Publ. Math. Debrecen
53 / 3-4 (1998), 293–307

On the geometry of curves in Finsler manifolds

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Abstract. By using the horizontal covariant derivative induced by the Cartan connection of a Finlser manifold $F^{(m+1)}$ on a curve C, we obtain the Frenet frame and write down the Frenet equations for C. Then we prove a fundamental theorem for curves in Finsler manifolds and obtain theorems on the reduction of codimension of C in $F^{(m+1)}$.

Introduction

In 1925, a few years after the dissertation of Finsler on generalized Riemannian manifolds (latter called Finsler manifolds), Taylor [9] has constructed a Frenet frame for a curve in a Finsler manifold. However, since then the theory was not embraced by many people, in spite of the fact that Finsler geometry was intensively studied in the last 30 years.

The purpose of the present paper is to initiate a study of geometry of curves in Finsler manifolds based on both the modern theory of Finsler connections (cf. Matsumoto [6]) and the theory of Finsler subspaces via the vertical vector bundle (cf. Bejancu [2], [3]). We are dealing with the geometry of a curve C in a Finsler manifold $F^{(m+1)}$ endowed with the Cartan connection. First we show that the vertical covariant derivative along C does not provide any Frenet frame for C. Then using the horizontal covariant derivative along C we obtain the Frenet frame and write down the Frenet equations. The main results are presented in the second section of the paper and they refer to the fundamental theorem for curves in Finsler manifolds (Theorem 2.1) and theorems on the reduction of the codimension of the imbedding of a curve in $F^{(m+1)} = (R^{m+1}, F)$ (Theorems 2.2 and 2.3).

Mathematics Subject Classification: 53B40, 53B25.
Key words and phrases: Frenet frame, Frenet equations, Finsler manifolds.

1. Frenet frame for a curve in a Finsler manifold

Let $F^{(m+1)} = (M, F)$ be a Finsler manifold, where M is a real $(m+1)$-dimensional smooth manifold and F is the fundamental function of $F^{(m+1)}$ (cf. Matsumoto [6], p. 101). Usually, F is not presumed to be smooth on the whole tangent bundle TM but on an open subset of TM.

Suppose that C is a curve (1-dimensional submanifold) of M, locally given by the equations

$$x^i = x^i(t), \quad t \in U,\ i \in \{0, \dots, m\},$$

where U is an open subset of R and $(\frac{dx^0}{dt}, \dots, \frac{dx^m}{dt}) \neq (0, \dots, 0)$ on U.

As it is well known (cf. Spivak [8]) the arc length parameter plays an important role in studying the geometry of curves in a Riemannian manifold. In our case, since $F(x, y)$ is positive homogeneous of degree 1, the arc length parameter s on C is defined by

$$s = \int_{t_0}^{t} F\left(x^i(t), \frac{dx^i}{dt}\right) dt.$$

From now on, we suppose that C is locally given by the equations

$$x^i = x^i(s); \ \left(\frac{dx^0}{ds}, \dots, \frac{dx^m}{ds}\right) \neq (0, \dots, 0), \tag{1.1}$$

where $s \in (-\varepsilon, \varepsilon)$ is the above arc length parameter.

We denote by i the imbedding of C in M and consider the differential mapping $di : TC \to TM$, where TC and TM are the tangent bundles of C and M respectively. Locally, a point of TC with coordinates (s, v) is carried by di into a point of TM with coordinates $(x^i(s), y^i(s, v))$, where we set

$$y^i(s, v) = v\frac{dx^i}{ds}. \tag{1.2}$$

Remark 1.1. As F is not presumed to be smooth on the zero section of TM, we are dealing with geometric objects along C at points $(x(s), y(s, v))$ with $v \neq 0$. Hence without loss of generality we may suppose $v > 0$ for any point of the coordinate neighbourhood in the study.

Next, using (1.1) and (1.2) we deduce that the natural fields of frames $\{\frac{\partial}{\partial s}, \frac{\partial}{\partial v}\}$ and $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ on manifolds TC and TM respectively, are related by

$$\frac{\partial}{\partial s} = \frac{dx^i}{ds}\,\frac{\partial}{\partial x^i} + v\frac{d^2x^i}{ds^2}\,\frac{\partial}{\partial y^i}, \tag{1.3}$$

and

$$\frac{\partial}{\partial v} = \frac{dx^i}{ds}\,\frac{\partial}{\partial y^i}. \tag{1.4}$$

Throughout the paper we use indices $i, j, k, \ldots \in \{0, \ldots, m\}$. Also we make use of Einstein convention, that is, repeated indices with one upper and one lower index denote the summation over their range. By $\Im(TM)$ and $\Gamma(E)$ we denote the algebra of smooth functions on TM and the $\Im(TM)$-module of smooth sections of a vector bundle E over TM. By VTM we denote the vertical vector bundle of M, that is, $VTM = \ker d\pi$, where π is the natural projection of TM on M.

