Publ. Math. Debrecen
54 / 1-2 (1999), 11–19

Exact controllability and spectrum assignment for infinite dimensional singular systems

By MOHAMED RABIE ABDALLAH MOUBARAK (El-Minia)

Abstract. In this paper we prove the equivalence of the two concepts of exact controllability and spectrum assignment property for infinite dimensional singular systems where the state space and control space are Hilbert spaces.

1. Introduction

We begin with the two infinite dimensional Hilbert spaces H and U as state space and control space, respectively. Let Y be a closed subspace of H, then H can be written in the decomposition form:

$$H = Y \oplus Y^{\perp} \tag{1}$$

where $Y^{\perp}$ is the orthogonal complement of Y.

Let $L(H)$ be the set of all bounded linear operators from H into H and $L(U,H)$ the set of all bounded linear operators from U into H. If $A \in L(H)$, then A can be represented in the matrix operator form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{2}$$

where $A_{11} : Y \longrightarrow Y$, $A_{12} : Y^{\perp} \longrightarrow Y$, $A_{21} : Y \longrightarrow Y^{\perp}$, $A_{22} : Y^{\perp} \longrightarrow Y^{\perp}$, see [3].

Mathematics Subject Classification: 46C, 49, 47.
Key words and phrases: exact controllability, spectrum assignment, infinite dimensional singular systems.

Consider the continuous control system:

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{3}$$

where $x(t) \in H$, $u(t) \in U$, the operators $E, A \in L(H)$ and $B \in L(U, H)$. When E is the identity operator I, the system (3) is called a normal system. In this work E is assumed to be singular and given in the form:

$$E = \begin{pmatrix} I_Y & 0 \\ 0 & 0 \end{pmatrix} \tag{4}$$

where I_Y is the identity operator on Y.

In this work, we want to prove the equivalence between the two concepts of exact controllability and spectrum assignment property of the infinite dimensional singular system (3)

We have chosen this kind of work in order to establish a tie between functional analysis and control theory. Also, the system (3) has been chosen because the study of nonlinear systems has become more active with an effort to overcome the analytical difficulties implied. Furthermore the system (3) is more general than the normal system which appears as a special case when E is the identity. Finally, the system (3) is important for many fields such as electrical networks, economics, robotics and aircraft dynamics.

In the case of a finite dimensional normal system Wonham [6] has proved the equivalence between controllability and pole assignment.

Some work has been done to generalize Wonham's result to infinite dimensional normal systems such as [2] and [4]. In our work, the proofs and the methods are completely different and independent of those used in [2] and [4] because in this work we have a problem, namely the singular operator E which disappears in the case of a normal system.

To make some developments easier, the system (3) will be transformed to what we call a transformed singular system by using a transformation which allows us to assume without loss of generality that $A = I$ as follows: Let $\alpha \in R$ be a real number such that $(\alpha E + A)^{-1}$ exists and let:

$$x(t) = e^{-\alpha t} y(t). \tag{5}$$

Differentiating both sides of (5) with respect to t yields

$$\dot{x}(t) = e^{-\alpha t}\dot{y}(t) - \alpha e^{-\alpha t} y(t). \tag{6}$$

Substituting from (5) and (6) in (3), we get

$$e^{-\alpha t}E\dot{y}(t) = (\alpha E + A)e^{-\alpha t}y(t) + Bu(t). \tag{7}$$

Multiplying both sides of (7) by $(\alpha E + A)^{-l}e^{\alpha t}$ on the left yields

$$E_1\dot{y}(t) = y(t) + B_1u_1(t) \tag{8}$$

where $E_1 = (\alpha E + A)^{-1}E$, $B_1 = (\alpha E + A)^{-1}B$, $u_1(t) = e^{\alpha t}u(t)$.

Consider the feedback law

$$u(t) = Fx(t) + v(t), \tag{9}$$

where $v(t)$ is an external input and $F \in L(H, U)$. Then the closed loop system takes the form:

$$E\dot{x}(t) = (A + BF)x(t) + Bv(t). \tag{10}$$

The systems (3), (8) and (10) will be denoted by (E, A, B), (E_1, B_1) and $(E, A + BF, B)$ respectively.

