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Fiberings on almost r-contact manifolds

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Abstract. The differentiable manifolds with almost contact structures were studied by many authors, for example: D.E. Blair [5], S. Sasaki [6], S.I. Goldberg and K. Yano [7], S. Ishihara [8], and others.

The purpose of this paper is to study strictly regular invariant r-contact manifolds related to an almost r–contact structure. This structure induces a principal bundle whose base manifold bears an almost complex structure. This paper is devoted to study the relations between the integrability conditions of almost r-contract structure and the induced complex structure. The motivation of this paper is to extend some results obtained by K. Oguie [4].

1. Introduction

Let V_n be a C^∞ real differentiable manifold of dimension $n(=2m+r)$. Let $\mathcal{J}(V_n)$ denote the ring of real valued differentiable functions on V_n and let $\mathcal{X}(V_n)$ be the module of the derivations of $\mathcal{J}(V_n)$. Let V_n be equipped with tensor field Φ which is a linear map $\Phi : \mathcal{X}(V_n) \to \mathcal{X}(V_n)$.

Let there be $r(C^\infty)$ 1-forms $A_1,\ A_2,\ \dots,\ A_r$ and $r(C^\infty)$ contravariant vector fields $T^1,\ T^2,\ \dots,\ T^r$ satisfying the following conditions [1]:

$$\text{(1.1)a} \qquad \Phi^2 X = -X + \sum_{p=1}^{r} A_p(X)T^p \quad \text{for all} \quad X \in \mathcal{X}(V_n).$$

Let us put

$$\text{(1.1)b} \qquad \bar{X} = \Phi(X),$$

$$\text{(1.2)} \qquad \Phi(T^p) = 0 \quad \text{for} \quad p = 1, 2, \dots, r,$$

$$\text{(1.3)} \qquad A_p(\bar{X}) = 0 \quad \text{for all} \quad X \in \mathcal{X}(V_n),$$

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(1.4) $$A_p(T^p) = 1.$$

Let a Riemannian metric g associated with Riemannian manifold V_n satisfy [1].

(1.5)
$$g(\bar{X}, \bar{Y}) = g(X, Y) - \sum_{p=1}^{r} A_p(X)A_p(Y), \quad \text{for all} \quad X,\ Y \in \mathcal{X}(V_n).$$

(1.6) $$A_q(X) = g(T^q, X), \quad \text{for} \quad q = 1,\ 2,\ \ldots, r.$$

Such a manifold satisfying all the conditions from (1.1)a to (1.6) is called an almost r-contact metric manifold. The structure endowed in V_n is called (Φ, A_p, T^p, g)-structure. Let ∇ be the Riemmanian connection given by [9]

(1.7) $$\nabla_X Y - \nabla_Y X = [X, Y]$$

and the Lie derivative L_X defined by

$$(L_X\Phi)\, Y = [X, \Phi Y] - \Phi[X, Y].$$

2. In this section we give some definitions and obtain some results:

Definition 2.1. An almost r-contact structure (Φ, A_p, T^p, g) is said to be invariant strictly regular if contravaviant vector fields T^1, T^2, $\ldots$, T^r are strictly regular vector fields, Φ and $r(C^\infty)$ 1-forms $A_1, A_2, \ldots, A_r$ are invariant under the action of G, generated by $T^1, T^2, \ldots, T^r$.

Definition 2.2. (V_n, M, G, Λ) is a principal G-bundle over M if M is an orbit space of $r(C^\infty)$ 1-forms and Λ is the natural projection of V_n onto M. If we define a (1,1) tensor field J and a (0,2) type tensor field $\overset{*}{g}$ on M by

$$J_{\Lambda(x)}\overset{*}{X} = d\Lambda\Phi_x\left(\overset{*}{X}{}^H_x\right)$$

where $\overset{*}{X}{}^H$ is the horizontal lift of $\overset{*}{X} \in \mathcal{X}(M)$ at $x \in M$ with respect to $A_1, A_2, \ldots, A_r$.