Further we recall that the fundamental function F of $F^{(m+1)}$ induces a Riemannian metric g on VTM, whose local components are given by (cf. Abate–Patrizio [1, p. 26], and Bejancu [2, p. 20])

$$g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}(x, y) = \frac{1}{2}\,\frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{1.5}$$

For any Finsler vector field X, that is $X \in \Gamma(VTM)$ we denote by $\|X\|$ the norm of X given by

$$\|X\| = \{g(X, X)\}^{\frac{1}{2}}\,.$$

As a consequence of (1.4) we deduce that the vertical vector bundle VTC of C (locally spanned by $\frac{\partial}{\partial v}$) is a vector subbundle of $VTM_{|TC}$. Moreover, using (1.4) and (1.5) we deduce that $\frac{\partial}{\partial v}$ is a unit Finsler vector field, that is, we have

$$\left\|\frac{\partial}{\partial v}\right\|^2 = g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = g_{ij}\big(x(s), x'(s)\big)\frac{dx^i}{ds}\frac{dx^j}{ds} = 1. \tag{1.6}$$

Denote by $VTC^{\perp}$ the orthogonal complementary vector bundle to VTC in $VTM_{|TC}$ and call it, as in the theory of Finsler submanifolds (cf. Bejancu [2, p. 47]), the *Finsler normal bundle* of the imbedding of C in

$F^{(m+1)}$. Thus we have the orthogonal decomposition

$$VTM_{|TC} = VTC \oplus VTC^{\perp}.$$

In the present paper we consider $F^{(m+1)}$ endowed with the Cartan connection $FC = (C^i_j, F^i_{jk}, C^i_{jk})$ with local coefficients

$$C^i_j = \frac{\partial G^i}{\partial y^j}, \quad \text{where } G^i = \frac{1}{4} g^{ih} \left\{ \frac{\partial^2 F^2}{\partial y^h \partial x^k} y^k - \frac{\partial F^2}{\partial x^h} \right\}, \tag{1.7}$$

$$F^i_{jk} = \frac{1}{2} g^{ih} \left\{ \frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h} \right\}, \tag{1.8}$$

$$C^i_{jk} = \frac{1}{2} g^{ih} \frac{\partial g_{hj}}{\partial y^k}, \tag{1.9}$$

where we use the differential operators

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - C^j_i \frac{\partial}{\partial y^j}, \tag{1.10}$$

and g^{ih} are the elements of the inverse matrix corresponding to $[g_{ih}]$. Clearly, both F^i_{jk} and C^i_{jk} are symmetric with respect to the pair of indices (i, j).

As a consequence of homogeneity of F it follows that g_{ij}, g^{ij} and F^i_{jk} are positive homogeneous of degree zero, while G^i, C^i_j and C^i_{jk} are positive homogeneous of degrees $2, 1,$ and -1 respectively. Thus we have:

$$g_{ij}(x(s), vx'(s)) = g_{ij}(x(s), x'(s)), \tag{1.11}$$

$$\begin{aligned} x'^j(s) F^i_{jk}(x(s), vx'(s)) &= x'^j(s) F^i_{jk}(x(s), x'(s)) \\ &= C^i_k(x(s), x'(s)), \end{aligned} \tag{1.12}$$

$$\begin{aligned} x'^j(s) C^i_j(x(s), vx'(s)) &= vx'^j(s) C^i_j(x(s), x'(s)) \\ &= 2vG^i(x(s), x'(s)), \end{aligned} \tag{1.13}$$

$$x'^j(s) C^i_{jk}(x(s), x'(s)) = 0, \tag{1.14}$$

and

$$C^i_{jk}(x(s), vx'(s)) = \frac{1}{v} C^i_{jk}(x(s), x'(s)). \tag{1.15}$$

Here and in the sequel we use the vector notation $x(s)$ and $x'(s)$ to represent the vectors $(x^0(s), \ldots, x^m(s))$ and $(x'^0(s), \ldots, x'^m(s))$, respectively, where the primes denote the derivatives with respect to s. Also if $T^{ij\ldots}_{kh\ldots}$ are the components of a geometric object T at the point $(x(s), x'(s))$ we denote them by $T^{ij\ldots}_{kh\ldots}(s)$.

