_n$. Let $t_1, \ldots, t_n$ be a list of terms. Then*

$$|A(t_1, \ldots, t_n)|^- = \|A([t_1], \ldots, [t_n])\| = |A(t_1, \ldots, t_n)|^+.$$

In particular, if A is a closed formula, then

$$|A|^- = \|A\| = |A|^+.$$

PROOF. 3.6.2., 3.5.2 $\square$

3.6.4. Fact. *If A is an axiom of our theory, then $\|A\| = T$.*

PROOF. 3.6.3., 3.5.3. $\square$

3.6.5. Fact. *If B is an arbitrary formula which is deducible in our theory and B' is an arbitrary evaluation of B in the model M, then $\|B'\| = T$. In particular, if B is a deducible closed formula then $\|B\| = T$.*

PROOF. By induction on the construction of the inference of B in our theory. Essentially this induction is trivial because $(\mathcal{N}, \subseteq)$ is a complete Boolean algebra, i.e. is in accordance with classical logic. A bit of accuracy is needed in checking the quantifier axioms. Namely, we use the substitution property 3.6.1. $\square$

The main feature of our model M is that we have also an *inverse* statement.

3.6.6. Theorem. *Let $A(x_1, \ldots, x_n)$ be an arbitrary formula with the parameters $x_1, \ldots, x_n$ and $t_1, \ldots, t_n$ being terms. If $\|A([t_1], \ldots, [t_n])\| = T$ then the formula $A(t_1, \ldots, t_n)$ is deducible is our theory. In particular, if A is a closed formula and $\|A\| = T$ then A is deducible.*

PROOF. Let $\|A([t_1], \ldots, [t_n])\|{=}T$. Then $|(At_1, \ldots, t_n)|^+ = T$ (3.6.2), i.e. $(R(A(t_1, \ldots, t_n)) \supset^\circ v) = T$ and hence $R(A(t_1, \ldots, t_n)) \subseteq v$. Let us choose the point $p \in T$, such that $h(p) = rA(t_1, \dots, t_n)$ (3.1.1.) Then $p \in R(A(t_1, \dots, t_n))$ and hence $p \in v$. So according to 3.3.1.(ii) $A(t_1, \dots, t_n)$ is deducible. $\square$

4.1. Thus we have constructed the designed model M for the theory Ar^c and now we are ready to imitate the classical reasoning of 1.3. Let us extend the model M by a new predicate symbol, putting: $\|F(a)\| = \bigvee\!\!\!\!\bigvee_{n\in\omega} \|a = \overline{n}\|$, where $a \in \mathcal{D}$ and ω is the set of all natural numbers $0, 1, 2, \dots, \bar{n}$ is the term $SS \dots SO$ representing the number n in our theory and the disjunction $\bigvee\!\!\!\!\bigvee$ is taken in the algebra $(\mathcal{N}, \subseteq)$. This way we get some interpretation MF for the language LF. Note that it is important here that $(\mathcal{N}, \subseteq)$ is a *complete* Boolean algebra: we need the existence of the disjunction in the definition of $\|F(a)\|$. So the usual Lindenbaum–Tarski algebra does not suit this situation.

We state that MF is a *model* for the theory ArF.

It is necessary to check the nonlogical axioms 1.2. (vii)–(ix). We begin with some auxiliary facts.

4.1.1. Fact.

(i) $\|F(\overline{n})\| = T$ *for any* $n \in \omega$*;*

(ii) $|F(c)\| = \mathbf{0}$.

Proof. By elementary checking. For instance, $\|\overline{n} < c\| = T$ (1.1.(vi)) and therefore $\|c = \overline{n}\| = \mathbf{0}$ for any $n \in \omega$. Hence,

$$\|F(c)\| = \bigvee\!\!\!\!\bigvee_{n\in\omega} \|c = \overline{n}\| = \mathbf{0}. \qquad \square$$

4.1.2. Fact.

(i) $\|a = b\| \cap \|F(a)\| \subseteq \|F(b)\|$*;*

(ii) $\|a = b\| \subseteq \|t(a) = t(b)\|$*;*

(iii) $\|a = b\| \cap \|A(a)\| \subseteq \|A(b)\|$*;*

where $a, b \in \mathcal{D}$*,* $t(x)$ *is an arbitrary term and* $A(x)$ *is an arbitrary formula of* ArF *all parameters of which, excepting* x*, are evaluated in* MF*.*

Proof. (i) By elementary checking according to the definition of $\|F(a)\|$; (ii) Because M is a model of Ar^c and the corresponding fact is deducible in Ar^c; (iii) By a straightforward induction on the construction of $A(x)$ using (i) and (ii). $\square$

