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On generalized pseudo-projective symmetric manifolds

By U. C. DE (Kalyani) and B. K. DE (Kalyani)

Abstract. The object of the present paper is to study a type of non-flat Riemannian manifold called generalized pseudo-projective symmetric manifold.

Introduction

The notions of weakly symmetric and weakly projective symmetric manifold were introduced by L. Tamássy and T. Q. Binh [1]

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ (these conditions will be supposed throughout this paper) is called weakly symmetric if there exist 1-forms α, β, γ, δ and a vector field F such that

$$\begin{aligned}(\nabla_X R)(Y,Z,V) &= \alpha(X)R(Y,Z,V)+\beta(Y)R(X,Z,V)+\gamma(Z)R(Y,X,V)\\ &+\delta(V)R(Y,Z,X)+g[R(Y,Z,V),X]F;\ \ X,Y,Z,V\in\chi(M^n)\end{aligned}\tag{1}$$

where R is the curvature tensor of (M^n, g). A Riemannian manifold (M^n, g) is called weakly projective symmetric if there exist 1-forms α, β, γ, δ and a vector field F such that

$$\begin{aligned}(\nabla_X W)(Y,Z,V) &= \alpha(X)W(Y,Z,V)+\beta(Y)W(X,Z,V)\\ &+\gamma(Z)W(Y,X,V)+\delta(V)W(Y,Z,X)+g[W(Y,Z,V),X]F\end{aligned}\tag{2}$$

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where W is the projective curvature tensor given by

$$W(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] \tag{3}$$

where S is the Ricci tensor of (M^n, g). Recently, Tamássy and Binh [3] further studied $(WS)_n$ with certain structures.

The object of the present paper is to study a type of non-flat Riemannian manifold (M^n, g) whose projective curvature tensor W satisfies the condition

$$\begin{aligned} &(\nabla_X W)(Y,Z,U) = 2A(X)W(Y,Z,U) + B(Y)W(X,Z,U) \\ &+ C(Z)W(Y,X,U) + D(U)W(Y,Z,X) + g[W(Y,Z,U),X]\rho \end{aligned} \tag{4}$$

where A, B, C, D are non-zero 1-forms and ρ is a vector field given by

$$g(X,\rho) = A(X) \quad \forall X. \tag{5}$$

Such a manifold will be called a generalized pseudo-projective symmetric manifold; A, B, C, D will be called its associated 1-forms and an n-dimensional manifold of this kind will be denoted by $G(PWS)_n$.

Let

$$\begin{aligned} g(X,\lambda) = B(X), \quad g(X,\mu) = C(X) \quad \text{and} \\ g(X,\nu) = D(X) \quad \forall X \in \chi(M^n). \end{aligned} \tag{6}$$

Then $\rho, \lambda, \nu, \in \chi(M^n)$ will be called the basic vector fields of $G(PWS)_n$ corresponding to the associated 1-forms A, B, C, D respectively. If, in particular, $A = B = C = D$, then the manifold defined by (4) reduces to a pseudo-projective symmetric manifold introduced by Chaki and Saha [2]. This justifies the name generalized pseudo-projective symmetric manifold. (4) and (5) together are a little stronger assumptions than (2). (2) gives (4) if α and F are related by

$$(X,F) = \alpha(X) \quad \forall X.$$

So the definition of a $G(PWS)_n$ is similar to that of a weakly projective symmetric manifold mentioned above.

A non-flat Riemannian manifold is called generalized pseudo-symmetric by M.C. Chaki [4] if the curvature tensor R satifies

$$\begin{aligned}(\nabla_X R)(Y,Z,U) &= 2A(X)R(Y,Z,U)+B(Y)R(X,Z,U)\\ &+C(Z)R(Y,X,U)+D(U)R(Y,Z,X)+g[R(Y,Z,U)X]\rho,\end{aligned} \tag{7}$$

where ρ is a vector field given by

$$g(X,2\rho)=A(X) \quad \forall X.$$

He denoted such a manifold by $G(PS)_n$. Recently, Chaki and Mondal [5] also studied $G(PS)_n$. Tamássy and Binh in their paper [1] find necessary and sufficient conditions for a weakly symmetric manifold to be a weakly projective symmetric manifold.