In another setting, the Cartan connection is thought of as the pair (HTM, ∇), where HTM is the complementary distribution to VTM in TTM locally represented by $\{\frac{\delta}{\delta x^0}, \ldots, \frac{\delta}{\delta x^m}\}$ and ∇ is the linear connection on VTM whose local coefficients are given by (1.8) and (1.9), that is, we have (cf. Bejancu [2, p. 23])

$$\nabla_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} = F^i_{jk} \frac{\partial}{\partial y^i}, \tag{1.16}$$

and

$$\nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = C^i_{jk} \frac{\partial}{\partial y^i}. \tag{1.17}$$

A Finsler vector field X on $F^{(m+1)}$ along C is said to be *projectible on C*, if locally at any point $(x(s), vx'(s)) \in TC$ it is expressed as follows

$$X(x(s), vx'(s)) = X^i(s) \frac{\partial}{\partial y^i}(x(s), vx'(s)), \tag{1.18}$$

or equivalently, the local components of X at any point of TC depend only on the arc length parameter s of C. The name is justified by the fact that X given by (1.18) on TC defines a vector field X^* on M at any point of C by the formula

$$X^*(x(s)) = X^i(s) \frac{\partial}{\partial x^i}(x(s)).$$

Thus $X^*(x(s))$ can be considered as the projection of the vector $X(x(s), vx'(s))$ on the tangent space $TM_{x(s)}$ of M at $x(s) \in C$. Clearly from (1.4) we deduce that $\frac{\partial}{\partial v}$ is a projectible Finsler vector field. Moreover, we shall see in this section that a Frenet frame for a curve in a Finsler manifold contains only projectible Finsler vector fields.

Proposition 1.1. *The vertical covariant derivative of a projectible Finsler vector field X vanishes identically on TC, that is we have*

$$\left(\nabla_{\frac{\partial}{\partial v}} X\right)(x(s), vx'(s)) = 0, s \in (-\varepsilon, \varepsilon). \tag{1.19}$$

Proof. Indeed, using (1.18), (1.4), (1.17) and (1.15) we infer

$$\left(\nabla_{\frac{\partial}{\partial v}} X\right)(x(s), vx'(s)) = X^i(s)x'^j(s)C^k_{ij}(x(s), vx'(s))\frac{\partial}{\partial y^k} = 0.$$

Thus in particular we have

$$\nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v} = 0, \tag{1.20}$$

which enables us to state that *the vertical covariant derivative along C does not provide any Frenet frame for C*. Hence we have to proceed with the horizontal covariant derivative along C. To this end we consider the induced non-linear connection HTC on TC which is locally represented by the vector field

$$\frac{\delta}{\delta s} = \frac{\partial}{\partial s} - f\frac{\partial}{\partial v}, \tag{1.21}$$

where f is obtained from the general formula (3.2) in Bejancu [3] for a Finsler manifold. More precisely, using (1.13) we obtain

$$f(s, v) = vg_{ij}(s)\left(x''^i(s) + 2G^i(s)\right)x'^j(s). \tag{1.22}$$

Then (1.20) and (1.21) yield

$$\nabla_{\frac{\delta}{\delta s}} \frac{\partial}{\partial v} = \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v}. \tag{1.23}$$

Next, using (1.3), (1.10) and (1.13) one obtains

$$\frac{\partial}{\partial s} = x'^i(s)\frac{\delta}{\delta x^i} + v\left(x''^i(s) + 2G^i(s)\right)\frac{\partial}{\partial y^i}. \tag{1.24}$$

Replacing $\frac{\partial}{\partial s}$ from (1.24) in (1.23) and taking into account (1.16), (1.17) and (1.12)–(1.14) we infer

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = \left(x''^i(s) + 2G^i(s)\right)\frac{\partial}{\partial y^i}. \tag{1.25}$$

On the other hand, using (1.6) and taking into account that ∇ is a metric connection we deduce

$$g\left(\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = 0. \tag{1.26}$$

Using (1.25) and (1.4) in (1.26) and comparing with (1.22) we arrive at $f = 0$. Hence we may state the following proposition.

Proposition 1.2. *The induced non-linear connection HTC on C is locally spanned by $\frac{\partial}{\partial s}$ given by equivalent formulas* (1.3) *and* (1.24).

