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A result on distributions and the change of variable

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Abstract. Let F be a distribution in $\mathcal{D}'$ and let f be a differentiable function such that $f^{(p+1)}$ is a locally summable function with $f'(x) > 0$, (or < 0), for all x in the interval (a, b). It is proved that if F is the p-th derivative of a continuous function $F^{(-p)}$ on the interval $(f(a), (f(b))$, (or $(f(b), f(a))$), then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle G, \varphi\rangle$$

for all φ in $\mathcal{D}$ with support contained in the interval (a, b), where $F_n(x) = (F * \delta_n)(x)$. This defines the distribution $F(f) = G$ on the interval (a, b). Some examples are given.

In the following, we let N be the neutrix, see van der CORPUT [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \qquad \lambda > 0, \; r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We now let $\varrho(x)$ be any infinitely differentiable function having the following properties:

(i) $\varrho(x) = 0$ for $|x| \geq 1$,
(ii) $\varrho(x) \geq 0$,
(iii) $\varrho(x) = \varrho(-x)$,
(iv) $\int_{-1}^{1} \varrho(x)dx = 1$.

Putting $\delta_n(x) = n\varrho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta–function $\delta(x)$.

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Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if F is an arbitrary distribution in $\mathcal{D}'$, we define

$$F_n(x) = (F * \delta_n)(x) = \langle F(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $F(x)$.

The following definition for the change of variable in distributions was given in [2].

Definition 1. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that distribution $F(f(x))$ exists and is equal to the distribution G on the interval (a, b) if

$$N - \lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle G, \varphi\rangle$$

for all test functions φ in $\mathcal{D}$ with support contained in the interval (a, b), where

$$F_n(x) = (F * \delta_n)(x)\,.$$

We now prove an existence theorem for a distribution $F(f(x))$ which in fact does not need a neutrix limit.

Theorem 1. *Let F be a distribution in $\mathcal{D}'$ and let f be a differentiable function such that $f^{(p+1)}$ is a locally summable function with $f'(x) > 0$, (or < 0), for all x in the interval (a, b). If F is the p-th derivative of a continuous function $F^{(-p)}$ on the interval $(f(a), f(b))$, (or $(f(b), f(a))$), then the distribution $F(f(x))$ exists on the interval (a, b) and*

$$\langle F(f(x)), \varphi(x)\rangle = (-1)^p \mathrm{sgn}.g' \int_{-\infty}^{\infty} F^{(-p)}(x) \frac{d^p}{dx^p}[g'(x)\varphi(g(x))]dx \tag{1}$$

$$= (-1)^p \int_{-\infty}^{\infty} F^{(-p)}(f(x))|f'(x)| \left[\frac{1}{f'(x)}\frac{d}{dx}\right]^p \left[\frac{\varphi(x)}{f'(x)}\right] dx \tag{2}$$

for all φ in $\mathcal{D}$ with support contained in the interval (a, b), where g is the inverse of f on the interval (a, b) and

$$\mathrm{sgn}.g' = \begin{cases} 1, & g'(x) > 0, \\ -1, & g'(x) < 0. \end{cases}$$

Alternatively, if $F^{(-p)}$ is only a locally summable function but either $f^{(p+1)}$ or $F^{(-p)}$ is a bounded, locally summable function on every bounded subset of (a, b) and $(f(a), f(b))$, (or $f(b), f(a))$), respectively, then $F(f(x))$ again exists and equations (1) *and* (2) *are satisfied.*

In particular, if f is infinitely differentiable, then $F(f(x))$ is a defined for every distribution F.

PROOF. Suppose first of all that $f^{(p+1)}$ is a locally summable function and $F^{(-p)}$ is a continuous function. Letting φ be an arbitrary function in $\mathcal{D}$ with support contained in the interval (a, b) and making the substitution $t = f(x)$, we have

$$\begin{aligned}\langle F_n(f(x)), \varphi(x)\rangle &= \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx \\ &= \operatorname{sgn}.g' \int_{-\infty}^{\infty} F_n(t)g'(t)\varphi(g(t))dt\end{aligned}$$

