Publ. Math. Debrecen
47 / 3-4 (1995), 349–375

Differential geometry of Cartan connections

By D. V. ALEKSEEVSKY (Moscow) and P. W. MICHOR (Wien)

1. Introduction

In this article a general theory of Cartan connections is developed and some applications are indicated. The starting idea is to consider a Cartan connection as a deformation of a local Lie group structure on the manifold, i.e. a 1-form λ with values in a Lie algebra $\mathfrak{h}$ which is non degenerate and satisfies the Maurer-Cartan equation. Such a Maurer-Cartan form λ may be considered as a flat Cartan connection. Many notions and results of the geometry of group manifolds are still valid in this more general setting.

More precisely, for a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{h}$ we define a Cartan connection of type $\mathfrak{h}/\mathfrak{g}$ on a manifold P of dimension $n = \dim \mathfrak{h}$ as a $\mathfrak{h}$-valued 1-form $\kappa : TP \to \mathfrak{h}$ which defines an isomorphism $\kappa_x : T_xP \to \mathfrak{h}$ for any $x \in P$ and such that

$$[\zeta_X, \zeta_Y] = \zeta_{[X,Y]}$$

holds for $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, where the linear mapping $\zeta : \mathfrak{h} \to \mathfrak{X}(P)$ from $\mathfrak{h}$ into the Lie algebra $\mathfrak{X}(P)$ of vector fields on P is given by $\zeta_X(x) = \kappa_x^{-1}(X)$. If $\mathfrak{g} = \mathfrak{h}$ then ζ defines a free transitive action of the Lie algebra $\mathfrak{h}$ on the manifold P in the sense of [5] and κ is the Maurer-Cartan form of the associated structure of the local Lie group structure on P. In the general case, when $\mathfrak{g} \neq \mathfrak{h}$, we only have a free action $\zeta|\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ on P. So we may think of the Cartan connection κ as a deformed Maurer-Cartan form, where the deformation is breaking the symmetry from $\mathfrak{h}$ to $\mathfrak{g}$. If the action of $\mathfrak{g}$ on P can be integrated to a free action of a corresponding Lie

---

Mathematics Subject Classification: 53B05, 53C10.
Key words and phrases: Cartan connection, G-structure, characteristic classes, prolongation.

group G on P with smooth orbit space $M = P/G$, the notion of Cartan connection reduces to the well known notion of a Cartan connection on the principal bundle $p : P \to M$.

In 2.3 and 2.4 we describe two situations when a Cartan connection arises naturally. First under a reduction of a principal bundle $p : Q \to M$ with a principal connection to a principal subbundle $p : P \to M$. Second when a G-structure with a connection is given: more precisely, if the Lie algebra admits a reductive decomposition $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{m}$ we may identify a Cartan connection of type $\mathfrak{h}/\mathfrak{g}$ on a principal G-bundle $p : P \to M$ with a G-structure on M together with a principal connection in $p : P \to M$.

Dropping the condition that the 1-form κ is non-degenerate we come to the notion of *generalized Cartan connection*. It is closely related with the the notion of a principal connection form on a $\mathfrak{g}$-manifold, defined in [5], see 2.6. In the end of section 2 we define for an arbitrary generalized Cartan connection κ such notions as the curvature 2-form

$$K = d\kappa + \tfrac{1}{2}[\kappa, \kappa]^\wedge,$$

the Bianchi identity

$$dK + [\kappa, K]^\wedge = 0,$$

the covariant exterior derivative

$$d_\kappa : \Omega^p_{\text{hor}}(P; W)^{\mathfrak{g}} \to \Omega^{p+1}_{\text{hor}}(M; V)^{\mathfrak{g}}, \quad d_\kappa(\Psi) = d\Psi + \rho^\wedge(\kappa)\Psi,$$

where $\Omega^p_{\text{hor}}(M; W)^{\mathfrak{g}}$ is the space of horizontal $\mathfrak{g}$-equivariant p-forms with values in the $\mathfrak{g}$-module defined by a representation $\rho : \mathfrak{h} \to \mathfrak{gl}(W)$.

In 2.9 we associate with a generalized Cartan connection κ of type $\mathfrak{h}/\mathfrak{g}$ the Chern-Weil homomorphism

$$\gamma : S(\mathfrak{h}^*)^{\mathfrak{h}} \to \Omega_{\text{hor}}(P)^{\mathfrak{g}}$$

of the algebra of $\mathfrak{h}$-invariant polynomials on $\mathfrak{h}$ into the algebra of $\mathfrak{g}$-invariant closed horizontal differential forms on P and prove that the characteristic cohomology class $[\gamma(f)]$ does not depend on the particular choice of the generalized Cartan connection.

In section 3 we study relations between principal Cartan connections of a principal G-bundle $p : P \to M$ and principal connections on the H-bundle $p : P[H] = P \times_G H \to M$, where $H \supset G$ is a Lie group associated to $\mathfrak{h}$. We also establish a canonical linear isomorphism

$$\Omega_{\text{hor}}(P; W)^G \to \Omega_{\text{hor}}(P[H]; W)^H$$

between the respective spaces of equivariant horizontal forms with values in a representation space W of H. As a corollary we obtain that the characteristic classes associated with Cartan connections in section 2 are the classical characteristic classes of the principal bundle $P[H] \to M$.

Section 4 deals with a flat Cartan connection of type $\mathfrak{h}/\mathfrak{h}$ on a manifold P. We give a simple conceptual proof of the result that any flat generalized Cartan connection on a simply connected manifold P, i.e. an $\mathfrak{h}$-valued 1-form κ on P which satifies the (left) Maurer-Cartan equation, is the left logarithmic derivative of a mapping $\varphi : P \to H$ into a Lie group H corresponding to $\mathfrak{h}$; so '$\kappa = \varphi^{-1}.d\varphi$'. Moreover the mapping φ is uniquely determined up to a left translation.

A generalized Cartan-connection $\kappa : TP \to \mathfrak{h}$ induces a homomorphism

$$\kappa^* : \Lambda(\mathfrak{h}^*) \to \Omega(P)$$
$$f \mapsto f \circ (\kappa \otimes_\wedge \cdots \otimes_\wedge \kappa)$$

of the complex of exterior forms on the Lie algebra $\mathfrak{h}$ into the complex of differential forms P and (following [10]) defines a a characteristic class of a flat generalized Cartan connection as the image of of cohomology classes of the Lie algebra $\mathfrak{h}$ under the induced homomorphism of cohomologies. This construction may also sometimes be applied for the infinite dimensional case.

In section 5 we describe a flat Cartan connection associated with a flat G-structure $p : P \to M$. It defines a Cartan connection on the total space P^∞ of the infinite prolongation $p^\infty : P^\infty \to M$, which consists of all infinite jets of holonomic sections of p.

In the last section 6 we review shortly the theory of prolongation of G-structures in the sense of [22]. Under some conditions we define a canonical Cartan connection of type $(V \oplus \mathfrak{g}^\infty)/\mathfrak{g}$ on the total space of the full prolongation of a G-structure of first or second order.

2. Cartan connections and generalized Cartan connections

2.1. Cartan connections. Let $\mathfrak{h}$ be a finite dimensional Lie algebra and let $\mathfrak{g}$ be a subalgebra of $\mathfrak{h}$. Let P be a smooth manifold with $\dim P = \dim \mathfrak{h}$. By an $\mathfrak{h}$-*valued absolute parallelism* on P we mean a 1-form $\kappa \in \Omega^1(P; \mathfrak{h})$ with values in $\mathfrak{h}$ which is non-degenerate in the sense that $\kappa_x : T_xP \to \mathfrak{h}$ is invertible for all $x \in P$. Thus its inverse induces a linear mapping $\zeta : \mathfrak{h} \to \mathfrak{X}(P)$ which is given by $\zeta_X(x) = (\kappa_x)^{-1}(X)$. Vector fields of the form ζ_X are called *parallel*. In general, ζ is not a Lie algebra homomorphism.

Definition. In this setting a *Cartan connection* of type $\mathfrak{h}/\mathfrak{g}$ on the manifold P is an $\mathfrak{h}$-valued absolute parallelism $\kappa : TP \to \mathfrak{h}$ such that

(1) $[\zeta_X, \zeta_Y] = \zeta_{[X,Y]}$ for $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

So the inverse mapping $\zeta : \mathfrak{h} \to \mathfrak{X}(P)$ preserves Lie brackets if one of the arguments is in $\mathfrak{g}$. In particular, the restriction of ζ to $\mathfrak{g}$ is a Lie algebra homomorphism, and in particular P is a free $\mathfrak{g}$-manifold.

2.2. Principal Cartan connections on a principal G-bundle. Let $p : P \to M$ be a principal bundle with structure group G whose Lie algebra is $\mathfrak{g}$. We shall denote by $r : P \times G \to P$ the principal right action and by $\zeta : \mathfrak{g} \to \mathfrak{X}(P)$ the fundamental vector field mapping, a Lie algebra homomorphism, which is given by $\zeta_X(x) = T_e(r_x).X$. Its 'inverse' is then defined on the vertical bundle VP, it is given by $\kappa_G : VP \to \mathfrak{g}$, $\kappa_G(\xi_x) = T_e(r_x)^{-1}(\xi_x)$; we call it the *vertical parallelism*.

Let us now assume that $\mathfrak{g}$ is a subalgebra of a Lie algebra $\mathfrak{h}$ with $\dim \mathfrak{h} = \dim P$. A $\mathfrak{h}/\mathfrak{g}$-Cartan connection $\kappa : TP \to \mathfrak{h}$ on P is called a *principal Cartan connection* of the principal bundle $p : P \to M$, if the following two conditions are satisfied:

(1) $\kappa|VP = \kappa_G$, i.e. κ is an extension of the natural vertical parallelism.

(2) κ is G-equivariant, i.e. $\kappa \circ T(r^g) = \mathrm{Ad}(g^{-1}) \circ \kappa$ for all $g \in G$. If G is connected this follows from 2.1,(1).

This is the usual concept of Cartan connection as used e.g. in [13], p. 127.

Remark. Let $\kappa \in \Omega^1(P; \mathfrak{h})$ be a $\mathfrak{h}/\mathfrak{g}$-Cartan connection on a manifold P. Assume that all parallel vector fields $\zeta(\mathfrak{g})$ are complete. Then they define a locally free action of a connected Lie group G with Lie algebra $\mathfrak{g}$. If this action is free and if the orbit space $M := P/G$ is a smooth manifold (this is the case if the action is also proper), then $p : P \to M$ is a principal G-bundle and κ is a principal Cartan connection on it.

2.3. Principal Cartan connections and a reduction of a principal bundle with a connection. Let H be a Lie group with Lie algebra $\mathfrak{h}$, let $p : Q \to M$ be a principal H-bundle, and let $\omega : TQ \to \mathfrak{h}$ be a principal connection form on Q. Let us denote by $\mathcal{H} = \ker \omega$ the horizontal distribution of the connection ω. Then we have

$$T_qQ = V_qQ \oplus \mathcal{H}_q, \tag{1}$$

where $VQ = \ker T(p) \subset TQ$ is the vertical subbundle. We assume now that G is a Lie subgroup of H and that the principal bundle $Q \to M$ admits a reduction of the structure group to a principal G-bundle $p = p|P : P \to M$.