From (1.26) we deduce that $\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} \in \Gamma(VTC^{\perp})$ and therefore we may set

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = \kappa_1 N_1, \tag{1.27}$$

where $N_1 \in \Gamma(VTC^{\perp})$ is a unit Finsler vector field and

$$\kappa_1 = \left\| \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} \right\|. \tag{1.28}$$

Using (1.25) in (1.28) we deduce that

$$\kappa_1(s) = \left\{ g_{ij}(s) \left(x''^i(s) + 2G^i(s) \right) \left(x''^j(s) + 2G^j(s) \right) \right\}^{\frac{1}{2}}. \tag{1.29}$$

We call κ_1 the *geodesic curvature* (*first curvature*) *function* of C. In order to justify the name geodesic curvature function we recall that C is a geodesic of $F^{(m+1)}$ if and only if the following differential equations are satisfied (cf. Rund [7, p. 79])

$$x''^i(s) + 2G^i(s) = 0. \tag{1.30}$$

Thus (1.29) and (1.30) enable us to state the following result (compare Rund [7, p. 152]).

Theorem 1.1. *A curve C in $F^{(m+1)}$ is a geodesic if and only if the geodesic curvature function vanishes identically on C.*

If $\kappa_1(s) \neq 0$, for all $s \in (-\varepsilon, \varepsilon)$ we call

$$N_1 = \frac{1}{\kappa_1(s)} \left(x''^i(s) + 2G^i(s) \right) \frac{\partial}{\partial y^i} = N_1^i(s) \frac{\partial}{\partial y^i}, \tag{1.31}$$

the *principal* (*first*) *normal* of C. From (1.31) we see that N_1 is a projectible Finsler vector field along C. This is also a consequence of the following general result.

Proposition 1.3. *The horizontal covariant derivative of a projectible Finsler vector field X with respect to $\frac{\partial}{\partial s}$ is a projectible Finsler vector field too given by*

$$\nabla_{\frac{\partial}{\partial s}} X = \left(\frac{dX^i}{ds} + X^j(s) B^i_j(s) \right) \frac{\partial}{\partial y^i}, \tag{1.32}$$

where we set

$$B^i_j(s) = C^i_j(s) + \left(x''^k(s) + 2G^k(s) \right) C^i_{jk}(s). \tag{1.33}$$

Proof. The assertion follows by direct calculations using (1.24), (1.16), (1.17), (1.12) and (1.15).

In case $m > 1$ from $g(N_1, N_1) = 1$ and $g(\frac{\partial}{\partial v}, N_1) = 0$ we deduce that $g(\nabla_{\frac{\partial}{\partial s}} N_1, N_1) = 0$ and $g(\nabla_{\frac{\partial}{\partial s}} N_1, \frac{\partial}{\partial v}) = -\kappa_1$. Thus we have

$$\nabla_{\frac{\partial}{\partial s}} N_1 = -\kappa_1 \frac{\partial}{\partial v} + N, \tag{1.34}$$

where $N \in \Gamma(VTC^{\perp})$ and it is orthogonal to N_1 at any point of TC. Next we define the *second curvature function* κ_2 by $\kappa_2 = \left\| \kappa_1(s) \frac{\partial}{\partial v} + \nabla_{\frac{\partial}{\partial s}} N_1 \right\|$. Using (1.32), (1.33) and (1.31) we derive

$$\kappa_2(s) = \left\{ g_{ij}(s) \left(\kappa_1 x'^i + N_1'^i + N_1^k B^i_k \right) \left(\kappa_1 x'^j + N_1'^j + N_1^h B^j_h \right) \right\}^{\frac{1}{2}} (s). \tag{1.35}$$

If $\kappa_2(s) \neq 0$ for any $s \in (-\varepsilon, \varepsilon)$ we define

$$N_2(s) = \frac{1}{\kappa_2(s)} \left(\kappa_1(s) \frac{\partial}{\partial v} + \nabla_{\frac{\partial}{\partial s}} N_1 \right).$$

Hence (1.34) becomes

$$\nabla_{\frac{\partial}{\partial s}} N_1 = -\kappa_1 \frac{\partial}{\partial v} + \kappa_2 N_2.$$

Now from the Proposition 1.3 it follows that N_2 is a projectible Finsler vector field along C. Finally, we suppose inductively for $1 \leq i \leq m$, that there exist orthonormal projectible vector fields $\left\{N_0 = \frac{\partial}{\partial v}, N_1, \dots, N_i\right\}$ and nowhere zero curvature functions $\{\kappa_1, \dots, \kappa_i\}$ such that

$$\begin{array}{llll}
(\mathrm{F}_1) & \nabla_{\frac{\partial}{\partial s}} N_0 & = & \kappa_1 N_1 \\
(\mathrm{F}_2) & \nabla_{\frac{\partial}{\partial s}} N_1 & = & -\kappa_1 N_0 + \kappa_2 N_1 \\
\dots & \dots & \dots & \dots \\
(\mathrm{F}_i) & \nabla_{\frac{\partial}{\partial s}} N_{i-1} & = & -\kappa_{i-1} N_{i-2} + \kappa_i N_i .
\end{array}$$