Integrating by parts p times we get

$$\begin{aligned}\langle F_n(f(x)), \varphi(x)\rangle &= (-1)^p \operatorname{sgn}.g' \int_{-\infty}^{\infty} F_n^{(-p)}(t)\frac{d^p}{dt^p}[g'(t)\varphi(g(t))]dt \\ &= (-1)^p \int_{-\infty}^{\infty} F_n^{(-p)}(f(x))|f'(x)| \left[\frac{1}{f'(x)}\frac{d}{dx}\right]^p \left[\frac{\varphi(x)}{f'(x)}\right] dx\,.\end{aligned}$$

Now $F_n^{(-p)}(t)$ and $F_n^{(-p)}(f(x))$ are continuous functions converging to the continuous functions $F^{(-p)}(t)$ and $F^{(-p)}(f(x))$ respectively as n tends to infinity and

$$\frac{d^p}{dt^p}[g'(t)\varphi(g(t))], \quad |f'(x)| \left[\frac{1}{f'(x)}\frac{d}{dx}\right]^p \left[\frac{\varphi(x)}{f'(x)}\right]$$

are locally summable functions. It follows that

$$\lim_{n\to\infty} \langle F_n(f(x)), \varphi(x)\rangle = (-1)^p \operatorname{sgn}.g' \int_{-\infty}^{\infty} F^{(-p)}(t)\frac{d^p}{dt^p}[g'(t)\varphi(g(t))]dt \tag{3}$$

$$= (-1)^p \int_{-\infty}^{\infty} F^{(-p)}(f(x))|f'(x)| \left[\frac{1}{f'(x)}\frac{d}{dx}\right]^p \left[\frac{\varphi(x)}{f'(x)}\right] dx \tag{4}$$

and equations (1) and (2) follow.

Now suppose that either $f^{(p+1)}$ or $F^{(-p)}$ are bounded functions on bounded subsets. Then

$$F^{(-p)}(t)\frac{d^p}{dt^p}[g'(t)\varphi(g(t))], \quad F^{(-p)}(f(x))|f'(x)| \left[\frac{1}{f'(x)}\frac{d}{dx}\right]^p \left[\frac{\varphi(x)}{f'(x)}\right]$$

will be locally summable functions, ensuring that equations (3) and (4) still hold.

In the corollary the product of $\delta^{(r)}$ and an r times continuously differentiable function f is defined by

$$f(x)\delta^{(r)}(x) = \sum_{i=0}^{r} (-1)^{(r+i)} \binom{r}{i} f^{(r-i)}(0)\delta^{(i)}(x)\,. \tag{5}$$

Corollary. *Let f be a continuously differentiable function having a single simple root at the point $x = \alpha$. Suppose that $f^{(p+2)}$ is a locally summable function on a neighbourhood of the point $x = \alpha$. Then*

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(\alpha)|} \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^r \delta(x-\alpha) \tag{6}$$

on the real line for $r = 0, 1, 2, \ldots, p$.

PROOF. We will suppose that $\alpha = 0$ and the general result will then follow by translation. Note that $\delta^{(p)}(f(x))$ is then equal to 0 on any interval not containing the origin. We therefore only have to prove the result on some neighbourhood of the origin. Since $x = 0$ is a simple root of f there exists a neighbourhood (a, b) containing the origin on which $f'(x) > 0$, (or < 0). Now $\delta^{(r)}$ is the $(r+1)$-th derivative of the bounded, locally summable function $H(x)$, where H denotes Heaviside's function, and so the conditions of the theorem are satisfied for the distribution $\delta^{(r)}(f(x))$ on the interval (a, b) for $r = 0, 1, 2, \ldots, p$. The existence of $\delta^{(r)}(f(x))$ follows by the theorem.

Now let φ is an arbitrary function in $\mathcal{D}$ with support contained in the interval (a, b). Then since $\delta^{(r)}(x)$ is the $(r+2)$-th derivative of the continuous function x_+ and supposing that $f'(x) > 0$, we have from equation (2)

$$\begin{aligned} \langle \delta^{(r)}(f(x)), \varphi(x) \rangle &= (-1)^{r+1} \int_0^\infty f'(x) \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^{r+1} \left[\frac{\varphi(x)}{f'(x)} \right] dx \\ &= -(-1)^{r+1} \int_0^\infty d \left\{ \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^r \left[\frac{\varphi(x)}{f'(x)} \right] \right\} \\ &= \frac{(-1)^r}{f'(0)} \frac{d}{dx} \left\{ \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^{r-1} \left[\frac{\varphi(x)}{f'(x)} \right] \right\} \Bigg|_{x=0} \end{aligned} \tag{7}$$