So the embedding $P \to Q$ is a principal bundle homomorphism over the group homomorphism $G \to H$:

$$\begin{array}{ccc} P & \longrightarrow & Q \\ {\scriptstyle p|P}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ M & = = & M \end{array}$$

Note that for the vertical bundles we have $T_uP \cap V_uQ = V_uP$, but the intersection $T_uP \cap \mathcal{H}_u$ may be arbitrary. We have the following characterization of the two extremal cases when this last intersection is maximal or minimal.

Proposition. (A) *In the situation above the following conditions are equivalent:*

(1) *For any $u \in P$ the horizontal subspace $\mathcal{H}_u = \ker \omega_u$ is contained in T_uP, and thus $T_uP = V_uP \oplus \mathcal{H}_u$.*
(2) *The connection ω on Q is induced from a principal connection on $P \to M$ on the associated bundle $Q = P \times_G H$, where G acts on H by conjugation.*
(3) *The holonomy group of the connection ω is contained in G.*

(B) *The restriction $\omega|P = incl^*\omega$ of ω on P is a Cartan connection of the principal bundle $p : P \to M$ if and only if $T_uP \cap \mathcal{H}_u = 0$ for each $u \in P$, and if* $\dim M = \dim \mathfrak{h}/\mathfrak{g}$. $\square$

2.4. Cartan connections as G-structures with connections. We establish here a bijective correspondence between principal Cartan connections and G-structures with a connection.

Let $G \subset GL(V)$, $V = \mathbb{R}^n$, be a linear Lie group. We recall that a G-structure on an n-dimensional manifold M is a principal G-bundle $p : P \to M$ together with a displacement form $\theta : TP \to V$, i.e. a V-valued 1-form which is G-equivariant and strictly horizontal in the sense that $\ker \theta = VP$.

We assume now that G is a reductive Lie subgroup of a Lie group H such that

$$\mathfrak{h} = V \oplus \mathfrak{g}, \quad [\mathfrak{g}, V] \subset V$$

is the reductive decomposition of the Lie algebra of H, and that the adjoint representation of G in V is faithful. Then we may identify G with a subgroup of $GL(V)$.

Proposition. *In this situation let $\kappa : TP \to \mathfrak{h} = V \oplus \mathfrak{g}$ be a Cartan connection on the principal G-bundle $p : P \to M$, and let $\theta = \mathrm{pr}_V \circ \kappa$ and $\omega = \mathrm{pr}_{\mathfrak{g}} \circ \kappa$ be its components in V and $\mathfrak{g}$, respectively.*

Then θ is a displacement form and ω is a connection form on $p : P \to M$, so that $(p : P \to M, \theta)$ is a G-structure with a connection form ω.

Conversely, if $(p : P \to M, \theta)$ is a G-structure with a connection ω, then $\kappa = \theta + \omega$ is a principal Cartan connection for the principal G-bundle $p : P \to M$. $\square$

2.5. Generalized Cartan connections. For a principal G-bundle $\pi : P \to M$ as in 2.2, if $\kappa \in \Omega^1(P; \mathfrak{h})^G$ is a G-equivariant extension of $\kappa_G : VP \to \mathfrak{g}$, we call it a *generalized principal $\mathfrak{h}/\mathfrak{g}$-Cartan connection*.

More general, let P be a smooth manifold, let $\mathfrak{h}$ be a Lie algebra with $\dim \mathfrak{h} = \dim P$. We then consider a free action of a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{h}$ on P, i.e. an injective Lie algebra homomorphism $\zeta : \mathfrak{g} \to \mathfrak{X}(P)$. A *generalized $\mathfrak{h}/\mathfrak{g}$-Cartan connection* κ on the $\mathfrak{g}$-manifold P is then a $\mathfrak{g}$-equivariant $\mathfrak{h}$-valued one form

$$\kappa \in \Omega^1(P; \mathfrak{h})^{\mathfrak{g}} := \{\varphi \in \Omega^1(P; \mathfrak{h}) : \mathcal{L}_{\zeta_X} \kappa = \mathrm{ad}(X) \circ \kappa \text{ for all } X \in \mathfrak{g}\}$$

which reproduces the generators of the $\mathfrak{g}$-fundamental vector fields on P: for all $X \in \mathfrak{g}$ we have $\kappa(\zeta_X(x)) = X$.

2.6. Generalized Cartan connections and principal connection forms. Let P be a smooth manifold with a free action of a Lie algebra $\mathfrak{g}$. In [5] we define the notion of a principal connection on P as follows: A principal connection form on P is a $\mathfrak{g}$-valued $\mathfrak{g}$-equivariant 1-form $\omega \in \Omega(P; \mathfrak{g})^{\mathfrak{g}}$ which reproduces the generators of the fundamental vector fields on P, so $\omega(\zeta_X) = X$ for $X \in \mathfrak{g}$.

As a generalization of proposition 2.3 we establish now relations between generalized Cartan connections and principal connection forms.

Proposition. *Let $\mathfrak{g}$ be a reductive subalgebra of a Lie algebra $\mathfrak{h}$ with reductive decomposition*

$$\mathfrak{h} = V \oplus \mathfrak{g}, \quad [\mathfrak{g}, V] \subset V.$$

Let $\kappa : TP \to \mathfrak{h}$ be a generalized Cartan connection on a $\mathfrak{g}$-manifold P with a free action of the Lie algebra $\mathfrak{g}$.

Then the $\mathfrak{g}$-component $\omega = \mathrm{pr}_{\mathfrak{g}} \circ \kappa$ is a principal connection form on the $\mathfrak{g}$-manifold P. In particular, κ defines a $\mathfrak{g}$-invariant horizontal distribution $\mathcal{H} := \kappa^{-1}(V) \subset TP$ which is a complementary subbundle of the 'vertical' distribution $\zeta_{\mathfrak{g}}(P) \subset P$ spanned by the $\mathfrak{g}$-action, and which is $\mathfrak{g}$-invariant:

$$[\zeta_{\mathfrak{g}}, \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H}),$$

where $\Gamma(\mathcal{H}) \subset \mathfrak{X}(P)$ is the space of section of the bundle $\mathcal{H}$.

Remark. It is a natural idea to consider the V-component $\theta = \mathrm{pr}_V \circ \kappa$ of κ as some analogon of the notion of displacement form. Clearly θ is $\mathfrak{g}$-equivariant and horizontal: $\ker\theta \supset \zeta_{\mathfrak{g}}(P)$. But it will be strictly horizontal ($\ker\theta = \zeta_{\mathfrak{g}}(P)$) if and only if κ is a Cartan connection. In general we only have $\ker\theta = \zeta_{\mathfrak{g}}(P) \oplus \mathcal{K}$, where $\mathcal{K} = \ker(\kappa|\mathcal{H})$ is a $\mathfrak{g}$-invariant distribution, possibly of non-constant rank.

2.7. Curvature and Bianchi identity. For a generalized Cartan connection $\kappa \in \Omega^1(P;\mathfrak{h})^{\mathfrak{g}}$ we define the *curvature* K by $K = d\kappa + \frac{1}{2}[\kappa,\kappa]^\wedge$, where we used the graded Lie bracket on $\Omega(P;\mathfrak{h})$ given in [5], 4.1 . From the graded Jacobi identity in $\Omega(P;\mathfrak{h})$ we get then easily the *Bianchi identity*

$$dK + [\kappa, K]^\wedge = 0.$$

Then K is *horizontal*, i.e. kills all ζ_X for $X \in \mathfrak{g}$, and is $\mathfrak{g}$-equivariant, $K \in \Omega^2_{\mathrm{hor}}(P;\mathfrak{h})^{\mathfrak{g}}$. If κ is a generalized principal Cartan connection on a principal G-bundle, then K is even G-equivariant, $K \in \Omega^2_{\mathrm{hor}}(P;\mathfrak{h})^G$.

If κ is a Cartan connection then an easy computation shows that

$$\zeta^\kappa(K(\zeta^\kappa_X, \zeta^\kappa_Y)) = [\zeta^\kappa_X, \zeta^\kappa_Y]_{\mathfrak{X}(P)} - \zeta^\kappa([X,Y]_{\mathfrak{h}}).$$

2.8. Covariant exterior derivative. For a generalized $\mathfrak{h}/\mathfrak{g}$-Cartan connection $\kappa \in \Omega^1(P;\mathfrak{h})^{\mathfrak{g}}$ and any representation $\rho : \mathfrak{h} \to GL(W)$ we define the *covariant exterior derivative*

$$d_\kappa : \Omega^p_{\mathrm{hor}}(P;W)^{\mathfrak{g}} \to \Omega^{p+1}_{\mathrm{hor}}(P;W)^{\mathfrak{g}}$$
$$d_\kappa \Psi = d\Psi + \rho^\wedge(\kappa)\Psi.$$

For a principal Cartan connection on a principal G-bundle we even have

$$d_\kappa(\Omega^p_{\mathrm{hor}}(P;W)^G) \subset \Omega^{p+1}_{\mathrm{hor}}(P;W)^G.$$

2.9. Chern-Weil forms. If $f \in L^k(\mathfrak{h}) := (\bigotimes^k \mathfrak{h}^*)$ is a k-linear function on $\mathfrak{h}$ and if $\psi_i \in \Omega^{p_i}(P;\mathfrak{h})$ we can construct the following differential forms

$$\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k \in \Omega^{p_1+\cdots+p_k}(P;\mathfrak{h}\otimes\cdots\otimes\mathfrak{h}),$$
$$f^{\psi_1,\dots,\psi_k} := f \circ (\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k) \in \Omega^{p_1+\cdots+p_k}(P).$$

The exterior derivative of the latter one is clearly given by

$$\begin{aligned} d(f \circ (\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k)) &= f \circ d(\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k) \\ &= f \circ \left(\textstyle\sum_{i=1}^k (-1)^{p_1+\cdots+p_{i-1}} \psi_1 \otimes_\wedge \cdots \otimes_\wedge d\psi_i \otimes_\wedge \cdots \otimes_\wedge \psi_k \right). \end{aligned}$$

Note that the form $f^{\psi_1,\dots,\psi_k}$ is $\mathfrak{g}$-invariant and horizontal if all $\psi_i \in \Omega^{p_i}_{\text{hor}}(P;\mathfrak{h})^{\mathfrak{g}}$ and $f \in L^k(\mathfrak{h})^{\mathfrak{g}}$ is invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{h}$. It is then the pullback of a form on M. For a principal Cartan connection one may replace $\mathfrak{g}$ by G.