2. Exact controllability and spectrum assignment property

Definition 2.1. The system (E, A, B) is exactly controllable if and only if:

$$\text{range}[sE - A, B] = H, \tag{11}$$

for all $s \in \mathcal{C}$ and,

$$\text{range}[E, B] = H. \tag{12}$$

Definition 2.2. The system (E_1, B_1) is exactly controllable if and only if

$$\text{range}[(s + \alpha)E_1 - I, B_1] = H, \tag{13}$$

for all $s \in \mathcal{C}$, $\alpha \in R$ and,

$$\text{range}[E_1, B_1] = H. \tag{14}$$

Theorem 2.1. *The system (E, A, B) is exactly controllable if and only if the system (E_1, B_1) is exactly controllable.*

PROOF.

$$\begin{aligned}
&[(s+\alpha)E_1 - I](H) + B_1(U) \\
&\quad = (\alpha E + A)^{-1}\{[(s+\alpha)E - (\alpha E + A)](H) + B(U)\} \\
&\quad = (\alpha E + A)^{-1}[(sE - A)(H) + B(U)]
\end{aligned} \tag{15}$$

and

$$E_1(H) + B_1(U) = (\alpha E + A)^{-1}[E(H) + B(U)]. \tag{16}$$

Since $(\alpha E + A)^{-1}$ is invertible, it follows from (15), (16), Definitions 2.1 and 2.2 that the system (E, A, B) is exactly controllable if and only if the system (E_1, B_1) is exactly controllable.

Theorem 2.2. *The system (E, A, B) is exactly controllable if and only if:*

$$\text{range}[E_1 - \lambda I, B_1] = H \tag{17}$$

for every $\lambda \in \mathcal{C}$.

For proof see [5].

Definition 2.3. The set

$$\sigma(E, A) = \{\lambda \in \mathcal{C} : \lambda E - A \text{ is singular}\} \tag{18}$$

is called the spectrum of (E, A) and $\sigma(I, A)$ is specified as $\sigma(A)$.

Definition 2.4. The system (E, A, B) has the spectrum assignment property if and only if for any non empty compact subset Λ of $\mathcal{C}$, there exists $F \in L(H, U)$ such that $\sigma(E, A + BF) = \Lambda$.

Theorem 2.3. *If the system (E, A, B) has the spectrum assignment property, then it is exactly controllable.*

PROOF. Let the system (E, A, B) have the spectrum assignment property, i.e. for any nonempty compact subset Λ of $\mathcal{C}$, there exists $F \in L(H, U)$ such that $\sigma(E, A + BF) = \Lambda$. Take Λ such that $(E + BF)^{-l}$ exists. Then,

$$(E + BF)(H) = H. \tag{19}$$

Multiplying both sides of (19) by $(\alpha E + A)^{-l}$ on the left yields

$$(E_1 + B_1 F)(H) = (\alpha E + A)^{-1}(H) = H,$$

which can be written as follows:

$$(E_1 - \lambda I + B_1 F)(H) = (I - \lambda I)(H) = (1 - \lambda) I(H) = H,$$

i.e.

$$H = (E_1 - \lambda I)(H) + (B_1 F)(H) \subseteq (E_1 - \lambda I)(H) + (B_1)(U).$$

Since $(E_1 - \lambda I)(H) + (B_1)(U) \subseteq H$, we get

$$(E_1 - \lambda I)(H) + (B_1)(U) = H,$$

which means that the system (E_l, B_l) is exactly controllable by Theorem 2.2. It follows from Theorem 2.1 that the system (E, A, B) is exactly controllable which completes the proof of Theorem 2.3.