In Section 2 of this paper it is shown that in a $G(PWS)_n$ the scalar curvature r of (M^n,g) is an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q defined by $g(Q,X)=A(X)+B(X)$, and also a necessary and sufficient condition for zero scalar curvature in a $G(PWS)_n$ is obtained. In Section 3 some properties of $G(PWS)_n$ have been proved. In Section 4 it is shown that an Einstein $G(PWS)_n$ reduces to a $G(PS)_n$ if $A(X)+nD(X)\neq 0$. Further, it is shown that if the vector field ρ defined by (5) is a paralled vector field, then an Einstein $G(PWS)_n$ reduces to a $G(PS)_n$ provided the vectors ρ and λ are not co-directional.

1. Preliminaries

In this section we derive some formulas which will be required in the study of a $G(PWS)_n$

Let

$$'W(X,Y,Z,U)=g[W(X,Y,Z,),U]. \tag{1.1}$$

Then form (3) we get

$$\begin{aligned}'W(X,Y,Z,U) &= {'R}(X,Y,Z,U)\\ &-\frac{1}{n-1}[g(X,U)S(Y,Z)-g(Y,U)S(X,Z)],\end{aligned} \tag{1.2}$$

where

$$'R(X,Y,Z,U) = g[R(X,Y,Z,),U]. \tag{1.3}$$

Let

$$P(X,U) = {'W}(X,e_i,e_i,U) \tag{1.4}$$

where $\{e_i\}$, $i = 1,2,\ldots,n$ is an orthonormal basis of the tangent space at a point and i is summed for $1 \le i \le n$. Then using (1.2) we get

$$P(X,U) = \frac{n}{n-1}S(X,U) - \frac{r}{n-1}g(X,U) \tag{1.5}$$

where S is the Ricci tensor and r is the scalar curvature of (M^n, g). Let ℓ and L be the symmetric endomorphisms of the tangent space at two points corresponding to the tensors P and S respectively, i.e.

$$g(\ell X, Y) = P(X,Y) \tag{1.6}$$

and

$$g(LX,Y) = S(X,Y). \tag{1.7}$$

Contracting (4) over Y, we get

$$B(W(X,Z,U)) + {'W}(\rho,Z,U,X) = 0$$

or

$$'W(X,Z,U,\lambda)) + {'W}(\rho,Z,U,X) = 0.$$

Putting $Z = U = e_i$ in the above relation and taking summation over i, $1 \le i \le n$, we have

$$P(X,\lambda) + P(X,\rho) = 0. \tag{1.8}$$

2. Nature of the scalar curvature of a $G(PWS)_n$ $(n > 2)$

From (1.5) and (1.8) it follows that $T(X)\frac{r}{n} = \bar{T}(X)$, where $T(X) = A(X) + B(X)$ and $\bar{T}(X) = A(LX) + B(LX)$. Hence

$$S(X,Q) = \frac{r}{n}g(X,Q), \tag{2.1}$$

where $g(X,Q) = T(X)$. This leads to the following

Theorem 1. *In a* $G(PWS)_n$, $\frac{r}{n}$ *is an eigenvalue of the Ricci tensor* S *corresponding to the eigenvector* Q *defined by* $g(X,Q) = T(X)$.

Now we obtain a necessary and sufficient condition for zero scalar curvature in a $G(PWS)_n$. First we suppose that $r = 0$ in a $G(PWS)_n$. Then from (2.1) we get $S(X,Q) = 0$. Therefore from (3) it follows that

$$W(X,Y,Q) = R(X,Y,Q). \tag{2.2}$$

Next we suppose that in a $G(PWS)_n$ the relation (2.2) holds, then from (3) we get

$$S(Y,Q)X = S(X,Q)Y. \tag{2.3}$$

Contraction of (2.3) gives $S(Y,Q) = 0$.

Hence from (2.1) we get $r = 0$, if $T(X) \neq 0$.

This leads to the following

Theorem 2. *A* $G(PWS)_n$ $(n > 2)$ *is of zero scalar curvature if and only if the relation* (2.2) *holds provided* $T \neq 0$.