Then following the proof in case of Riemannian manifolds (cf. Spivak [8, Vol. IV, p. 92]) and taking into account the Proposition 1.3 for any $i < m$ we obtain

$$(\mathrm{F}_{i+1}) \quad \nabla_{\frac{\partial}{\partial s}} N_i = -\kappa_i N_{i-1} + \kappa_{i+1} N_{i+1},$$

where

$$N_{i+1} = \left(\kappa_i N_i^j + N_i'^j + N_i^k B_k^j\right) \frac{\partial}{\partial y^j},$$

and

$$\kappa_{i+1} = \|N_{i+1}\| .$$

Moreover, $\{N_0, \dots, N_i, N_{i+1}\}$ is an orthonormal set of projectible vector fields along C. If $i = m$ then $\{N_0, \dots, N_m\}$ is an orthonormal basis of $\Gamma(VTM_{|TC})$ whose elements are projectible Finsler vector fields. As $\nabla_{\frac{\partial}{\partial s}} N_m$ is orthogonal to N_m, (F_{i+1}) becomes

$$(\mathrm{F}_{m+1}) \quad \nabla_{\frac{\partial}{\partial s}} N_m = -\kappa_m N_{m-1}.$$

From equations (F_1)–(F_{m+1}) we see that each N_i belongs to the subspace $V_i = \mathrm{Span}\{\xi_0, \dots, \xi_i\}$ where we set $\xi_0 = \frac{\partial}{\partial v}$, $\xi_1 = \nabla_{\frac{\partial}{\partial s}} \xi_0$, $\dots$, $\xi_i = \nabla^{(i)}_{\frac{\partial}{\partial s}} \xi_0$. Hence $\{N_0, \dots, N_m\}$ is just an orthonormal basis obtained from $\{\xi_0, \dots, \xi_m\}$ provided all the curvature functions are nowhere zero on $(-\varepsilon, \varepsilon)$.

If $\kappa_i = 0$ on $(-\varepsilon, \varepsilon)$ for some $i < m$, then we can not define N_i and (F_i) becomes

$$(\mathrm{F}_i)' \quad \nabla_{\frac{\partial}{\partial s}} N_{i-1} = -\kappa_{i-1} N_{i-2}.$$

Therefore, on summing up the above results, we may say that in case there exist nowhere zero curvature functions $\{\kappa_1, \dots, \kappa_{i-1}\}$ as κ_i is everywhere zero on $(-\varepsilon, \varepsilon)$, we have constructed the Frenet frame $\{N_0, \ldots, N_{i-1}\}$ that satisfying the Frenet equations $(\mathrm{F}_1), \ldots, (\mathrm{F}_{i-1}), (\mathrm{F}'_i)$.

In case M is an oriented manifold and all curvature functions are nowhere zero on $(-\varepsilon, \varepsilon)$, as in case of Riemannian manifolds we may choose N_m as the cross product $N_0 \times \cdots \times N_{m-1}$ with respect to the metric g and taking into account the orientation of M. In this case κ_m could be positive, negative or zero.

Remark 1.2. As in Finsler geometry there are many Finsler connections, some of them being intensively investigated, we ask whether the above theory applies for them. First, we have to note that ∇ should be at least h-parallel. Therefore Rund connection is a good candidate for the theory, while Berwald connection is not. Moreover, we remark that the only changes in case we use Rund connection is in our B^i_j from (1.33) which becomes C^i_j.

We close this section with a formula for calculation of geodesic curvature of a curve in $F^{(2)}$. In this special case we define it in a slightly different way from the other cases. First we define the Finsler vector N_1 along C by

$$N_1 = \frac{1}{\sqrt{\Delta}} \left\{ -(g_{01}x'^0 + g_{11}x'^1)\frac{\partial}{\partial y^0} + (g_{00}x'^0 + g_{01}x'^1)\frac{\partial}{\partial y^1} \right\}, \tag{1.36}$$

where $\Delta = \det[g_{ij}]$. It is easy to check that N_1 is a unit projectible Finsler vector field along C orthogonal to $N_0 = \frac{\partial}{\partial v}$. Moreover, the basis $\{N_0, N_1\}$ and $\left\{\frac{\partial}{\partial y^0}, \frac{\partial}{\partial y^1}\right\}$ have the same orientation since

$$\begin{vmatrix} x'^0 & -g_{01}x'^0 - g_{11}x'^1 \\ x'^1 & g_{00}x'^0 + g_{01}x'^1 \end{vmatrix} = 1.$$