4.1.3. Fact. *If* $A(x)$ *is a formula of* ArF *with the parameters evaluated in* MF *(excepting maybe* x*) then*

$$\|\forall x(F(x) \supset A(x))\| = \bigwedge_{n\in\omega} \|A(\overline{n})\|;$$

$$\|\exists x(F(x) \wedge A(x))\| = \bigvee_{n\in\omega} \|A(\overline{n})\|$$

PROOF. By elementary calculations in $(\mathcal{N}, \subseteq)$ using 4.1.2. For instance, (4.1.2.(iii)): $\|A(\overline{n})\| \wedge \|a = \overline{n}\| \subseteq \|A(a)\|$, hence $\|A(\overline{n})\| \subseteq \|a = \overline{n}\| \Rightarrow \|A(a)\|$ and further

$$\bigwedge_{n} \|A(\overline{n})\| \subseteq \bigwedge_{n} (\|a = \overline{n}\| \Rightarrow \|A(a)\|) =$$

$$((\bigvee_{n} \|a = \overline{n}\|) \Rightarrow \|A(a)\|), \text{ i.e. } \bigwedge_{n} \|A(\overline{n})\| \subseteq (\|F(a)\| \Rightarrow \|A(a)\|),$$

$$\bigwedge_{n\in\omega} \|A(\overline{n})\| \subseteq \bigwedge_{a\in\mathcal{D}} (\|F(a)\| \Rightarrow \|A(a)\|). \quad \square$$

4.2. Now the checking of 1.2. (vii)–(ix) can be reduced to some elementary calculations. For example, 1.2.(viii):

$$\|\forall xy(F(x) \wedge F(y) \supset F(x+y))\| = \bigwedge \|F(\overline{m} + \overline{n})\| =$$

$$\bigwedge_{m,n\in\omega} \bigvee_{k\in\omega} \|\overline{k} = \overline{m} + \overline{n}\|,$$

and the last expression has trivially the value T.

The induction principle 1.2.(ix) could be rewritten by (4.1.3.) as

$$\|A(0)\| \wedge \bigwedge_{n\in\omega} \|A(\overline{n}) \subset A(S\overline{n})\| \subseteq \bigwedge_{n\in\omega} \|A(\overline{n})\|.$$

We prove this with the help of

$$\|A(0)\| \wedge \bigwedge \|A(\overline{n}) \supset A(S\overline{n})\| \subseteq \|A(\overline{n})\|$$

for any $n \in \omega$. This last statement could be proven by metamathematical induction on $n \in \omega$.

4.3. Let us prove now the conservativity of ArF over Ar,

Let A be a closed formula in the language L and let us suppose $ArF \vdash A$. Then $MF \models A$ (i.e. $\|A\| = T$ in the model MF). But A does not contain the predicate F, so $M \models A$ and, hence, $Ar^c \vdash A$ (3.6.6.). Finally, as A does not contain the constant c, we have $Ar \vdash A$, because Ar^c is conservative over Ar (see 1.1.(vi)).

5.*1.* Finally let us discuss shortly the possible investigation of algorithms giving the proof of $Ar \vdash A$ for the given proof $ArF \vdash A$ for a formula A in L. Our explicit construction for models M and MF enables us to give a lot of algorithms for classes of input data, but here we give only a rather rough estimation founded on a proof-theoretic technique.

For every concrete formula $A(x_1, \ldots, x_n)$ of the language LF the corresponding predicate $p \in \|A([t_1], \ldots, [t_n])\|$ of argumments $p, t_1, \ldots, t_n$ could be expressed by an arithmetical formula of L. It could be viewed by metamathematical induction on the construction of A in LF. In fact, one can arrage the primitive recursive function giving the formula $p \in \|A([t_1], \ldots, [t_n])\|$ with the variables $p, t_1, \ldots, t_n$ for the given input formula A (to be a bit more pedantic, this function works not with the formulas but with their Gödel numbers, of course).

The constructive character of our metamathematics provides a constructive proof for the implication

$$ArF \vdash A \Longrightarrow (\forall p \in T)(p \in \|A\|)$$

for any (metamathematically given) formula A in LF. This proof could be executed in the constructive system of arithmetic, say, in HA (see, for instance, [8], par 2, sect 2) formalizing the reasoning of 4.1.

Further, we reproduce in HA the proof of the implication

$$(\forall p \in T)(p \in \|A\|) \Longrightarrow Ar^c \vdash A$$

for any A in L^c (3.6.6.), and, finally, the proof

$$Ar^c \vdash A \Longrightarrow Ar \vdash A$$

for $A \in L$ (1.1.(vi)).

So we can prove in HA

$$ArF \vdash A \Longrightarrow Ar \vdash A$$

for any, metamathematically given, closed formula A in the language L. This last statement could be reformulated in the $\forall\,\exists$ form:

$$\forall x \exists y((\text{ Proof } x \text{ for } A \text{ in } ArF) \Longrightarrow (\text{ Proof } y \text{ for } A \text{ in } Ar))$$

which is provable in HA.

By the finite type realizability technique (see, for example, [9]) for HA, one can exclude from the last proof the ε_0-*recursive* function $y = G(x)$ which gives the proof y for A in Ar for the input proof x for A in ArF.

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