Let us put

$$\overset{*}{g}(\overset{*}{X}, \overset{*}{Y}) \overset{\text{def}}{=} g(\overset{*}{X}{}^H, \overset{*}{Y}{}^H)$$

then $(J, \overset{*}{g})$ is an almost Hermitian structure on M.

Theorem 2.1. *If we define a* (1,1) *tensor field J on M by*

(2.1) $$J_{\Lambda(x)}(\overset{*}{X}) = d\Lambda\Phi_x\left(\overset{*}{X}{}^H_x\right)$$

where $\overset{*}{X}{}^H_x$ *denotes the horizontal lift of* $\overset{*}{X} \in \mathcal{X}(M)$ *at the point* $x \in M$ *with respect to* $A_1, A_2, \ldots, A_r$*, then* J *is almost complex structure on* M*.*

Proof. For an $\overset{*}{X} \in \mathcal{X}(M)$ we have

$$\begin{aligned} J^2_{\Lambda(x)}(\overset{*}{X}) =& J_{\Lambda(x)}\left(J_{\Lambda(x)}(\overset{*}{X})\right) = J_{\Lambda(x)}\left(d\Lambda\Phi_x\left(\overset{*}{X}{}^H_x\right)\right) \\ =& d\Lambda\Phi_x\left(d\Lambda\Phi_x\left(\overset{*}{X}{}^H_x\right)\right)^{*H} = d\Lambda\Phi\Phi\left(\overset{*}{X}{}^H_x\right) \end{aligned}$$

On making use of (1.1)a we obtain

$$=d\Lambda\left[-\overset{*}{X}{}^H + \sum_{p=1}^{r} A_p\left(\overset{*}{X}{}^H\right)T^P\right] = -d\Lambda\left(\overset{*}{X}{}^H\right) = -\overset{*}{X}.$$

Hence J is an almost complex structure on M.

3. Integrability and normality

We define tensor fields $\overset{*}{N}$ and $\overset{*}{\Phi}$ on V_n as follows:

$$\overset{*}{N}\left(\overset{*}{X},\overset{*}{Y}\right) = \left[J\overset{*}{X}, J\overset{*}{Y}\right] - J\left[J\overset{*}{X},\overset{*}{Y}\right] - J\left[\overset{*}{X}, J\overset{*}{Y}\right] + J^2\left[\overset{*}{X},\overset{*}{Y}\right], \quad \forall \overset{*}{X},\overset{*}{Y} \in \mathcal{X}(M). \tag{3.1}$$

$$\overset{*}{J}\left(\overset{*}{X},\overset{*}{Y}\right) = \overset{*}{N}\left(\overset{*}{X},\overset{*}{Y}\right) + 2\sum_{p=1}^{r} dA_p\left(\overset{*}{X}{}^H,\overset{*}{Y}{}^H\right)T^p \tag{3.2}$$

and $N(X,Y)$ the Nijenhuis tensor of Φ is given by [2].

$$N(X,Y) = [\bar{X},\bar{Y}] + [X,Y] - [X,\bar{Y}] - [\bar{X},Y] \tag{3.3}$$

we now prove the following:

Lemma 3.1. *We have the following*

$$\overset{*}{N}\left(\overset{*}{X},\overset{*}{Y}\right) = d\Lambda N\left(\overset{*}{X}{}^H,\overset{*}{Y}{}^H\right)$$

and

$$\overset{*}{N}\left(\overset{*}{X},\overset{*}{Y}\right) = d\Lambda\overset{*}{\Phi}\left(\overset{*}{X}{}^H,\overset{*}{Y}{}^H\right)$$

where $\overset{*}{X}^H$ denotes the lift of X with respect to A's.