As $\nabla_{\frac{\partial}{\partial s}} N_0$ is orthogonal to N_0 we set

$$\nabla_{\frac{\partial}{\partial s}} N_0 = \kappa N_1, \tag{1.37}$$

where κ is a differentiable function on $(-\varepsilon, \varepsilon)$ that we name as the *curvature* of C. Clearly κ could be positive, negative or zero at points of C. By direct calculations using (1.6), (1.25), and (1.37) we obtain

$$\kappa = g(\nabla_{\frac{\partial}{\partial s}} N_0, N_1) = \sqrt{\Delta} \begin{vmatrix} x'^0 & x''^0 + 2G^0 \\ x'^1 & x''^1 + 2G^1 \end{vmatrix}. \tag{1.38}$$

Finally the (F_2)-Frenet equation becomes

$$\nabla_{\frac{\partial}{\partial s}} N_1 = -\kappa N_0. \tag{1.39}$$

We are now interested to get a formula for κ in case C is given by equations $x^i = x^i(t)$, where t is an arbitrary parameter on C. First, from the definition of arc length one obtains

$$x'^i = \frac{\dot{x}^i}{F}; \quad x''^i = \frac{1}{F^2}\left(\ddot{x}^i - \dot{x}^i \frac{d(\log F)}{dt}\right), \tag{1.40}$$

where $\dot{x}^i$ and $\ddot{x}^i$ denote the first and second derivative of x^i with respect to t.

Taking into account that G^i are positive homogeneous of degree 2 with respect (y^i) and using (1.7) we deduce

$$\begin{aligned} 2G^i(x(s), x'(s)) &= \frac{2}{F^2} G^i(x(t), \dot{x}(t)) \\ &= \frac{1}{F^2} g^{ij} F_{\dot{x}^j} F_{x^h} \dot{x}^h + \frac{g^{ij}}{F}\left(F_{\dot{x}^j x^h}\dot{x}^h - F_{x^j}\right), \end{aligned}$$

where F_{x^j} and $F_{\dot{x}^j x^h}$ are the partial derivatives of first order and second order of F respectively. As we have $F_{\dot{x}^j} = \frac{1}{F} g_{jk}\dot{x}^k$ one obtains

$$2G^i(x(s), x'(s)) = \frac{\dot{x}^i}{F^3} F_{x^h}\dot{x}^h + \frac{g^{ij}}{F}\left(F_{\dot{x}^j x^h}\dot{x}^h - F_{x^j}\right). \tag{1.41}$$

Now using (1.40) and (1.41) we derive

$$\kappa = \frac{\sqrt{\Delta}}{F^3}\left\{\ddot{x}^1\dot{x}^0 - \ddot{x}^0\dot{x}^1 - F(g^{0j}\dot{x}^1 - g^{1j}\dot{x}^0)\left(F_{\dot{x}^j x^h}\dot{x}^h - F_{x^j}\right)\right\}. \tag{1.42}$$

As F_{x^h} are positive homogeneous of degree 1 with respect to $(\dot{x}^h)$ we have

$$F_{x^h \dot{x}^j}\dot{x}^j = F_{x^h}. \tag{1.43}$$

Replace $F_{x^0\dot{x}^0}\dot{x}^0$ and $F_{x^1\dot{x}^1}\dot{x}^1$ from (1.43) in (1.42) and deduce

$$\kappa = \frac{\sqrt{\Delta}}{F^3}\left(\ddot{x}^1\dot{x}^0 - \ddot{x}^0\dot{x}^1\right) - \frac{1}{\sqrt{\Delta}}\left(F_{\dot{x}^0 x^1} - F_{\dot{x}^1 x^0}\right),$$

where we used the equality $g_{ij}\dot{x}^i\dot{x}^j = F^2$. As it is well known (cf. Berwald [4, p. 196]) we have $\Delta = F^3F_1$, where $F_1 = \frac{1}{(\dot{x}^1)^2}F_{\dot{x}^0\dot{x}^0}$. Therefore the curvature κ of a curve C $(x^i = x^i(t))$ in $F^{(2)}$ is given by

$$\kappa = \frac{1}{\sqrt{F^3F_1}}\left(F_1\left(\ddot{x}^1\dot{x}^0 - \ddot{x}^0\dot{x}^1\right) + F_{\dot{x}^0x^1} - F_{\dot{x}^1x^0}\right). \tag{1.44}$$

If in particular, $F^{(2)}$ is a Minkowski plane, that is $M = R^2$ and F depends only on (y^0, y^1) we deduce

$$\kappa = \frac{1}{F}\sqrt{\frac{F_1}{F}}\left(\ddot{x}^1\dot{x}^0 - \ddot{x}^0\dot{x}^1\right). \tag{1.45}$$

If moreover, $F = \left\{(\dot{x}^0)^2 + (\dot{x}^1)^2\right\}^{\frac{1}{2}}$ then $F_1 = \frac{1}{F^3}$ and we obtain the well known expression for the curvature of a curve in an Euclidean plane (cf. Spivak [8, Vol. II, p. 1–11]).