Theorem 2.4. *The system (E, A, B) is exactly controllable if and only if for each $F \in L(H, U)$, the system $(E, A + BF, B)$ is exactly controllable.*

Theorem 2.5. *The system (E, A, B) has the spectrum assignment property if and only if for each $F \in L(H, U)$, the system $(E, A + BF, B)$ has the spectrum assignment property.*

Proof. Let the system (E, A, B) have the spectrum assignment property, i.e. for any nonempty compact subset Λ of $\mathcal{C}$, there exists $F_1 \in L(H, U)$ such that:

$$\sigma(E, A + BF_1) = \Lambda$$

$$\begin{aligned} \sigma(E, A + BF_1) &= \sigma(E, A + BF_1 + BF - BF) \\ &= \sigma(E, A + BF + BF_2) \end{aligned}$$

where $F_2 = F_1 - F$. Then

$$\sigma(E, A + BF + BF_2) = \Lambda,$$

which means that for given Λ there exists $F_2 \in L(H, U)$ such that:

$$\sigma(E, A + BF + BF_2) = \Lambda,$$

i.e. $(E, A + BF, B)$ has the spectrum assignment property.

Conversely let $(E, A+BF, B)$ have the spectrum assignment property, i.e. for given nonempty compact subset Λ' of $\mathcal{C}$, there exists $F_3 \in L(H, U)$ such that:

$$\sigma(E, A + BF + BF_3) = \Lambda'$$

$$\begin{aligned}\sigma(E, A + BF + BF_3) &= \sigma(E, A + B(F + F_3)) \\ &= (E, A + BF_4),\end{aligned}$$

where $F_4 = F_3 - F$. Then for given Λ', there exists $F_4 \in L(H, U)$ such that:

$$\sigma(E, A + BF_4) = \Lambda'$$

i.e. (E, A, B) has the spectrum assignment property.

Theorem 2.6. *If the system (E, A, B) is exactly controllable, then it has the spectrum assignment property.*

Proof. The system (3) can be written in the form:

$$\begin{pmatrix} I_Y & 0 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} (t) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(t). \tag{20}$$

Choosing $F = [0F_2]$ such that $(A_{22} + B_2F_2)^{-1}$ exists and using (9), the closed loop system takes the from:

$$\begin{pmatrix} I_Y & 0 \\ 0 & 0 \end{pmatrix} \dot{x}(t) = \begin{pmatrix} A_{11} & A_{12} + B_1F_2 \\ A_{21} & A_{22} + B_2F_2 \end{pmatrix} x(t) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} v(t). \tag{21}$$

Define the two operators

$$S = \begin{pmatrix} I_Y & -(A_{12} + B_1F_2)(A_{22} + B_2F_2)^{-1} \\ 0 & I_{Y^\perp} \end{pmatrix} \tag{22}$$

and

$$T = \begin{pmatrix} I_Y & 0 \\ -(A_{22} + B_2F_2)^{-1}A_{21} & (A_{22} + B_2F_2)^{-1} \end{pmatrix}. \tag{23}$$

It is clear that S, T are invertible operators. Setting $x(t) = Tz(t)$, (21) becomes:

$$\begin{pmatrix} I_Y & 0 \\ 0 & 0 \end{pmatrix} T\dot{z}(t) = \begin{pmatrix} A_{11} & A_{12} + B_1F_2 \\ A_{21} & A_{22} + B_2F_2 \end{pmatrix} Tz(t) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} v(t). \tag{24}$$

Multiplying both sides of (24) on the left by S yields

$$S\begin{pmatrix} I_Y & 0 \\ 0 & 0 \end{pmatrix} T\dot{z}(t) = S\begin{pmatrix} A_{11} & A_{12}+B_1F_2 \\ A_{21} & A_{22}+B_2F_2 \end{pmatrix} Tz(t) + S\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} v(t). \tag{25}$$

Now,

$$S\begin{pmatrix} I_Y & 0 \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} I_Y & 0 \\ 0 & 0 \end{pmatrix} \tag{26}$$

$$\begin{aligned} &S\begin{pmatrix} A_{11} & A_{12}+B_1F_2 \\ A_{21} & A_{22}+B_2F_2 \end{pmatrix} T \\ &= \begin{pmatrix} A_{11}-(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}A_{12} & 0 \\ 0 & I_{Y^\perp} \end{pmatrix} \end{aligned} \tag{27}$$

$$S\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} B_1-(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}B_2 \\ B_2 \end{pmatrix} \tag{28}$$

i.e. (25) becomes

$$\begin{aligned} \begin{pmatrix} I_Y & 0 \\ 0 & 0 \end{pmatrix}\dot{z} &= \begin{pmatrix} A_{11}-(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}A_{12} & 0 \\ 0 & I_{Y^\perp} \end{pmatrix} z \\ &+ \begin{pmatrix} B_1-(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}B_2 \\ B_2 \end{pmatrix} v. \end{aligned} \tag{29}$$