PROOF. We have from (3.1)

$$\overset{*}{N}\left(\overset{*}{X},\overset{*}{Y}\right)=[J\overset{*}{X},J\overset{*}{Y}]-J[J\overset{*}{X},\overset{*}{Y}]-J[\overset{*}{X},J\overset{*}{Y}]+J^2[\overset{*}{X},\overset{*}{Y}]$$

Making use of (2.1), we obtain

$$\begin{aligned}
&=\left[d\Lambda\Phi\overset{*}{X}^H,d\Lambda\Phi\overset{*}{Y}^H\right]-d\Lambda\Phi\left[d\Lambda\Phi\overset{*}{X}^H,d\Lambda\Phi\overset{*}{Y}^H\right]^{*H}\\
&\quad-d\Lambda\Phi\left[d\Lambda\overset{*}{X}^H,d\Lambda\Phi\overset{*}{Y}^H\right]^{*H}+d\Lambda\Phi\left(d\Lambda\Phi\left[\overset{*}{X},\overset{*}{Y}\right]^{*H}\right)^{*H}\\
&=d\Lambda\left[\Phi\overset{*}{X}^H,\Phi\overset{*}{Y}^H\right]-d\Lambda\Phi\left[\Phi\overset{*}{X}^H,\overset{*}{Y}^H\right]-d\Lambda\Phi\left[\overset{*}{X}^H,\Phi\overset{*}{Y}^H\right]\\
&\quad+d\Lambda\Phi^2\left[\overset{*}{X},\overset{*}{Y}\right]^{*H}\\
&=d\Lambda\left\{\left[\Phi\overset{*}{X}^H,\Phi\overset{*}{Y}^H\right]-\Phi\left[\Phi\overset{*}{X}^H,\overset{*}{Y}^H\right]-\Phi\left[\overset{*}{X}^H,\Phi\overset{*}{Y}^H\right]+\Phi^2\left[\overset{*}{X},\overset{*}{Y}\right]\right\}\\
&=d\Lambda N\left[\overset{*}{X}^H,\overset{*}{Y}^H\right]
\end{aligned}$$

The second relation follows immediately from first and from the fact that $T^1,T^2,\dots,T^r$ are vertical.

Theorem 3.1. $\overset{*}{N}\left(\overset{*}{X},\overset{*}{Y}\right)=0$ *if and only if* $N\left(\overset{*}{X}^H,\overset{*}{Y}^H\right)$ *is vertical for all* $\overset{*}{X},\overset{*}{Y}\in\mathcal{X}(M)$.

PROOF. In view of Lemma 3.1, this follows in an obvious manner.

Definition 3.1. An almost r-contact structure is said to be integrable if $N=0$ and is said to be normal if $\overset{*}{\Phi}=0$.

Definition 3.2. The connection forms $A_1,A_2,\dots,A_r$ are involutive if $[X,Y]$ is horizontal for all horizontal vector fields X and Y.

Theorem 3.3. *The almost r-contact structure is integrable if J is integrable and the connection forms $A_1,A_2,\dots,A_r$ are involutive.*

PROOF. By Theorem 3.1 $\overset{*}{N}\left(\overset{*}{X},\overset{*}{Y}\right)=0$ implies that

$$N\left(\overset{*}{X}^H,\overset{*}{Y}^H\right)\quad\text{is vertical for all}\quad X,Y\in\mathcal{X}(M).$$

On the other hand from (3.1) we have

$$\sum_{p=1}^{r} A_p\left(N\left[\overset{*}{X}^H,\overset{*}{Y}^H\right]\right)=\sum_{p=1}^{r} A_p\left(\left[\Phi\overset{*}{X}^H,\Phi\overset{*}{X}^H\right]\right)=0,$$

since $\Phi\overset{*}{X}^H$, and $\Phi\overset{*}{X}^H$ are horizontal. This shows that $N\left(\overset{*}{X}^H,\overset{*}{Y}^H\right)$ is horizontal. Hence we have $N\left(\overset{*}{X}^H,\overset{*}{Y}^H\right)=0$.