Further

$$\begin{aligned} &\left\langle \frac{1}{f'(0)} \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^r \delta(x), \varphi(x) \right\rangle = \\ &= -\frac{1}{f'(0)} \left\langle \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^{r-1} \delta(x), \frac{d}{dx} \frac{\varphi(x)}{f'(x)} \right\rangle \end{aligned} \tag{8}$$

$$= \frac{(-1)^r}{f'(0)} \left\langle \delta(x), \frac{d}{dx}\left\{\left[\frac{1}{f'(x)}\frac{d}{dx}\right]^{r-1}\left[\frac{\varphi(x)}{f'(x)}\right]\right\}\right\rangle .$$

Comparing equations (7) and (8) we see that equation (6) is proved for the case $f'(x) > 0$ and $r = 0, 1, 2, \ldots, p$. The case $f'(x) < 0$ follows similarly.

Note that this corollary can be extended to a function f having any number of simple roots at the points $x = \alpha_1, \alpha_2, \ldots$. If then $f^{(p_i+2)}$ is a locally summable function on a neighbourhood of the point $x = \alpha_i$, we have

$$\delta^{(r)}(f(x)) = \sum_i \frac{1}{|f'(\alpha_i)|}\left[\frac{1}{f'(x)}\frac{d}{dx}\right]^r \delta(x - \alpha_i)$$

on the real line for $r = 1, 2, \ldots, p$, where $p = \min\{p_i : i = 1, 2, \ldots\}$. This is in agreement with Gel'fand and Shilov's definition of $\delta^{(r)}(f(x))$, see [3], but their definition was only given for infinitely differentiable f.

Example 1.

$$(x + x_+^p)^{-1} = x^{-1} - \frac{x^{p-2}H(x)}{1 + x^{p-1}} \tag{9}$$

on the real line for $p = 2, 3, \ldots$.

Proof. The distribution x^{-1} is the first derivative of the locally summable function $\ln|x|$ and $(x + x_+^p)'' = p(p-1)x_+^{p-2}$ is bounded on every bounded set. Using equation (2) we have

$$\langle (x + x_+^p)^{-1}, \varphi(x)\rangle = -\int_{-\infty}^{\infty} \ln|x + x_+^p| \left[\frac{\varphi(x)}{1 + px_+^{p-1}}\right]' dx$$

$$= -\int_{-\infty}^{0} \ln|x|\varphi'(x)dx - \int_0^{\infty} \ln(x + x^p)\left[\frac{\varphi(x)}{1 + px^{p-1}}\right]' dx$$

$$= -\langle x_-^{-1}, \varphi(x)\rangle - \int_0^1 \ln(x + x^p)\left[\frac{\varphi(x) - \varphi(0)}{1 + px^{p-1}}\right]' dx +$$

$$-\int_1^{\infty} \ln(x + x^{p-1})\left[\frac{\varphi(x)}{1 + px^{p-1}}\right]' dx +$$

$$-\varphi(0)\int_0^1 \ln(x + x^p)[(1 + px^{p-1})^{-1}]' dx .$$

Now

$$\int_0^1 \ln(x + x^p)\left[\frac{\varphi(x) - \varphi(0)}{1 + px^{p-1}}\right]' dx = \ln 2\frac{\varphi(1) - \varphi(0)}{1 + p} - \int_0^1 \frac{\varphi(x) - \varphi(0)}{x + x^p} dx$$

$$= \ln 2 \frac{\varphi(1) - \varphi(0)}{1+p} - \int_0^1 \frac{\varphi(x) - \varphi(0)}{x} dx + \int_0^1 \frac{x^{p-2}\varphi(x)}{1+x^{p-1}} dx - \ln 2 \frac{\varphi(0)}{p-1}\,,$$

$$\int_1^\infty \ln(x+x^p) \left[\frac{\varphi(x)}{1+px^{p-1}} \right]' dx = -\ln 2 \frac{\varphi(1)}{1+p} - \int_1^\infty \frac{\varphi(x)}{x+x^p} dx$$
$$= \ln 2 \frac{\varphi(1)}{1+p} - \int_1^\infty \frac{\varphi(x)}{x} dx + \int_1^\infty \frac{x^{p-2}\varphi(x)}{1+x^{p-1}} dx\,,$$