2.10. Lemma. *Let κ be a generalized $\mathfrak{h}/\mathfrak{g}$-Cartan connection on P. Let $f \in L^k(\mathfrak{h})^{\mathfrak{h}}$ be $\mathfrak{h}$-invariant under the adjoint action then the differential form $f^K := f^{K,\dots,K}$ is closed in $\Omega^{2k}_{\text{hor}}(M)^{\mathfrak{g}}$.*

Proof. The same computation as in the proof of [5], 7.4 with ω and Ω replaced by κ and K. □

2.11. Proposition. *Let κ_0 and κ_1 be two generalized $\mathfrak{h}/\mathfrak{g}$-Cartan connections on P with curvature forms $K_0, K_1 \in \Omega^2(P;\mathfrak{h})^{\mathfrak{g}}$, and let $f \in L^k(\mathfrak{h})^{\mathfrak{h}}$. Then the cohomology classes of the two closed forms f^{K_0} and f^{K_1} in $H^{2k}(\Omega^*_{\text{hor}}(P)^{\mathfrak{g}})$ agree.*

If $P \to M$ is a principal G-bundle and if κ_1 and κ_2 are principal generalized Cartan connections on it, then the cohomology classes of the two closed forms f^{K_0} an f^{K_1} agree in $H^{2k}(M)$.

Proof. Literally the same proof as for [5], 7.5 applies, with ω and Ω replaced by κ and K. □

3. The relation between principal Cartan connections and principal connections

In this section we follow the notation and concepts of [14], Chapter III, which we also explain here.

3.1. Extension of the structure group. Given a principal bundle $\pi : P \to M$ with structure group G and $G \subset H$ we consider the left action of G on H (by left translation) and the associated bundle $\pi : P[H] = P \times_G H \to M$. Recall from [14], 10.7 the G-bundle projection $q : P \times H \to P[H] = P \times_G H$. Since $q(u.g,h) = q(u,gh)$ we get $Tq(Tr(X_u,Z_g),Y_h) = Tq(X_u, T\lambda_g.Y_h + T\rho_h.Z_g)$. This is then a principal H-bundle with principal H-action $\tilde{r} : P[H] \times H \to P[H]$ given by $\tilde{r}(q(u,h),h') = q(u,hh')$. Since $G \subset H$ is G-equivariant we get a homomorphism of principal bundles over $G \subset H$

$$\begin{array}{ccc} P & \longrightarrow & P[H] \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array}$$

3.2. Lemma. *In the situation of 3.1 the generalized Cartan connections in the space $\Omega^1(P;\mathfrak{h})^G$ correspond canonically and bijectively to the H-principal connections in $\Omega^1(P[H];\mathfrak{h})^H$.*

Proof. For $Y\in\mathfrak{h}$ the fundamental vector field $\zeta_Y^{P[H]}$ on PH is given by

$$\zeta_Y^{P[H]}(q(u,h)) = T(\tilde r)(Tq(0_u,0_h),Y) = T_{(u,h)}q(0_u,T\lambda_h.Y).$$

For a generalized Cartan connection $\kappa\in\Omega^1(P;\mathfrak{h})^G$ we define for $X_u\in T_uP$ and $Y\in\mathfrak{h}$:

$$(q^\flat\kappa): TP[H]\to\mathfrak{h}$$

$$(q^\flat\kappa)(T_{(u,h)}q(X_u,T_e\lambda_h.Y)) := Y + \operatorname{Ad}(h^{-1})\kappa_u(X_u) \tag{1}$$

We claim that $(q^\flat\kappa)\in\Omega^1(P[H];\mathfrak{h})^H$ is well defined and is a principal connection.

It is well defined: for $g\in G$ we have

$$\begin{aligned}(q^\flat\kappa)(T_{(u,h)}q(Tr^g.X_u,T\lambda_{g^{-1}}T_e\lambda_h.Y)) &= Y+\operatorname{Ad}((g^{-1}h)^{-1})\kappa_u(Tr^g.X_u)\\ &= Y+\operatorname{Ad}(h^{-1})\operatorname{Ad}(g)\operatorname{Ad}(g^{-1})\kappa_u(X_u)\\ &= (q^\flat\kappa)(T_{(u,h)}q(X_u,T_e\lambda_h.Y))\end{aligned}$$

Moreover $T_{(u,h)}q(X_u,Y_h)=0$ if and only if $(X_u,Y_h)=(\zeta_X^P(u),-T\rho_h.X)$ for some $X\in\mathfrak{g}$, but then

$$\begin{aligned}(q^\flat\kappa)(\zeta_X^P(u),-T\rho_h.X) &= (q^\flat\kappa)(\zeta_X^P(u),-T_e\lambda_h.\operatorname{Ad}(h^{-1})X)\\ &= -\operatorname{Ad}(h^{-1})X+\operatorname{Ad}(h^{-1})\kappa_u(\zeta_X^P(u))=0.\end{aligned}$$

We check that it is H-equivariant:

$$\begin{aligned}(q^\flat\kappa)(T(\tilde r^k).T_{(u,h)}q.(X_u,T_e\lambda_h.Y)) &= (q^\flat\kappa)(T_{(u,hk)}q.(X_u,T\rho_k.T_e\lambda_h.Y))\\ &= (q^\flat\kappa)(T_{(u,hk)}q.(X_u,T_e\lambda_{hk}.T\rho_k.T\lambda_{k^{-1}}.Y))\\ &= (q^\flat\kappa)(T_{(u,hk)}q.(X_u,T_e\lambda_{hk}.\operatorname{Ad}(k^{-1})Y))\\ &= \operatorname{Ad}(k^{-1})Y+\operatorname{Ad}(k^{-1}.h^{-1})\kappa_u(X_u)\\ &= \operatorname{Ad}(k^{-1})(q^\flat\kappa)(T_{(u,h)}q.(X_u,T_e\lambda_h.Y)).\end{aligned}$$

Next we check that it reproduces the infinitesimal generators of fundamental vector fields:

$$(q^\flat\kappa)(\zeta_Y^{P[H]}(q(u,h)) = (q^\flat\kappa)(T_{(u,h)}q(0_u,T\lambda_h.Y)) = Y$$

Now let $\omega \in \Omega^1(P[H];\mathfrak{h})^H$ be a principal connection form. Then the pull back $(q^\flat)^{-1}\omega$ of ω to the G-subbundle $P \subset p[H]$ is in $\Omega^1(P;\mathfrak{h})^G$ and clearly reproduces the infinitesimal generators of G-fundamental vector fields, so it is a generalized Cartan connection. Explicitely we have $((q^\flat)^{-1}\omega)(X_u) := \omega(T_{(u,e)}q(X_u, 0_e))$ and with this formula it is easy to check that the two construction are inverse to each other. □

3.3. Theorem. *Let $\pi : P \to M$ be a principal bundle with structure group G, let H be a Lie group containing G and let $\rho : H \to GL(W)$ be a finite dimensional linear representation of H.*

Then there is a canonical linear isomorphism

$$q^\flat : \Omega^p_{\text{hor}}(P;W)^G \to \Omega^p_{\text{hor}}(P[H];W)^H$$

which intertwines the covariant exterior derivatives of any generalized Cartan connection κ on P with values in $\mathfrak{h}$ and of its canonically associated principal connection $q^\flat\kappa$ on $P[H]$:

$$d_{q^\flat\kappa} \circ q^\flat = q^\flat \circ d_\kappa : \Omega^p_{\text{hor}}(P;W)^G \to \Omega^{p+1}_{\text{hor}}(P[H];W)^G$$

If $K \in \Omega^2_{\text{hor}}(P;\mathfrak{h})^G$ is the curvature of a generalized Cartan connection κ in the sense of 2.7 then $q^\flat K \in \Omega^2_{\text{hor}}(P[H];\mathfrak{h})^H$ is the principal curvature of the principal connection $q^\flat\kappa$ on $P[H]$.

Proof. For $\Psi \in \Omega^p_{\text{hor}}(P;\mathfrak{h})^G$ we define $q^\flat\Psi \in \Omega^p_{\text{hor}}(P[H];\mathfrak{h})^H$ by

$$\begin{aligned}(q^\flat\Psi)_{q(u,h)}(Tq(\xi^1_u, T_e\lambda_h.Y^1), Tq(\xi^2_u, T_e\lambda_h.Y^2), \dots) &= \\ = \operatorname{Ad}(h^{-1})\Psi_u(\xi^1_u, \xi^2_u, \dots).\end{aligned} \tag{1}$$

This is well defined and horizontal, since a vector $Tq(\xi_u, T\lambda_h.Y)$ is vertical in $P[H]$ if and only if it is of the form $Tq(\zeta^P_X(u), T_e\lambda_h(Z - \operatorname{Ad}(h^{-1})X))$ for some $Z \in \mathfrak{h}$ and $X \in \mathfrak{g}$, and the right hand side vanishes if one such vector appears in the left hand side. Note that $q^\flat\Psi$ is well defined only if Ψ is horizontal. It is easily seen that $q^\flat\Psi$ is H-equivariant.

If $\Phi \in \Omega^p_{\text{hor}}(P[H];\mathfrak{h})^H$ then the pull back of Φ to the subbundle P gives a form $(q^\flat)^{-1}\Phi \in \Omega^p_{\text{hor}}(P;\mathfrak{h})^G$. We have the explicit formula $((q^\flat)^{-1}\Phi)(\xi^1_u, \xi^2_u, \dots) = \Phi(Tq(\xi^1_u, 0_e), Tq(\xi^2_u, 0_e), \dots)$, and using this it is easy to show that the two constructions are inverse to each other:

$$\begin{aligned}((q^\flat)(q^\flat)^{-1}\Phi)_{q(u,h)}(Tq(\xi^1_u, T\lambda_h.Y^1), \dots) &= \operatorname{Ad}(h^{-1})((q^\flat)^{-1}\Phi)_u(\xi^1_u, \xi^2_u, \dots) \\ &= \operatorname{Ad}(h^{-1})\Phi(T_{(u,e)}q(\xi^1_u, 0_e), \dots) = \Phi(T(\tilde r^h).T_{(u,e)}q(\xi^1_u, 0_e), \dots) \\ &= \Phi(T_{(u,h)}q(\xi^1_u, T\lambda_h.0_e), \dots) = \Phi(T_{(u,h)}q(\xi^1_u, T\lambda_h.Y), \dots),\end{aligned}$$

since Φ is horizontal, and

$$((q^\flat)^{-1}(q^\flat)\Psi)_u(\xi_u^1,\xi_u^2,\dots)=((q^\flat)^{-1}\Psi)_{q(u,e)}(Tq(\xi_u^1,0_e),\dots)=\Psi_u(\xi_u^1,\dots).$$

Claim 1: $d_{q^\flat\kappa}\circ q^\flat = q^\flat\circ d_\kappa : \Omega^p_{\text{hor}}(P;W)^G\to\Omega^{p+1}_{\text{hor}}(P[H];W)^G$ holds for a generalized Cartan connection κ on P. Here $d_{q^\flat\kappa}$ is given by $d_{q^\flat\kappa}\Phi=\chi^*d\Phi$ for any form $\Phi\in\Omega(P[H],V)$ with values in a vector space V, where χ is the horizontal projection induced by $q^\flat\kappa$. In [4], 1.4 it is proved that for $\Phi\in\Omega_{\text{hor}}(P[H];W)^H$ the formula $d_{q^\flat\kappa}\Phi=d\Phi+[q^\flat\kappa,\Phi]^\wedge$ holds. On the other hand we have $d_\kappa\Psi=d\Psi+\rho^\wedge(\kappa)\Psi$ for $\Psi\in\Omega_{\text{hor}}(P;W)^G$ by Definition 2.8.