Remark 1.3. The curvature κ given by (1.44) has appeared first under the name *extremal curvature* in Landsberg [5] with respect to a variational problem.

2. Main results in theory of curves in a Finsler manifold

In the present section we show that the Frenet frame we defined in the first section is a good tool in studying the geometry of curves in a Finsler manifold. More precisely we prove a fundamental theorem and two theorems on the reduction of the codimension of a curve in a Finsler manifold.

Theorem 2.1 (Fundamental theorem for curves in $F^{(m+1)}$). *Let $F^{(m+1)} = (M, F)$, be a Finsler manifold, $(x_0, y_0) = (x_0^i, y_0^i)$ be a fixed point of TM, $\{V_0, \dots, V_m\}$ be an orthonormal basis of $VTM_{(x_0,y_0)}$ and $\kappa_1, \dots, \kappa_m : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be everywhere positive differentiable functions. Then there exists a unique curve C on M given by the equations $x^i = x^i(s), s \in (-\varepsilon, \varepsilon)$, where s is the arc length parameter of C such that $x^i(0) = x_0^i$ and $\kappa_1, \dots, \kappa_m$ are the curvature functions of C with respect to the Frenet frame $\{N_0, \dots, N_m\}$ that satisfies $N_h(0) = V_h, h \in \{0, \dots, m\}$.*

Proof. Taking into account the local expressions of Frenet formulas from previous section, we consider the system of differential equations

$$
\begin{aligned}
(2.1)\qquad &(\mathrm{F}_0^*) && x'^i(s) &&= N_0^i(s)\\
&(\mathrm{F}_1^*) && N_0'^i &&= -2G^i(s) + \kappa_1(s)N_1^i(s)\\
&\ldots && \ldots && \ldots\\
&(\mathrm{F}_h^*) && N_{h-1}'^i(s) &&= -N_{h-1}^j(s)B_j^i(s) - \kappa_{h-1}(s)N_{h-2}^i(s)\\
& && && \quad + \kappa_h(s)N_h^i(s) \quad (1 < h \leq m)\\
&\ldots && \ldots && \ldots\\
&(\mathrm{F}_{m+1}^*) && N_m'^i(s) &&= -N_m^j(s)B_j^i(s) - \kappa_m(s)N_{m-1}^i(s),
\end{aligned}
$$

where $G^i(s)$ and $B_j^i(s)$ are the functions from (1.7) and (1.33) calculated at points $(x(s), x'(s))$ and $(x^i(s), N_0^i(s), \ldots, N_m^i(s))$ are the unknown functions. Then using a similar argument as in the case of Riemannian manifolds (see Spivak [8, Vol. IV, p. 35]) we get that there exists a unique solution $(x^i(s), N_0(s), \ldots, N_m(s))$ on $(-\varepsilon, \varepsilon)$, where we set $N_h(s) = N_h^i(s)\frac{\partial}{\partial y^i}$ satisfying the initial conditions $x^i(0) = x_0^i$ and $N_h(0) = V_h$ for any $h \in \{0, \ldots, m\}$. As N_h are projectible Finsler vector fields along C $(x^i = x^i(s))$, by Proposition 1.3 we see that equations (F_h^*) are just local expressions of (F_h), $1 \leq h \leq m+1$. It remains only to prove that $\{N_0(s), \ldots, N_m(s)\}$ is an orthonormal basis of $VTM_{(x(s),x'(s))}$. To this end we set

$$g_{hr}^*(s) = g(x(s), x'(s))(N_h(s), N_r(s)), \quad h, r \in \{0, \ldots, m\},$$

and taking into account that ∇ is a metric connection on VTM we infer

$$\frac{dg_{hr}^*}{ds} = g\left(\nabla_{\frac{\partial}{\partial s}} N_h, N_r\right) + g\left(N_h, \nabla_{\frac{\partial}{\partial s}} N_r\right). \tag{2.2}$$