3. The case of $G(PWS)_n$ satisfying $A(W(X,Y,Z)) = 0$

Contracting (4) over X, we get

$$(\operatorname{div} W)(Y,Z,U) = 3A(W(Y,Z,U)), \tag{3.1}$$

where 'div' denotes divergence. It is known that in a Riemannian manifold (M^n, g) $(n > 2)$

$$(\operatorname{div} W)(X,Y,Z) = \frac{n-2}{n-1}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)]. \tag{3.2}$$

Since $A(W(X,Y,Z)) = 0$ we get from (3.1) $(\operatorname{div} W)(X,Y,Z) = 0$.
Hence from (3.2) it follows that $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$.

Thus we can state the following

Theorem 3. *In a* $G(PWS)_n$ *satisfying* $A(W(X,Y,Z)) = 0$ *the Ricci tensor* S *is of Codazzi type.*

Lemma. *In order that* $(\nabla_X W)(Y,Z,U) + (\nabla_Y W)(Z,X,U) + (\nabla_Z W)(X,Y,U){=}0$, *it is necessary and sufficient that* $(\operatorname{div} W)(X,Y,Z){=}0$.

Proof of the Lemma. First suppose that

$$(\nabla_X W)(Y,Z,U) + (\nabla_Y W)(Z,X,U) + (\nabla_Z W)(X,Y,U) = 0. \tag{3.3}$$

Contracting (3.3) over Z, we get $(\operatorname{div} W)(X,Y,U) = 0$.
Next suppose that $(\operatorname{div} W)(X,Y,U) = 0$. Hence from (3.2) we get

$$\frac{n-2}{n-1}\left[(\nabla_X S)(Y,U) - (\nabla_Y S)(X,U)\right] = 0. \tag{3.4}$$

Again from the Bianchi indentity we get

$$(\nabla_X R)(Y,Z,U) + (\nabla_Y R)(Z,X,U) + (\nabla_Z R)(X,Y,U) = 0. \tag{3.5}$$

Hence from (3.4) and (3.5) it follows that

$$(\nabla_X W)(Y,Z,U) + (\nabla_Y W)(Z,X,U) + (\nabla_Z W)(X,Y,U) = 0. \qquad \square$$

Now from (4) we get

$$\begin{aligned}(\nabla_X W)&(Y,Z,U)+(\nabla_Y W)(Z,X,U)+(\nabla_Z W)(X,Y,U)\\ &= [2A(X) - B(X) - C(X)]W(Y,Z,U) + [2A(Y) - B(Y)\\ &\quad - C(Y)]W(Z,X,U) + [2A(Z) - B(Z) - C(Z)]W(X,Y,U)\\ &= G(X)W(Y,Z)U - G(Y)W(X,Z)U - G(Z)W(Y,X)U,\end{aligned} \tag{3.6}$$

since

$$W(X,Y,Z) = -W(Y,X)Z \quad \text{and}$$
$$W(X,Y,Z) + W(Y,Z,X) + W(Z,X,Y) = 0,$$

where

$$G(X) = 2A(X) - B(X) - C(X).$$

Since we assume that $A(W(X,Y,Z)) = 0$, it follows from (3.1) that $(\operatorname{div} W)\ (X,Y,Z) = 0$. On the other hand, from the above Lemma and (3.6), it follows that

$$G(X)W(Y,Z)U - G(Y)W(X,Z)U - G(Z)W(Y,X)U = 0. \tag{3.7}$$

Putting $X=\rho$ in (3.7) and applying $A(W(X,Y,Z))=0$ we get

$$G(\rho)W(Y,Z)U=0.$$

Then either $G(\rho)=0$ or the manifold is projectively flat.