To compute $d(q^\flat\Psi)$ we need vector fields. So let $\xi_i\in\mathfrak{X}(P)^G$ be G-equivariant vector fields on P, and for $Y_i\in\mathfrak{h}$ let L_{Y_i} denote the left invariant vector field on H, $L_{Y_i}(h)=T\lambda_h.Y_i$. Then the vector field $\xi_i\times L_{Y_i}$ is G-equivariant and factors thus to a vector field on the associated bundle as indicated in the following diagram:

$$\begin{array}{ccc} P\times H & \xrightarrow{\ \xi_i\times L_{Y_i}\ } & TP\times TH \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Tq} \\ P[H]=P\times_G H & \xrightarrow{\ \widetilde{\xi_i\times L_{Y_i}}\ } & TP\times_{TG}TH=T(P[H]) \end{array}$$

So the vector fields $\xi_i\times L_{Y_i}$ on $P\times H$ and $\widetilde{\xi_i\times L_{Y_i}}$ on $P[H]$ are q-related and thus we have

$$[\widetilde{\xi_i\times L_{Y_i}},\widetilde{\xi_j\times L_{Y_j}}]=\widetilde{[\xi_i,\xi_j]\times L_{[Y_i,Y_j]}} \tag{2}$$

Now we compute

$$\begin{aligned}
&d(q^\flat\Psi)\left(\widetilde{\xi_0\times L_{Y_0}},\dots,\widetilde{\xi_p\times L_{Y_p}}\right)\\
&=\sum_{i=0}^p(-1)^i(\widetilde{\xi_i\times L_{Y_i}})\left(q^\flat\Psi\left(\widetilde{\xi_0\times L_{Y_0}},\dots,\widehat{\ \ }_i,\dots,\widetilde{\xi_p\times L_{Y_p}}\right)\right)\\
&+\sum_{i<j}(-1)^{i+j}(q^\flat\Psi)\left(\left[\widetilde{\xi_i\times L_{Y_i}},\widetilde{\xi_i\times L_{Y_i}}\right],\widetilde{\xi_0\times L_{Y_0}},\dots,\widehat{\ \ }_i,\dots,\widehat{\ \ }_j,\dots\right).
\end{aligned}$$

Since we have

$$\begin{aligned}
&(q^\flat\Psi)_{q(u,h)}\left(\widetilde{\xi_1\times L_{Y_1}},\dots,\widetilde{\xi_p\times L_{Y_p}}\right)\\
&=(q^\flat\Psi)_{q(u,h)}(Tq(\xi_1(u),T_e\lambda_h.Y_1),\dots,Tq(\xi_p(u),T_e\lambda_h.Y_p))\\
&=\operatorname{Ad}(h^{-1}).\Psi_u(\xi_1(u),\dots,\xi_p(u))\in\mathfrak{h}
\end{aligned} \tag{3}$$

we get

$$\begin{aligned}&(\widetilde{\xi_0 \times L_{Y_0}}) \left(q^\flat \Phi \left(\widetilde{\xi_1 \times L_{Y_1}}, \dots, \widetilde{\xi_p \times L_{Y_p}}\right)\right)(q(u,h)) \\ &= (T_h(\mathrm{Ad}\circ \mathrm{Inv}).T_e\lambda_h.Y_0).\Psi_u(\xi_1,\dots,\xi_p) + \mathrm{Ad}(h^{-1})(\xi_0\Psi(\xi_1,\dots,\xi_p)) \\ &= -[Y_0, \mathrm{Ad}(h^{-1}).\Psi_u(\xi_1,\dots,\xi_p)] + \mathrm{Ad}(h^{-1})(\xi_0\Psi(\xi_1,\dots,\xi_p)).\end{aligned}$$

Inserting we get

$$\begin{aligned}d(q^\flat\Psi)&\left(\widetilde{\xi_0 \times L_{Y_0}}, \dots, \widetilde{\xi_p \times L_{Y_p}}\right)(q(u,h)) \\ &= -\sum_{i=0}^p (-1)^i [Y_i, \mathrm{Ad}(h^{-1}).\Psi_u(\xi_0,\dots,\widehat{\xi_i},\dots \xi_p)]+ \\ &\quad + \mathrm{Ad}(h^{-1}).(d\Psi)_u(\xi_0,\dots,\xi_p).\end{aligned}$$

Next we compute

$$\begin{aligned}&[q^\flat\kappa, q^\flat\Psi]^\wedge \left(\widetilde{\xi_0 \times L_{Y_0}}, \dots, \widetilde{\xi_p \times L_{Y_p}}\right)(q(u,h)) \\ &= \sum_{i=0}^p (-1)^i \left[(q^\flat\kappa)_{q(u,h)}\left(\widetilde{\xi_i \times L_{Y_i}}\right), (q^\flat\Psi)_{(q(u,h))}\left(\widetilde{\xi_0 \times L_{Y_0}}, \dots, \widehat{}_i, \dots\right)\right] \\ &= \sum_{i=0}^p (-1)^i [Y_i + \mathrm{Ad}(h^{-1})\kappa_u(\xi_i), \mathrm{Ad}(h^{-1})\Psi_u(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_p)]_{\mathfrak{h}} \\ &= \sum_{i=0}^p (-1)^i [Y_i, \mathrm{Ad}(h^{-1}).\Psi_u(\xi_0,\dots,\widehat{\xi_i},\dots \xi_p)] \\ &\quad + \mathrm{Ad}(h^{-1}).[\kappa,\Psi]^\wedge(\xi_0,\dots,\xi_p)(u).\end{aligned}$$

On the other hand we have

$$\begin{aligned}&(q^\flat \circ d_\kappa\Psi)_{(q(u,h))}\left(\widetilde{\xi_0 \times L_{Y_0}}, \dots, \widetilde{\xi_p \times L_{Y_p}}\right) \\ &\quad = \mathrm{Ad}(h^{-1}).(d_\kappa\Psi)_u(\xi_0,\dots,\xi_p) = \mathrm{Ad}(h^{-1}).(d\Psi + [\kappa,\Psi])_u(\xi_0,\dots,\xi_p),\end{aligned}$$

so the claim follows by comparing the last three expressions.

Claim 2: $q^\flat K = d(q^\flat\kappa) + \frac{1}{2}[q^\flat\kappa, q^\flat\kappa]^\wedge$ for a generalized Cartan connection κ on P with curvature $K = d\kappa + \frac{1}{2}[\kappa,\kappa]^\wedge$.

We have $K \in \Omega^2_{\mathrm{hor}}(P;\mathfrak{h})^G$ but κ is not horizontal, so we must redo parts of the above computations. We use the same vector fields as in the proof of Claim 1. Since by 3.2,(1) we have

$$(4)\qquad \begin{aligned}(q^\flat\kappa)_{q(u,h)}\left(\widetilde{\xi_1\times L_{Y_1}}\right) &= (q^\flat\kappa)_{q(u,h)}(Tq(\xi_1(u),T_e\lambda_h.Y_1))\\ &= Y_1+\operatorname{Ad}(h^{-1}).\kappa_u(\xi_1)\end{aligned}$$

we get again the same formula as for Ψ

$$\begin{aligned}&\widetilde{\xi_0\times L_{Y_0}}(q^\flat\kappa)_{q(u,h)}\left(\widetilde{\xi_1\times L_{Y_1}}\right)\\ &\qquad = -[Y_0,\operatorname{Ad}(h^{-1})\kappa_u(\xi_1)]+\operatorname{Ad}(h^{-1})(\xi_0\kappa(\xi_1))(u).\end{aligned}$$

This leads to

$$\begin{aligned}&d(q^\flat\kappa)\left(\widetilde{\xi_0\times L_{Y_0}},\widetilde{\xi_1\times L_{Y_1}}\right)\\ &\quad = -[Y_0,\operatorname{Ad}(h^{-1})\kappa_u(\xi_1)]+[Y_1,\operatorname{Ad}(h^{-1})\kappa_u(\xi_0)]-[Y_0,Y_1]\\ &\qquad +\operatorname{Ad}(h^{-1})(d\kappa_u(\xi_0,\xi_1)).\end{aligned}$$

Again from (4) we get

$$\begin{aligned}&\tfrac12[q^\flat\kappa,q^\flat\kappa]^\wedge\left(\widetilde{\xi_0\times L_{Y_0}},\widetilde{\xi_1\times L_{Y_1}}\right)(q(u,h))\\ &\qquad = \tfrac12[Y_0+\operatorname{Ad}(h^{-1})\kappa_u(\xi_0),Y_1+\operatorname{Ad}(h^{-1})\kappa_u(\xi_1)]\\ &\qquad\quad -\tfrac12[Y_1+\operatorname{Ad}(h^{-1})\kappa_u(\xi_1),Y_0+\operatorname{Ad}(h^{-1})\kappa_u(\xi_0)]\\ &\qquad = [Y_0,\operatorname{Ad}(h^{-1})\kappa_u(\xi_1)]-[Y_1,\operatorname{Ad}(h^{-1})\kappa_u(\xi_0)]+[Y_0,Y_1]\\ &\qquad\quad +\tfrac12\operatorname{Ad}(h^{-1}).[\kappa,\kappa]^\wedge(\xi_0,\xi_1)(u)\end{aligned}$$

from which now the result follows. □

3.4. Corollary. *The characteristic class for an invariant $f\in L^k(\mathfrak{h})^H$ constructed in Proposition 2.11 with the help of generalized Cartan connections on P is exactly the characteristic class of the principal bundle $P[H]$ associated to f. Since $P[H]$ admits a reduction of the structure group to G, this class is a characteristic class of P, associated to $f|\mathfrak{g}\in L^k(\mathfrak{g})^G$. If $f|\mathfrak{g}=0$ then the form f^K of Proposition 2.11 is exact.*

Proof. This follows from well known properties of characteristic classes of principal bundles. □

4. Flat Cartan connections

4.1. Flat Cartan connections. Let P be a smooth manifold. A Cartan connection $\kappa : TP \to \mathfrak{h}$ is said to be flat if its curvature $K = d\kappa + \frac{1}{2}[\kappa, \kappa]^\wedge$ (see 2.7) vanishes. In this case the subalgebra $\mathfrak{g} \subset \mathfrak{h}$ does not play any role. The inverse mapping $\zeta : \mathfrak{h} \to \mathfrak{X}(P)$, given by $\zeta_X(x) = (\kappa_x)^{-1}(X)$ is then a homomorphism of Lie algebras, and it defines a free transitive action of the Lie algebra $\mathfrak{h}$ on the manifold P in the sense of [5], 2.1. The inverse statement is also valid, see [5], 5.1.