$$\varphi(0) \int_0^1 \ln(x+x^p)[(1+px^{p-1})^{-1}]' dx =$$
$$= \varphi(0) \int_0^1 \ln(x+x^p) d[(1+px^{p-1})^{-1} - 1]$$
$$= -\ln 2 \frac{p\varphi(0)}{1+p} + \varphi(0) \int_0^1 \frac{px^{p-2}}{1+x^{p-1}} dx = -\ln 2 \frac{p\varphi(0)}{1+p} + \ln 2 \frac{p\varphi(0)}{p-1}\,.$$

Thus

$$\langle (x+x_+^p)^{-1}, \varphi(x) \rangle = -\langle x_-^{-1}, \varphi(x) \rangle + \int_0^\infty x^{-1}[\varphi(x) - \varphi(0)H(1-x)]dx +$$
$$- \int_0^\infty \frac{x^{p-2}\varphi(x)}{1+x^{p-1}} dx = \langle x^{-1}, \varphi(x) \rangle - \left\langle \frac{x^{p-2}H(x)}{1+x^{p-1}}, \varphi(x) \right\rangle$$

and equation (9) follows.

Example 2.

$$\delta(x+x_+^p) = \delta(x) \tag{10}$$

on the real line for $p = 3, 4, \ldots$ and

$$\delta'(x+x_+^p) = \delta'(x) - \delta(x) \tag{11}$$

on the real line for $p = 2, 3, \ldots$.

Proof. The distribution $\delta(x)$ is the first and the distribution $\delta'(x)$ is the second derivative of the continuous function x_+. Using equations (5) and (6), equation (10) follows immediately for $p = 2, 3, \ldots$.

Using equations (5) and (6) again have

$$\delta'(x+x_+^p) = \frac{1}{1+px_+^{p-1}} \delta(x) = \delta'(x) - \delta(x)\,,$$

giving equation (11) for $p = 3, 4, \ldots$.

Example 3.

$$\delta(x + x^2 + x_+^p) = \delta(x+1) + \delta(x) \tag{12}$$

on the real line for $p = 2, 3, \dots$ and

$$\delta'(x + x^2 + x_+^p) = \delta'(x+1) + 2\delta(x+1) + \delta'(x) + 2\delta(x) \tag{13}$$

on the real line for $p = 3, 4, \dots$.

Proof. The function $x + x^2 + x_+^p$ has zeros at the points $x = -1, 0$. Using equations (5) and (6) on neighbourhoods of these points, equation (12) follows immediately for $p = 2, 3, \dots$.

Using equations (5) and (6) again neighbourhoods of these points we have

$$\begin{aligned}\delta'(x + x^2 + x_+^p) &= \frac{1}{1 + 2x + px_+^{p-1}}\delta'(x+1) + \frac{1}{1 + 2x + px_+^{p-1}}\delta'(x)\\ &= \delta'(x+1) + 2\delta(x+1) + \delta'(x) + 2\delta(x)\,,\end{aligned}$$

giving equation (13) for $p = 3, 4, \dots$.

Example 4.

$$(x + x_+^p + i0)^{-1} = (x + i0)^{-1} - \frac{x^{p-2}H(x)}{1 + x^{p-1}} \tag{14}$$

on the real line for $p = 2, 3, \dots$.

Proof. The distribution $(x + i0)^{-1}$ is defined by

$$(x + i0)^{-1} = x^{-1} - i\pi\delta(x)\,,$$

see Gel'fand and Shilov [3]. Using equations (9) and (10) it follows that

$$\begin{aligned}(x + x_+^p + i0)^{-1} &= (x + x_+^p)^{-1} - i\pi\delta(x + x_+^p)\\ &= x^{-1} - \frac{x^{p-2}H(x)}{1 + x^{p-1}} - i\pi\delta(x) = (x + i0)^{-1} - \frac{x^{p-2}H(x)}{1 + x^{p-1}}\,,\end{aligned}$$

giving equation (14) for $p = 2, 3, \dots$.

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