Now on the other hand it is clear that $\left[T^p,\overset{*}{X}^H\right]=0$. Since $\overset{*}{X}^H$ is invariant under the action of the parameter group G generated by $T^1,T^2,\dots,T^p$. Hence it is easily seen that $N\left(T^p,\overset{*}{X}^H\right)=0$. We have thus proved that $N(X,Y)=0$ is valid for the lifts of vector fields on M and the vertical vector fields. Since N is a tensor field, $N(X,Y)=0$ holds for any vector fields X and Y. The converse is clear.

Theorem 3.4. *The almost r-contact structure on V_n is normal if and only if J is integrable and*

$$\sum_{p=1}^{r}\left(J\overset{*}{X},J\overset{*}{Y}\right)=\sum_{p=1}^{r}\left(\overset{*}{X},\overset{*}{Y}\right)\quad \textit{for all}\quad \overset{*}{X},\overset{*}{Y}\in\mathcal{X}(M),$$

where

$$\sum_{p=1}^{r} dA_p\left(\Phi\overset{*}{X},\Phi\overset{*}{Y}\right)=\left(\Lambda\sum_{p=1}^{r}\right)\left(\overset{*}{X},\overset{*}{Y}\right).$$

Proof. By Theorem 3.2, $N(X,Y)=0$ implies that $\overset{*}{\Phi}\left(\overset{*}{X}^H,Y^H\right)$ is vertical for all $\overset{*}{X},\overset{*}{Y}\in\mathcal{X}(M)$. Also from (3.2) we have

$$\begin{aligned}\sum_{p=1}^{r} A_p\left(\overset{*}{\Phi}\left(\overset{*}{X}^H\overset{*}{Y}^H\right)\right)&=\sum_{p=1}^{r} A_p\left(\left[\Phi\overset{*}{X}^H,\Phi\overset{*}{Y}^H\right]\right)+2\sum_{p=1}^{r} dA_p\left(\overset{*}{X}^H,\overset{*}{Y}^H\right)\\
&=-2\sum_{p=1}^{r} A_p\left(\Phi\overset{*}{X}^H,\Phi\overset{*}{Y}^H\right)+2\sum_{p=1}^{r} dA_p\left(\overset{*}{X}^H,\overset{*}{Y}^H\right)\\
&=-2\left(\Lambda\sum_{p=1}^{r}\right)\left(\Phi\overset{*}{X}^H,\Phi\overset{*}{Y}^H\right)+2\left(\Lambda\sum_{p=1}^{r}\right)\left(\overset{*}{X}^H,\overset{*}{Y}^H\right)\end{aligned}$$

$$= -2\sum_{p=1}^{r}\left(J\overset{*}{X}, J\overset{*}{Y}\right) + 2\sum_{p=1}^{r}\left(\overset{*}{X}, \overset{*}{Y}\right) = 0.$$

This shows that $\overset{*}{\Phi}$ is horizontal. Hence we have $\overset{*}{\Phi} = 0$. Now it is clear that $\sum_{p=1}^{r} dA_p\left(T^p, \overset{*}{X}^H\right) = 0$, since

$$2\sum_{p=1}^{r} dA_p\left(T^p, \overset{*}{X}^H\right)$$
$$= \sum_{p=1}^{r} T^p \cdot A_p\left(\overset{*}{X}^H\right) - \overset{*}{X}^H \sum_{p=1}^{r} A_p\left(T^p\right) - \sum_{p=1}^{r} A_p\left(\left[T^p, \overset{*}{X}^H\right]\right)$$

Hence we have

$$\overset{*}{\Phi}\left(T^p, \overset{*}{X}^H\right) = N\left(T^p, \overset{*}{X}^H\right) + 2\sum_{p=1}^{r} dA_p\left(T^p, \overset{*}{X}^H\right) T^p = 0$$

Thus $\overset{*}{\Phi}(X, Y) = 0$ is valid for the lifts of vector fields on M. The converse is clear.

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