A flat generalized Cartan connection is then a form $\kappa : TP \to \mathfrak{h}$ which satisfies the Maurer-Cartan equation $d\kappa + \frac{1}{2}[\kappa, \kappa]^\wedge$ (without the assumtion that it is non-degenerate).

4.2. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$, multiplication $\mu : H \times H \to H$, and for $g \in H$ let $\mu_g, \mu^g : H \to H$ denote the left and right translation, $\mu(g, h) = g.h = \mu_g(h) = \mu^h(g)$. For a smooth mapping $\varphi : P \to H$ let us use the left trivialization of TH and consider the *left logarithmic derivative* $\delta^l\varphi \in \Omega^1(P; \mathfrak{h})$, given by $\delta^l\varphi_x := T(\mu_{\varphi(x)^{-1}}) \circ T_x\varphi : T_xP \to T_{\varphi(x)}H \to \mathfrak{h}$. Similarly we consider the *right logarithmic derivative* $\delta^r\varphi \in \Omega^1(P; \mathfrak{h})$ which is given by $\delta^r\varphi_x := T(\mu^{\varphi(x)^{-1}}) \circ T_x\varphi : T_xP \to T_{\varphi(x)}H \to \mathfrak{h}$. The following result can be found in [17], [18], [19], or in [11] (proved with moving frames); see also [5], 5.2. We include a simple conceptual proof and we consider all variants.

Proposition. *For a smooth mapping $\varphi : P \to H$ the left logarithmic derivative $\delta^l\varphi \in \Omega^1(P; \mathfrak{h})$ satisfies the (right) Maurer-Cartan equation $d\delta^l\varphi + \frac{1}{2}[\delta^l\varphi, \delta^l\varphi]^\wedge = 0$.*

If conversely a 1-form $\kappa \in \Omega^1(P; \mathfrak{h})$ satisfies $d\kappa + \frac{1}{2}[\kappa, \kappa]^\wedge = 0$ then for each simply connected subset $U \subset P$ there exists a smooth function $\varphi : U \to H$ with $\delta^l\varphi = \kappa|U$, and φ is uniquely detemined up to a right translation in H.

For a smooth mapping $\varphi : P \to H$ the right logarithmic derivative $\delta^r\varphi \in \Omega^1(P; \mathfrak{h})$ satisfies the (left) Maurer-Cartan equation $d\delta^r\varphi - \frac{1}{2}[\delta^r\varphi, \delta^r\varphi]^\wedge = 0$.

If a 1-form $\kappa \in \Omega^1(P; \mathfrak{h})$ satisfies $d\kappa - \frac{1}{2}[\kappa, \kappa]^\wedge = 0$ then for each simply connected subset $U \subset P$ there exists a smooth function $\varphi : U \to H$ with $\delta^r\varphi = \kappa|U$, and φ is uniquely determined up to a left translation in H.

Proof. Let us treat first the right logarithmic derivative since it leads to a principal connection for a bundle with right principal action. We consider the trivial principal bundle $\mathrm{pr}_1 : P \times H \to P$ with right principal action. Then the submanifolds $\{(x,\varphi(x).g) : x \in P\}$ for $g \in H$ form a foliation of $P \times G$ whose tangent distribution is transversal to the vertical bundle $P \times TH \subset T(P \times H)$ and is invariant under the principal right H-action. So it is the horizontal distribution of a principal connection on $P \times H \to H$. For a tangent vector $(X_x, Y_g) \in T_xP \times T_gH$ the horizontal part is the right translate to the foot point (x, g) of $(X_x, T_x\varphi.X_x)$, so the decomposition in horizontal and vertical parts according to this distribution is

$$(X_x, Y_g) = (X_x, T(\mu^g).T(\mu^{\varphi(x)^{-1}}).T_x\varphi.X_x) \\ + (0_x, Y_g - T(\mu^g).T(\mu^{\varphi(x)^{-1}}).T_x\varphi.X_x).$$

Since the fundamental vector fields for the right action on H are the left invariant vector fields, the corresponding connection form is given by

$$\omega^r(X_x, Y_g) = T(\mu_{g^{-1}}).(Y_g - T(\mu^g).T(\mu^{\varphi(x)^{-1}}).T_x\varphi.X_x),$$
$$\omega^r_{(x,g)} = T(\mu_{g^{-1}}) - \mathrm{Ad}(g^{-1}).\delta^r\varphi_x,$$
$$\omega^r = \kappa^l_H - (\mathrm{Ad}\circ\mathrm{Inv}).\delta^r\varphi, \tag{1}$$

where $\kappa^l_H : TH \to \mathfrak{h}$ is the left Maurer-Cartan form on H (the left trivialization), given by $(\kappa^l_H)_g = T(\mu_{g^{-1}})$. Note that κ^l_H is the principal connection form for the (unique) principal connection $p : H \to$ point with right principal action, which is flat so that the right (from right action) Maurer-Cartan equation holds in the form

$$d\kappa^l + \tfrac{1}{2}[\kappa^l, \kappa^l]^\wedge = 0. \tag{2}$$

The principal connection ω^r is flat since we got it via the horizontal leaves, so the principal curvature form vanishes:

$$\begin{aligned} 0 &= d\omega^r + \tfrac{1}{2}[\omega^r, \omega^r]^\wedge \\ &= d\kappa^l_H + \tfrac{1}{2}[\kappa^l_H, \kappa^l_H]^\wedge - d(\mathrm{Ad}\circ\mathrm{Inv}) \wedge \delta^r\varphi - (\mathrm{Ad}\circ\mathrm{Inv}).d\delta^r\varphi \\ &\quad - [\kappa^l_H, (\mathrm{Ad}\circ\mathrm{Inv}).\delta^r\varphi]^\wedge + \tfrac{1}{2}[(\mathrm{Ad}\circ\mathrm{Inv}).\delta^r\varphi, (\mathrm{Ad}\circ\mathrm{Inv}).\delta^r\varphi]^\wedge \\ &= -(\mathrm{Ad}\circ\mathrm{Inv}).(d\delta^r\varphi - \tfrac{1}{2}[\delta^r\varphi, \delta^r\varphi]^\wedge), \end{aligned} \tag{3}$$

where we used (2) and since for $X \in \mathfrak{g}$ we have:

$$d(\mathrm{Ad}\circ\mathrm{Inv})(T(\mu_g)X) = \tfrac{\partial}{\partial t}\big|_0 \mathrm{Ad}(\exp(-tX).g^{-1}) = -\mathrm{ad}(X)\,\mathrm{Ad}(g^{-1})$$
$$= -\mathrm{ad}(\kappa^l_H(T(\mu_g)X))(\mathrm{Ad}\circ\mathrm{Inv})(g),$$
$$d(\mathrm{Ad}\circ\mathrm{Inv}) = -(\mathrm{ad}\circ\kappa^l_H)(\mathrm{Ad}\circ\mathrm{Inv}). \tag{4}$$

So we have $d\delta^r\varphi - \frac{1}{2}[\delta^r\varphi, \delta^r\varphi]^\wedge$ as asserted.

If conversely we are given a 1-form $\kappa^r \in \Omega^1(P;\mathfrak{h})$ with $d\kappa^r - \frac{1}{2}[\kappa^r, \kappa^r]^\wedge = 0$ then we consider the 1-form $\omega^r \in \Omega^1(P \times H;\mathfrak{h})$, given by the analogon of (1),

$$\omega^r = \kappa_H^l - (\mathrm{Ad} \circ \mathrm{Inv}).\kappa^r \ . \tag{5}$$

Then ω^r is a principal connection form on $P \times H$, since it reproduces the generators in $\mathfrak{h}$ of the fundamental vector fields for the principal right action, i.e. the left invariant vector fields, and ω^r is H-equivariant:

$$\begin{aligned}((\mu^g)^*\omega^r)_h &= \omega^r_{hg} \circ (Id \times T(\mu^g)) = T(\mu_{g^{-1}.h^{-1}}).T(\mu^g) - \mathrm{Ad}(g^{-1}.h^{-1}).\kappa^r \\ &= \mathrm{Ad}(g^{-1}).\omega^r_h.\end{aligned}$$

The computation in (3) for κ^r instead of $\delta^r\varphi$ shows that this connection is flat. So the horizontal bundle is integrable, and $\mathrm{pr}_1 : P \times H \to P$, restricted to each horizontal leaf, is a covering. Thus it may be inverted over each simply connected subset $U \subset P$, and the inverse $(Id, \varphi) : U \to P \times H$ is unique up to the choice of the branch of the covering, and the choice of the leaf, i.e. φ is unique up to a right translation by an element of H. The beginning of this proof then shows that $\delta^r\varphi = \kappa^r|U$.

For the left logarithmic derivative $\delta^l\varphi$ the proof is similar, and we discuss only the essential deviations. First note that on the trivial principal bundle $\mathrm{pr}_1 : P \times H \to P$ with left principal action of H the fundamental vector fields are the right invariant vector fields on H, and that for a principal connection form ω^l the curvature form is given by $d\omega^l - \frac{1}{2}[\omega^l, \omega^l]^\wedge$. Look at the proof of [14], 11.2 to see this. The connection form is then given by

$$\omega^l = \kappa_H^r - \mathrm{Ad}\,.\delta^l\varphi, \tag{1'}$$

where the right Maurer-Cartan form $(\kappa_H^r)_g = T(\mu^{g^{-1}}) : T_gH \to \mathfrak{h}$ now satifies the left Maurer-Cartan equation

$$d\kappa_H^r - \frac{1}{2}[\kappa_H^r, \kappa_H^r]^\wedge = 0. \tag{2'}$$

Flatness of ω^l now leads to the computation

$$\begin{aligned}0 &= d\omega^l - \tfrac{1}{2}[\omega^l, \omega^l]^\wedge \\ &= d\kappa_H^r - \tfrac{1}{2}[\kappa_H^r, \kappa_H^r]^\wedge - d\,\mathrm{Ad} \wedge \delta^l\varphi - \mathrm{Ad}\,.d\delta^l\varphi \\ &\quad + [\kappa_H^r, \mathrm{Ad}\,.\delta^l\varphi]^\wedge - \tfrac{1}{2}[\mathrm{Ad}\,.\delta^l\varphi, \mathrm{Ad}\,.\delta^l\varphi]^\wedge \\ &= -\mathrm{Ad}\,.(d\delta^l\varphi + \tfrac{1}{2}[\delta^l\varphi, \delta^l\varphi]^\wedge),\end{aligned} \tag{3'}$$

where we used

$$d\,\mathrm{Ad}(T(\mu^g)X) = \tfrac{\partial}{\partial t}\big|_0 \mathrm{Ad}(\exp(tX).g) = \mathrm{ad}(X)\,\mathrm{Ad}(g)$$
$$= \mathrm{ad}(\kappa^r_H(T(\mu^g)X))\,\mathrm{Ad}(g),$$

(4')
$$d\,\mathrm{Ad} = (\mathrm{ad}\circ\kappa^r_H)\,\mathrm{Ad}\,.$$

The rest of the proof is obvious. □

4.3. Characteristic classes for flat Cartan connections. A generalized Cartan connection $\kappa : TP \to \mathfrak{h}$ on the manifold P induces a homomorphism

$$\kappa^* : \Lambda(\mathfrak{h}^*) \to \Omega(P),$$
$$f \mapsto f^\kappa = f \circ (\kappa \otimes_\wedge \cdots \otimes_\wedge \kappa)$$

of the algebra of exterior forms on $\mathfrak{h}$ into the algebra of differential forms on P. Let us assume now that Cartan connection κ is flat. Then κ^* commutes with the exterior differentials and is a homomorphism of differential complexes, since we have by 2.9

$$d(f(\kappa,\dots,\kappa)) = \sum_{i=1}^{k}(-1)^{i-1} f(\kappa,\dots,d\kappa,\dots,\kappa)$$
$$= \sum_{i=1}^{k}(-1)^{i-1} f(\kappa,\dots,-\tfrac{1}{2}[\kappa,\kappa]^\wedge,\dots,\kappa)$$
$$= (df)(\kappa,\dots,\kappa) \quad k+1 \text{ times.}$$

Thus we have an associated homomorphism $\kappa^* : H^*(\mathfrak{h},\mathbb{R}) \to H^*(P)$. The nontrivial elements of its image are called *characteristic classes* for the flat Cartan connection κ. In [6] a similar construction is applied even for infinite dimensional manifolds.