Replace the covariant derivative in (2.2) by their expressions from (F_{h+1}) and (F_{r+1}) respectively, and obtain

$$
\begin{aligned}
\frac{dg_{hr}^*}{ds} = {}& \kappa_h\kappa_r g_{(h-1)(r-1)}^* - \kappa_{h+1}\kappa_r g_{(h+1)(r-1)}^*\\
& - \kappa_h\kappa_{r+1} g_{(h-1)(r+1)}^* + \kappa_{h+1}\kappa_{r+1} g_{(h+1)(r+1)}^*.
\end{aligned} \tag{2.3}
$$

By direct calculation it is easy to check that (δ_{hr}) is a solution of the system (2.3). Since $g^*_{hr}(0) = g(x(0), x'(0))(V_h, V_r) = \delta_{hr}$, we conclude that $g^*_{hr} = \delta_{hr}$, which concludes the proof of the theorem.

Next, we consider a curve C in $F^{(m+1)}$ given locally by (1.1) and a vector subbundle D of rank p in $VTM_{|TC}$. Denote by $D^\perp$ the orthogonal complementary vector bundle to D in $VTM_{|TC}$. Then we say that D is a *parallel vector bundle along* C if for any $s \in (-\varepsilon, \varepsilon)$ we have

$$g(x(s), x'(s))(V(s), W) = 0, \quad \forall V(s) \in D(s),\ W \in D^\perp(0), \tag{2.4}$$

where $D(s)$ and $D^\perp(0)$ are the fibers of D and $D^\perp$ at the points $(x(s), x'(s))$ and $(x(0), x'(0))$ respectively.

In particular for $M = R^{m+1}$ we may state two theorems on the reduction of the codimension of the embedding of a curve in $F^{(m+1)}$.

Theorem 2.2. *Let C be a curve in $F^{(m+1)} = (R^{m+1}, F)$ and D be a parallel vector bundle of rank $p < m+1$ along C such that $\frac{\partial}{\partial v} \in \Gamma(D)$. Then C lies in some p-dimensional plane of R^{m+1}.*

Proof. Without loss of generality we may assume that $D(0) = \mathrm{Span}\left\{\frac{\partial}{\partial y^0}, \dots, \frac{\partial}{\partial y^{p-1}}\right\}$ at $(x(0), x'(0))$. As D is parallel along C we deduce that $D(s) = \mathrm{Span}\left\{\frac{\partial}{\partial y^0}, \dots, \frac{\partial}{\partial y^{p-1}}\right\}$ at $(x(s), x'(s))$. Now suppose there exists a point $x(s) \in C$ such that $x'^k(s) = 0$ for certain $k > p-1$. Then by (1.4) it follows that $\frac{\partial}{\partial v}$ does not lie in $D(s)$, which is a contradiction. Hence at any point of C we have $x'^p = \cdots = x'^m = 0$, which means that C lies in a p-dimensional plane of R^{m+1}.

Theorem 2.3. *Let C be a curve in $F^{(m+1)} = (R^{m+1}, F)$ such that the curvature functions $\kappa_1, \dots, \kappa_{p-1}$ are nowhere zero and κ_p is everywhere zero on $(-\varepsilon, \varepsilon)$. Then C lies in some p-dimensional plane of R^{m+1}.*

Proof. Denote by D the vector bundle spanned by the Frenet frame $\left\{N_0 = \frac{\partial}{\partial v}, N_1, \dots, N_{p-1}\right\}$ along C. From Frenet equations (F_1)–(F_{p-1}) and $(\mathrm{F}_p)'$ we obtain

$$\nabla_{\frac{\partial}{\partial s}} N_j(s) = \sum_{h=0}^{p-1} A^h_j(s) N_h(s), \quad j \in \{0, \dots, p-1\}. \tag{2.5}$$

Then consider an arbitrary $W \in D^{\perp}(0)$ and denote by the same letter the constant vector field $W(s) = W$ at any $(x(s), x'(s))$. Taking into account that ∇ is a metric connection and using (2.5) we deduce

$$\frac{d}{ds}\left(g(N_j, W)(s)\right) = g(\nabla_{\frac{\partial}{\partial s}} N_j, W)(s) = \sum_{h=0}^{p-1} A_j^h(s) g(N_h, W)(s). \tag{2.6}$$

As $g(N_h, W)(0) = 0$, the uniqueness of solutions of the system (2.6) implies $g(N_h, W)(s) = 0$, for any $s \in (-\varepsilon, \varepsilon)$. Hence D is parallel along C. As $\frac{\partial}{\partial v} \in \Gamma(D)$, the assertion follows from Theorem 2.2.

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(Received May 27, 1997)