Now $G(\rho)=0$ implies $g(\rho,\tilde{\rho})=0$, where $\tilde{\rho}$ is a vector field defined by

$$g(X,\tilde{\rho})=G(X). \tag{3.8}$$

Thus we have the following

Theorem 4. *If a $G(PWS)_n$ satisfies $A(W(X,Y,Z))=0$, then either the manifold is of constant curvature or the associated vector ρ is orthogonal to the vector $\tilde{\rho}$ defined by (3.8).*

4. Einstein $G(PWS)_n$ $(n > 3)$

In this section we assume that a $G(PWS)_n$ defined by (4) is an Einstein manifold. Then the Ricci tensor S satisfies

$$S(X,Y)=\frac{r}{n}g(X,Y) \tag{4.1}$$

from which it follows that

$$dr(X)=0 \quad \text{and} \quad (\nabla_Z S)(X,Y)=0. \tag{4.2}$$

By using (3), (4.1) and (4.2) we get from (4)

$$\begin{aligned}(\nabla_X R)(Y,Z)U = {} & 2A(X)\Big[R(Y,Z)U-\frac{r}{n(n-1)}(g(Z,U)Y-(-g(Y,U)Z)\Big]\\ & + B(Y)\Big[R(X,Z)U-\frac{r}{n(n-1)}(g(Z,U)X-g(X,U)Z)\Big]\\ & +C(Z)\Big[R(Y,X)U-\frac{r}{n(n-1)}(g(X,U)Y-g(Y,U)X)\Big]\\ & + D(U)\Big[R(Y,Z)X-\frac{r}{n(n-1)}(g(Z,X)Y-g(Y,X)Z)\Big]\\ & + g\Big[R(Y,Z)U-\frac{r}{n(n-1)}(g(Z,U)Y-g(Y,U)Z)\Big].\end{aligned} \tag{4.3}$$

From the Bianchi identity and (4.3) it follows that

$$
\begin{aligned}
&3A(R(Y,Z)U) + B(R(Y,Z)U) + C(R(Y,Z)U) \\
&\quad + \Big[2S(Z,U) - \frac{2r}{n-1}g(Z,U)\Big]A(Y) \\
&\quad + \Big[\frac{(n+1)}{n(n-1)}rg(Y,U) - 2S(Y,U)\Big]A(Z) \\
&\quad - \frac{r}{n(n-1)}B(Y)g(Z,U) + \frac{r}{n(n-1)}B(Z)g(Y,U) \\
&\quad - \frac{r}{n(n-1)}C(Y)g(Z,U) + \frac{r}{n(n-1)}C(Z)g(Y,U) \\
&\quad - \frac{r}{n}D(U)g(Z,Y) = 0.
\end{aligned} \tag{4.4}
$$

Putting $Y = Z = e_i$ in (4.4) and taking summation over i, we get

$$r[A(U) + nD(U)] = 0.$$

Hence

$$r = 0, \quad \text{if } A(U) + nD(U) \neq 0.$$

Putting $r = 0$ in (4.3), it follows that a $G(PWS)_n$ is a $G(PS)_n$. Hence we can state the following

Theorem 5. *An Einstein $G(PWS)_n$ is a $G(PS)_n$ if $A(X)+nD(X)\neq 0$.*

Next we suppose that in an Einstein $G(PWS)_n$ the vector field ρ defined by (5) is parallel:

$$\nabla_X \rho = 0 \quad \forall X \in \chi G(PWS)_n. \tag{4.5}$$

Applying the Ricci identity we get

$$R(X,Y,\rho) = 0. \tag{4.6}$$

From (4.6) we get

$$S(Y,\rho) = 0. \tag{4.7}$$

Now by (4.5) and (4.7) it follows that

$$(\nabla_X S)(Y,\rho) = 0. \tag{4.8}$$

From (4.4) we get

$$(\nabla_X S)(Z,U) = B(R(X,Z,U)) - \frac{r}{n(n-1)}\big[g(Z,U)B(X) - g(X,U)B(Z)\big]. \tag{4.9}$$

Putting $U = \rho$ in (4.9) and using (4.6), (4.7), (4.8) we get $r = 0$, if $A(X)B(Z) \neq A(Z)B(X)$

Hence we can state the following

Theorem 6. *If the vector field ρ is a paralled vector field in an Einstein $G(PWS)_n$, then $G(PWS)_n$ reduces to a $G(PS)_n$ provided the vector fields ρ and λ corresponding to the 1-forms A and B are not co-directional.*

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U. C. DE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI – 741 235
WEST BENGAL
INDIA

B. K. DE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI – 741 235
WEST BENGAL
INDIA

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