4.4. Let $\kappa : TP \to \mathfrak{h}$ be a flat generalized Cartan connection on the manifold P, so $d\kappa + \frac{1}{2}[\kappa,\kappa]^\wedge = 0$ holds. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$. Suppose that there exists a smooth mapping $\varphi : P \to H$ with $\delta^l\varphi = \kappa$. By Proposition 4.2 such a φ is unique up to a right translation in H, and it exists if e.g. P is simply connected. Clearly φ is a local diffeomorphism if and only if κ is a Cartan connection (non degenerate). Such a mapping φ is called *Cartan's developing*, see also [5], 5.2. It gives a convenient way to express the characteristic classes of 4.3, by the following easy results.

Lemma. *Let $\kappa : TP \to \mathfrak{h}$ be a flat generalized Cartan connection such that a Cartan's developing $\varphi : P \to H$ exists. Then the following diagram commutes*

$$\begin{array}{ccc} \Lambda^k(\mathfrak{h}^*) & \xrightarrow{\kappa^*} & \Omega^k(P) \\ & {\scriptstyle L}\searrow \quad \nearrow{\scriptstyle \varphi^*} & \\ & \Omega^k(H) & \end{array}$$

where L is the extension to left invariant differential forms.

Proof. Plug in the definitions. □

4.5. Cartan connections with constant curvature. Let $\kappa : TP \to \mathfrak{h}$ be a Cartan connection of type $\mathfrak{h}/\mathfrak{g}$ on a manifold P. Then its curvature belongs to the space $\Omega^2(P;\mathfrak{h}) \cong \mathfrak{h}\otimes\Omega^2(P)$. Using the absolute parallelism on P defined by κ we may associate with κ the function

$$\begin{aligned} k :&P \to \mathfrak{h}\otimes\Lambda^2\mathfrak{h}^* \\ k(u)(X,Y) :&= K(\zeta_X(u),\zeta_Y(u)) \text{ for } u\in P \text{ and } X,Y\in\mathfrak{h}. \end{aligned}$$

We say that the Cartan connection κ has *constant curvature* if this function k is constant.

5. Flat Cartan connections associated with a flat G-structure

5.1. G-structures. By a G-structure on a smooth finite dimensional manifold M we mean a principal fiber bundle $p : P \to M$ together with a representation $\rho : G \to GL(V)$ of the structure group in a real vector space V of dimension $\dim M$ and a 1-form σ (called the *soldering form*) on M with values in the associated bundle $P[V,\rho] = P\times_G V$ which is fiber wise an isomorphism and identifies T_xM with $P[V]_x$ for each $x\in M$. Then σ corresponds uniquely to a G-equivariant 1-form $\theta\in\Omega^1_{\text{hor}}(P;V)^G$ which is strongly horizontal in the sense that its kernel is exactly the vertical bundle VP. The form θ is called the *displacement form* of the G-structure.

If the representation $\rho : G\to GL(V)$ is faithful so that $G\subset GL(V)$ is a linear Lie group, then a G-structure (P,p,M,G,V,θ) is a subbundle of the linear frame bundle $GL(V;TM)$ of M. To see this recall the projection $q : P\times V \to P\times_G V = P[V]$ onto the associated bundle and the mapping $\tau = \tau^V : P\times_M P[V] \to V$ which is uniquely given by $q(u_x,\tau(u_x,v_x)) = v_x$

and which satisfies $\tau(u_x, q(u_x, v)) = v$ and $\tau(u_x.g, v_x) = \rho(g^{-1})\tau(u_x, v_x)$, see [14], 10.7. Then we have the smooth mapping over the identity on M,

$$\begin{aligned} (1) \qquad & P \to GL(V; P[V]) \xrightarrow{\sigma^{-1}} GL(V; TM), \\ & u_x \mapsto \tau(u_x, \quad)^{-1}, \end{aligned}$$

which is G-equivariant and thus an embedding.

Let $p : P \to M$ and $p' : P' \to M'$ be two G-structures with the same representation $\rho : G \to GL(V)$, with soldering forms σ, σ', and with displacement forms θ, θ', respectively. We are going to define the notion of an *isomorphism of G-structures*.

For $G \subset GL(V)$ an isomorphism of G-structures is a diffeomorphism $\varphi : M \to M'$ such that the natural prolongation $GL(V; T\varphi) : GL(V; TM) \to GL(V; TM')$ to the frame bundles maps the subbundle $P \subset GL(V; TM)$ to the subbundle $P' \subset GL(V; TM')$.

In the general case an isomorphism of G-structures with the same representation $\rho : G \to GL(V)$ is an isomorphism of principal G-bundles

$$(2) \qquad \begin{array}{ccc} P & \xrightarrow{\bar\varphi} & P' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow[\varphi]{} & M \end{array}$$

whose induced isomorphism $\bar\varphi \times_G \mathrm{Id}_{\rho(G)}$ of of principal $\rho(G)$-bundles coincides on $P \times_G \rho(G) \subset GL(V, TM)$ with the restriction of the natural prologation $GL(V; T\varphi)$ of φ to the linear frame bundle, and which preserves the displacement forms: $\bar\varphi^*\theta' = \theta' \circ T\bar\varphi = \theta : TP \to V$.

A G-structure (P, p, M, G, V, θ) is called *flat* if it is locally (in a neighborhood of any point $x \in M$) isomorphic to the standard flat G-structure $\mathrm{pr}_1 : V \times G \to V$ with displacement form $d\,\mathrm{pr}_1 : T(V \times G) \to V$. Then the soldering form is just the identity $\sigma = \mathrm{Id} : TV = V \times V \to V \times V$, the linear frame bundle is $GL(V; TV) = V \times GL(V)$ and the associated $\rho(G)$-bundle is the subbundle $V \times \rho(G) \subset V \times GL(V)$.

The standard examples of flat G-structures are foliations with structure group

$$\begin{pmatrix} GL(p) & * \\ 0 & GL(n-p) \end{pmatrix},$$

and symplectic structures.

Suppose that $G = \rho(G) \subset GL(V)$ and let us consider a local diffeomorphism defined near 0 and respecting 0 in V (we write $\varphi : V, 0 \to V, 0$).

It is a local automorphism of the standard flat G-structure $\mathrm{pr}_1 : V \times G \to V$ with displacement form $d\,\mathrm{pr}_1 : T(V \times G) \to V$ if and only if the following condition holds:

(3) $d\varphi(x) : V \to V$ is in $G \subset GL(V)$ for all x in the domain of φ,

because only then its natural prolongation $GL(V; T\varphi)$ to the linear frame bundle maps G-frames to G-frames.

5.2. The infinite dimensional Lie group $GL^\infty(V)$. Let V be a real vector space of dimension n, and let $J^\infty(V,V)$ be the linear space of all infinite jets of smooth mappings $V \to V$, equipped with the initial topology with respect to all projections $J^\infty(V,V) \to J^k(V,V)$, which is a nuclear Fréchet space topology.

We shall use the calculus of Frölicher and Kriegl in infinite dimensions, see [9], [15], [16], where a smooth mapping is one which maps smooth curves to smooth curves. On the spaces which we are going to use here the smooth mappings with values in finite dimensional spaces are just those which locally factor over some finite dimensional quotient like $J^k(V,V)$ and are smooth there.

Then we consider the closed linear subspace $\mathfrak{gl}_\infty(V) \subset J^\infty(V,V)$ of infinite jets of smooth mappings $V \to V$ which map the origin to the origin. Note that composition is defined on $\mathfrak{gl}_\infty(V)$ and is smooth, but is linear only in one (the left) component. Then we consider the open subset $GL_\infty(V)$ of all infinite jets of local diffeomorphisms of V, defined near and respecting 0. This is a smooth Lie group in the sense that composition and inversion are smooth. Its Lie algebra is $\mathfrak{gl}_\infty(V)$ which we may view as the full prolongation

(1) $$\mathfrak{gl}_\infty(V) = \mathfrak{gl}(V) \times \mathfrak{gl}^1(V) \times \mathfrak{gl}^2(V) \times \mathfrak{gl}^3(V) \times \dots ,$$

where $\mathfrak{gl}^k(V) = S^k V^* \otimes V$ is the space of homogeneous polynomials $V \to V$ of order k. One may view $\mathfrak{gl}_\infty(V)$ also as the vector space $\{j^\infty_0 X : X \in \mathfrak{X}(V), X(0) = 0\}$ with the bracket

$$[j^\infty_0 X, j^\infty_0 Y] = -j^\infty_0 [X,Y]$$

and with the smooth (unique) exponential mapping $\exp : \mathfrak{gl}_\infty(V) \to GL_\infty(V)$ given by

$$\exp(j^\infty_0 X) = j^\infty_0(\mathrm{Fl}^X_1),$$

where Fl_t^X is the flow of the vector field X on V. It is well known that $\exp : \mathfrak{gl}_\infty(V) \to GL_\infty(V)$ is not surjective onto any neighborhood of the identity, see [23].

See [14], Section 13, for a detailed discussion of the finite jet groups $GL_k(V)$; the book [16] will contain a thorough discussion of $GL_\infty(V)$.

5.3. The infinite prolongation of a linear Lie group G and its Lie algebra. Let $G \subset GL(V)$ be a closed linear Lie group. We denote by $G_\infty \subset GL_\infty(V)$ the subgroup of all infinite jets $j_0^\infty \varphi$ of local automorphisms φ of the standard flat G-structure $\mathrm{pr}_1 : V \times G \to V$, defined near 0 and respecting 0. Note that these φ are exactly the local diffeomorphisms $\varphi : V, 0 \to V, 0$ such that $d\varphi(x) \in G \subset GL(V)$ for all $x \in V$ near 0, by the discussion in 5.1. Then G_∞ is a group with multiplication and inversion

$$\begin{aligned} j_0^\infty \varphi \circ j_0^\infty \psi &:= j_0^\infty (\varphi \circ \psi), \\ (j_0^\infty \varphi)^{-1} &= j_0^\infty (\varphi^{-1}), \end{aligned}$$

respectively. We will not address the question here in which sense G_∞ is a Lie group. We continue just on a formal level.

The infinitesimal automorphisms respecting 0 of the standard flat G-structure on V are then those local vector fields X defined near 0 and vanishing at 0 in V whose local flows Fl_t^X consist of automorphisms of the G-structure.

Lemma. *The infinitesimal automorphisms are exactly the vector fields X defined near 0 and vanishing near 0 such that $dX(x) \in \mathfrak{g} \in L(V, V)$, where $\mathfrak{g}$ is the Lie algebra of G.*

Proof. Namely, $c(t) = d(\mathrm{Fl}_t^X)(x)$ is a curve in $G \subset GL(V)$ if and only if the following expression lies in $\mathfrak{g}$:

$$\begin{aligned} c'(t).c(t)^{-1} &= \frac{d}{dt}(d(\mathrm{Fl}_t^X)(x)).d(\mathrm{Fl}_t^X)(x)^{-1} = d(\frac{d}{dt}\,\mathrm{Fl}_t^X)(x).d(\mathrm{Fl}_t^X)(x)^{-1} \\ &= d(X \circ \mathrm{Fl}_t^X)(x).d(\mathrm{Fl}_t^X)(x)^{-1} \\ &= d(X(\mathrm{Fl}_t^X)(x)). \qquad \square \end{aligned}$$

We consider now the infinite jets $j_0^\infty X$ of all these infinitesimal automorphisms respecting 0. These jets form a sub vector space $\mathfrak{g}_\infty \subset \mathfrak{gl}_\infty(V)$ which we may view as the full prolongation

$$\mathfrak{g}_\infty = \mathfrak{g} \times \mathfrak{g}^2 \times \mathfrak{g}^3 \times \mathfrak{g}^4 \times \dots , \tag{1}$$

where $\mathfrak{g}^k = \mathfrak{g}_\infty \cap S^k V^* \otimes V$ is the space of homogeneous polynomials $V \to V$ of order k in $\mathfrak{g}_\infty$. Then $\mathfrak{g}_\infty$ is a Lie algebra with the bracket

$$[j_0^\infty X, j_0^\infty Y] = -j_0^\infty [X,Y]$$

and with the smooth (unique) exponential mapping $\exp : \mathfrak{g}_\infty \to G_\infty$ given by

$$\exp(j_0^\infty X) = j_0^\infty (\mathrm{Fl}_1^X),$$

where Fl_t^X is the flow of the vector field X on V. We expect that in general $\exp : \mathfrak{g}_\infty \to G_\infty$ is not surjective onto any neighborhood of the identity.

Now we consider the Lie algebra of all infinitesimal automorphisms of the standard flat G-structure $\mathrm{pr}_1 : V \times G \to V$, i.e. all local vector fields X defined near 0 in V such that the local flows Fl_t^X are automorphisms. As above one sees that these are the vector fields X with $dX(x) \in \mathfrak{g} \subset \mathfrak{gl}(V)$ for all x, without the restriction that they should vanish at 0. Let $\mathfrak{a}_\infty$ be the Lie algebra of all infinite jets $j_0^\infty X$ of such fields, again with bracket

$$[j_0^\infty X, j_0^\infty Y] = -j_0^\infty [X,Y].$$

By decomposing into monomials we have again

$$\mathfrak{a}_\infty = V \times \mathfrak{g} \times \mathfrak{g}^2 \times \mathfrak{g}^3 \times \mathfrak{g}^4 \times \ldots = V \oplus \mathfrak{g}_\infty. \tag{2}$$

We have an adjoint representation $\mathrm{Ad} : G_\infty \to \mathrm{Aut}(\mathfrak{a}_\infty)$ which is given by

$$\mathrm{Ad}(j_0^\infty \varphi) j_0^\infty X = j_0^\infty (\varphi^* X) = j_0^\infty (T\varphi^{-1} \circ X \circ \varphi)$$

In a formal sense we have also the left Maurer-Cartan form on G_∞. First let us define the tangent bundle TG_∞ as the set of all $(j_0^\infty \varphi_0, j_0^\infty \frac{d}{dt}|_0 \varphi_t)$ where φ_t is a smooth curve of local automorphisms of the standard flat G-structure, respecting 0, smooth in the sense that $(t,x) \mapsto \varphi_t(x)$ is smooth. Then we define the *left* Maurer-Cartan form $\kappa^l_{G_\infty}$ by

$$\kappa^l_{G_\infty}(j_0^\infty \varphi_0, j_0^\infty \tfrac{d}{dt}|_0 \varphi_t) := j_0^\infty (T\varphi_0^{-1} \circ \tfrac{d}{dt}|_0 \varphi_t). \tag{3}$$

5.4. Proposition. *Let $\theta_0 = d\,\mathrm{pr}_1 : T(V \times G) \to V$ be the displacement form of the standard flat G-structure. Then*

$$\kappa_0 := \theta_0 \oplus \kappa^l_{G_\infty} : T(V \times G_\infty) = TV \times TG_\infty \to V \oplus g_\infty = \mathfrak{a}_\infty$$

is a flat Cartan connection on the manifold $V \times G_\infty$ with values in the Lie algebra $\mathfrak{a}_\infty$.

PROOF. Note first that the left Maurer-Cartan form $\kappa^l_{G_\infty}$ given in 5.3.(3) is really a trivialization of the tangent bundle TG_∞ because of the lemma in 5.3, and we show that it satisfies the Maurer-Cartan equation:

Let X (and later also Y) be a local vector field defined near 0 and vanishing at 0, which is an infinitesimal automorphism of the standard flat G-structure, so that $j_0^\infty X$ is a typical element in $\mathfrak{g}_\infty$. Let $j_0^\infty \varphi \in G_\infty$ be a typical element, so $\varphi : V, 0 \to V, 0$ is a local automorphism. Then

$$L_{j_0^\infty X} : G_\infty \ni j_0^\infty \varphi \mapsto T(j_0^\infty \varphi) \circ j_0^\infty X = j_0^\infty (T\varphi \circ X)$$

is the left invariant vector field on G_∞ generated by $j_0^\infty X$. We have

$$[L_{j_0^\infty X}, L_{j_0^\infty Y}] = L_{[j_0^\infty X, j_0^\infty Y]} = L_{-j_0^\infty [X,Y]},$$

and the Maurer-Cartan equation follows as usual:

$$\begin{aligned}(d\kappa^l_{G_\infty})(L_{j_0^\infty X}, L_{j_0^\infty Y}) &= 0 - 0 - \kappa^l_{G_\infty}(L_{[j_0^\infty X, j_0^\infty Y]}) = -[j_0^\infty X, j_0^\infty Y] = \\ &= -[\kappa^l_{G_\infty}(L_{j_0^\infty X}), \kappa^l_{G_\infty}(L_{j_0^\infty Y})] = -\frac{1}{2}[\kappa^l_{G_\infty}, \kappa^l_{G_\infty}]^\wedge(L_{j_0^\infty X}, L_{j_0^\infty Y}).\end{aligned}$$

One may now easily carry over to κ_0 this result. □

5.5. The infinite prolongation of a flat G-structure. Let again $G \subset GL(V)$ be a linear Lie group, and let (P, p, M, G, V, θ) be a flat G-structure. We denote by $p_\infty : P^\infty \to M$ the *infinite prolongation* of this G-structure which is defined as follows:

The total space P^∞ is the space of all infinite jets $j_0^\infty \varphi$ of local isomorphisms $\varphi : V, 0 \to M$ of the standard flat G-structure onto the given one. Then obviously the group G_∞ acts freely from the right on P^∞, and also transitive on the fiber. We have the mapping

$$\begin{gathered}\tau^{P^\infty} : P^\infty \times_M P^\infty \to G_\infty \\ \tau^{P^\infty}(j_0^\infty \varphi, j_0^\infty \psi) := j_0^\infty (\varphi^{-1} \circ \psi)\end{gathered}$$

(see [14], 10.2) describing the principal G_∞-bundle structure, which is locally isomorphic to the trivial bundle $V \times G_\infty$. The local isomorphisms $\varphi : V, 0 \to M$ induce on P^∞ a flat Cartan connection

$$\kappa : TP^\infty \to \mathfrak{a}_\infty = V \oplus \mathfrak{g}_\infty$$

which locally is just given as the push forward via $j_0^\infty \varphi$ of the canonical flat Cartan connection κ_0 on $V \times G_\infty$.

6. The canonical Cartan connection for a G-structure of first or second order

6.1. Let $G \subset GL(V)$ be a linear Lie group with Lie algebra $\mathfrak{g}$. We assume that the G-module $V \otimes \Lambda^2 V^*$ admits a decomposition

$$V \otimes \Lambda^2 V^* = \delta(\mathfrak{g} \otimes V^*) \oplus \mathfrak{d} \tag{1}$$

where $\delta : V \otimes V^* \otimes V^* \to V \otimes \Lambda^2 V^*$ is the Spencer operator of alternation, and where $\mathfrak{d}$ is a G-submodule.

We now recall the definition of the torsion function of a G-structure and the construction of its first prolongation in the sense of [22]. Let $p : P \to M$ be a G-structure on a manifold M with a displacement form $\theta : TP \to V$, see 5.1. The 1-jet $j^1_x s$ of a local section $s : M \supset U \to P$ near $x \in M$ may be identified with its image $j^1_x s(T_x M) = H$, a horizontal linear subspace $H \subset T_{s(x)} P$. So the first jet bundle $J^1 P \to P$ may be identified with the space of all horizontal linear subspaces in fibers of $TP \to P$.

Then the restriction $\theta|H$ to such a horizontal space of the displacement form θ is a linear isomorphism $\theta|H : H \to V$ and we may use it to define the torsion function

$$t : J^1(P) \to V \otimes \Lambda^2 V^*$$
$$t(H)(v, w) := d\theta((\theta|H)^{-1}(v), (\theta|H)^{-1}(w))$$

We consider $P^1 := t^{-1}(\mathfrak{d})$. It is a sub fiber bundle of $J^1(P)$ and the abelian vector group $G^1 := \mathfrak{g} \otimes V^* \cap V \otimes S^2 V^* \subset \operatorname{Hom}(V, \mathfrak{g})$ acts on P^1 freely by $g^1 : P^1 \ni H \mapsto g^1(H) := \{h + \zeta^P_{g^1(\theta(h))}(p(H)) : h \in H\}$ where $\zeta^P : \mathfrak{g} \to \mathfrak{X}(P)$ is the fundamental vector field mapping. The orbits of P^1 under this G^1-action are fibers of the natural projection $p^1 : P^1 \to P$, hence $p^1 : P^1 \to P$ becomes a principal G^1-bundle.

Moreover, there exists a natural displacement form θ^1 on P^1. In order to define it we denote by $\Phi_H : T_u P \to V_u P$ the projection onto the vertical bundle $V_u P$ along the horizontal subspace $H \subset T_u P$. Then we have a well defined $\mathfrak{g}$-valued p^1-horizontal 1-form $\omega \in \Omega^1(P^1; \mathfrak{g})^G$ given by $\zeta^P_{\omega_H(X)}(p^1(H)) = \Phi_H(T_H(p^1).X)$. It is part of the *universal connection form* on the bundle of all connections $J^1(P)$. The 1-form

$$\theta^1 = \omega + \theta \circ T(p^1) : TP^1 \to \mathfrak{g} \ltimes V$$

with values in the semidirect product $\mathfrak{g} \ltimes V$ is the desired displacement form. It is equivariant with respect to the free action of the semidirect product $G \ltimes G^1$, where G^1 acts on $\mathfrak{g} \times V$ by $(g^1, (X, v)) \mapsto (X + g^1(v), v)$. So we have proved the main parts of

6.2. Lemma. *The fibration $p_1 : P^1 \to M$ is a principal fiber bundle with structure group $G \ltimes G^1$.*

The fibration $p^1 : P^1 \to P$ is a principal bundle with structure group G^1 and a G^1-structure on P with the displacement form θ^1. Moreover the form $\theta^1 : TP^1 \to \mathfrak{g} \ltimes V$ is $G \ltimes G^1$-equivariant.

6.3. ***G*-structures of type 1.** We assume now that the first prolongation G^1 of the group G is trivial. Any G-structure with such a structure group G is then called a *G-structure of type 1.* In this case the projection $p^1{:}P^1{\to}P$ is a diffeomorphism and the displacement form $\theta^1 = \omega + (p^1)^*\theta$ may be identified with a G-equivariant 1-form on P with values in $V_1 := \mathfrak{g} \ltimes V$. This form is a Cartan connection in the principal bundle $p : P \to M$ of type $(\mathfrak{g} \ltimes V)/\mathfrak{g}$.

Proposition. *Let $G \subset GL(V)$ be a linear Lie group of type 1 satisfying condition 6.1.(1). Then for any G-structure (P, p, M, G, θ) the displacement form θ^1 of the first prolongation $p^1 : P^1 \to P$ defines a Cartan connection on P of type $(\mathfrak{g} \ltimes V)/\mathfrak{g}$.*

6.4. ***G*-structures of type 2.** We say that $G \subset GL(V)$ is a *linear Lie group of type 2*, if the first prolongation $G^1 \subset GL(V_1)$ is not trivial, but the second prolongation $(G^1)^1$ is trivial. So we have

$$\mathfrak{g} \otimes V^* \cap V \otimes S^2V^* \neq 0, \quad \mathfrak{g} \otimes S^2V^* \cap V \otimes S^3V^* = 0.$$

We assume also that the Condition 6.1.(1) is satisfied.

Let (P, p, M, G, θ) be a G-structure and let $p^1 : P^1 \to P$ be its first prolongation with the displacement form $\theta^1 : TP^1 \to V_1 = \mathfrak{g} \ltimes \mathfrak{g}$. We note that the G^1-submodule $\delta(\mathfrak{g}^1 \otimes V_1^*)$ of $V_1 \otimes \Lambda^2 V_1^*$ is not a direct summand. However, we may assume (at least when G is reductive) that there exists a G-submodule $\mathfrak{d}^1$ such that

$$V_1 \otimes \Lambda^2 V_1^* = \delta(\mathfrak{g}^1 \otimes V_1^*) \oplus \mathfrak{d}^1. \tag{1}$$

This is the case if there exist G-submodules $\mathfrak{d}_1$, $\mathfrak{d}_2$, $\mathfrak{d}_3$ such that

$$\begin{aligned} \mathfrak{g} \otimes \Lambda_2 V_* &= R(\mathfrak{g}) \oplus \mathfrak{d}_1, \\ R(\mathfrak{g}) &= \delta(\mathfrak{g}^1 \otimes V^*) \oplus \mathfrak{d}_2, \\ \mathfrak{g} \otimes V^* &= \mathfrak{g}^1 \oplus \mathfrak{d}_3, \end{aligned}$$

where $R(\mathfrak{g})$ is the space of curvature tensors of type $\mathfrak{g}$, i.e. the space of closed (with respect to the Spencer differential) $\mathfrak{g}$-valued 2-forms, and where $\mathfrak{d}_2$ may be identified with the second Spencer cohomology space.

We denote by $t_1 : J^1P^1 \to V_1 \otimes \Lambda^2 V_1^*$ the torsion function of the G^1-structure $p^1 : P^1 \to P$. The inverse image $P^2 = t_1^{-1}(\mathfrak{d}^1)$ defines a submanifold of J^1P^1. The natural projection $p^2 : P^2 \to P^1$ is a diffeomorphism since by assumption the second prolongation G^2 of the group G is trivial. In other words, we have a canonical field of horizontal subspaces in TP^1. Note that it is not a principal connection since it is not invariant under the group G^1. Using this field of horizontal subspaces in TP^1 we may extend the canonical vertical parallelism $VP^1 \to \mathfrak{g} \ltimes \mathfrak{g}^1$ of the principal $G \ltimes G^1$-bundle $p_1 : P^1 \to M$ to a G-quivariant 1-form ω_1 on P^1. The $\mathfrak{g}$-component of ω_1 is the $\mathfrak{g}$-valued 1-form ω from 6.1.

The form

$$\theta^2 = (p^1)^*\theta + \omega_1 : TP^1 \to V \times \mathfrak{g} \times \mathfrak{g}^1 = \mathfrak{a}_\infty$$

is non degenerate, G-equivariant, and it prolongs the vertical parallelism $VP^1 \to \mathfrak{g} \ltimes \mathfrak{g}^1$ of the principal $G \ltimes G^1$-bundle $p_1 : P^1 \to M$. Hence it is a Cartan connection of type $\mathfrak{a}_\infty/(\mathfrak{g} \ltimes \mathfrak{g}^1)$, where $\mathfrak{a}_\infty = V \times \mathfrak{g} \times \mathfrak{g}^1$ is the full prolongation of the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$.

Summarizing we have

6.5. Proposition. *Let $p : P \to M$ be a G-structure of type 2 satisfying the Conditions 6.1.*(1) *and 6.4.*(1)*.*

Then on the total space P^1 of the first prolongation $p^1 : P^1 \to P$ of the bundle $p : P \to M$ there exists a canonically defined Cartan connection of type $\mathfrak{a}_\infty/\mathfrak{g}$, where $\mathfrak{a}_\infty = V \times \mathfrak{g} \times \mathfrak{g}^1$ is the full prolongation of the Lie algebra $\mathfrak{g}$ of G.

References

[1] D. V. Alekseevsky, Conformal mappings of G-structures, *Funct. Anal. Appl.* **22** (1988), 311–312.

[2] D. V. Alekseevsky and M. M. Graev, Twistors and Cartan connections, (*Preprint*).

[3] D. V. Alekseevsky and S. Marchiafava, Quaternionic structures on a manifold, *Rend. Mat. Accad. Lincei, Serie 9* **4** (1993).

[4] D. V. Alekseevsky and P. W. Michor, Characteristic classes for G-structures, *Diff. Geom. Appl.* **3** (1993), 323–329.

[5] D. V. Alekseevsky and P. W. Michor, Differential Geometry of $\mathfrak{g}$-manifolds, *Diff. Geom. Appl.*

[6] I. N. Bernstein and B. I. Rozenfeld, Homogeneous spaces of infinite dimensional Lie algebras and characteristic classes of foliations, *Uspechi Mat. Nauk* **284** (1973), 101–138.

[7] E. Cartan, Les groupes des transformations continues, infinie, simple, *Ann. Sci. École Norm. Sup.* **26** (1909), 93–161.

[8] E. Cartan, Les espaces à connexion conforme, *Ann. Soc. Polon. Math.* **2** (1923), 171–221.

[9] A. Frölicher and A. Kriegl, Linear spaces and differentiation theory, Pure and Applied Mathematics, *J. Wiley*, *Chichester*, 1988.

[10] I. M. Gelfand and D. N. Kazhdan, Some questions of differential geometry and the computation of the cohomology of the Lie algebras of vector fields, *Dokl. Akad. Nauk SSSR* **200** (1971), 269–272; English, *Sov. Math. Dokl.* **12** (1971), 1367–1370.

[11] P. Griffiths, On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry, *Duke Math. J.* **41** (1974), 775–814.

[12] V. Guillemin, The integrability problem for G-structures, *Trans. AMS* **116** (1965), 544–560.

[13] S. Kobayashi, Transformation groups in differential geometry, Ergebnisse der Mathematik, 70, *Springer-Verlag*, *Berlin, Heidelberg, New York*, 1972.

[14] I. Kolář, J. Slovák and P. W. Michor, Natural operators in differential geometry, *Springer-Verlag*, *Heidelberg, Berlin, New York*, 1993.

[15] A. Kriegl and P. W. Michor, Aspects of the theory of infinite dimensional manifolds, *Diff. Geom. Appl.* **1** (1991), 159–176.

[16] A. Kriegl and P. W. Michor, Foundations of Global Analysis, (*A book in preparation*).

[17] A. L. Onishchik, On the classification of fiber spaces, *Sov. Math. Doklady* **2** (1961), 1561–1564.

[18] A. L. Onishchik, Connections with zero curvature and the de Rham theorem, *Sov. Math. Doklady* **5** (1964), 1654–1657.

[19] A. L. Onishchik, Some concepts and applications of non-abelian cohomology theory, *Trans. Moscow Math. Soc.* **17** (1967), 49–98.

[20] R. Palais, On the existence of slices for actions of non-compact Lie groups, *Ann. of Math. (2)* **73** (1961), 295–323.

[21] P. Stefan, Accessible sets, orbits and, foliations with singularities, *Proc. London Math. Soc.* **29** (1974), 699–713.

[22] S. Sternberg, Lectures on differential geometry, *Prentice-Hall*, *Englewood Cliffs, N. J.*, 1964.

[23] S. Sternberg, Infinite Lie groups and formal aspects of dynamical systems, *J. Math. Mech.* **10** (1961), 451–474.

[24] H. J. Sussman, Orbits of families of vector fields and integrability of distributions, *Trans. AMS* **180** (1973), 171–188.

D. V. ALEKSEEVSKY
GEN. ANTONOVA 2 KV 99
117279 MOSCOW, B–279
RUSSIA

P. W. MICHOR
INSTITUT FÜR MATHEMATIK
UNIVERSITÄT WIEN
STRUDLHOFGASSE 4, A–1090 WIEN
AUSTRIA

(Received December 21, 1994)