Suppose that the system (E, A, B) is exactly controllable, then by Theorem 2.4, for each $F \in L(H,U)$ the system $(E, A+BF, B)$ is exactly controllable, i.e. the system in (24) is exactly controllable which implies that the normal subsystem (G, D) is exactly controllable, where

$$G = A_{11} - (A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1},$$

$$D = B_1 - (A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}B_2.$$

It follows from [4] that this normal subsystem has the spectrum assignment property, i.e. for any nonempty compact subset Λ of C, there exists an operator $K = [F_1 0]$ such that $\sigma(G+DK) = \Lambda$. Put $v(t) = [F_1 0]z(t)+\omega(t)$. The overall feedback control is

$$u(t) = \overline{F}x(t) + \omega(t),$$

where $\overline{F} = F + KT^{-1} = [0F_2] + [F_1 0]T^{-1}$. Now for such F, we get:

$$\begin{aligned}
\sigma(E, A + B\overline{F}) &= \sigma(S(E, A + B\overline{F})T) \\
&= \sigma(SET, S(A + B\overline{F})T) \\
&= \{s \in \mathcal{C} : sSET - S(A + B\overline{F})T \text{ is singular}\} \\
&= \{s \in \mathcal{C} : sSET - S(A + BF)T + SBK \text{ is singular}\} \\
&= \{s \in \mathcal{C} : \begin{pmatrix} sI_Y & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} G + DK & 0 \\ B_2 & I_{Y^\perp} \end{pmatrix} \text{ is singular}\} \\
&= \{s \in \mathcal{C} : \begin{pmatrix} sI_Y - (G + DK) & 0 \\ B_2 & I_{Y^\perp} \end{pmatrix} \text{ is singular}\}.
\end{aligned}$$

The matrix operator $\begin{pmatrix} sI_Y - (G + DK) & 0 \\ B_2 & I_{Y^\perp} \end{pmatrix}$ is singular if and only if the operator $sI_Y - (G + DK)$ is singular. Then

$$\begin{aligned}
\sigma(E, A + B\overline{F}) &= \{s \in \mathcal{C} : sI_Y - (G + DK) \text{ is singular}\} \\
&= \sigma(I_Y, G + DK) = \sigma(G + DK) = \Lambda
\end{aligned}$$

which means that the system (E, A, B) has the spectrum assignment property. This completes the proof.

Theorem 2.7. *The set*

$$\{(E, A, B) \in L(H) \times L(H) \times L(H, U) : (E, A, B) \text{ is exactly controllable}\} \tag{30}$$

is open.

Proof. If (E, A, B) is exactly controllable, then the system (E_1, B_1) is exactly controllable i.e.

$$\operatorname{range}[E_1 - \lambda I, B_1] = H \quad \text{for every } \lambda \in \mathcal{C} \tag{31}$$

which means that the operator $[E_1 - \lambda I, B_1]$ is right invertible. Since the set of all right invertible operators is open [1], the theorem is obvious.

If (E_1, B_1) satisfies (31), then for (E_2, B_2) in a neighbourhood of (E_1, B_1), (31) holds.

Theorem 2.8. *The set*

$$\{(E,A,B)L(H)\times L(H)\times L(H,U): \quad (E,A,B) \text{ has the spectrum assignment property}\} \tag{32}$$

is open.

PROOF. From Theorems 2.3 and 2.6 we have the equivalence between exact controllability and spectrum assignment property. Then the elements of the set in (32) are exactly controllable, so by Theorem 2.7 we have the result.

References

[1] A. BROWN and C. PEAREY, Introduction to Operator Theory, Elements of Functional Analysis, *Springer Verlag*, 1977.

[2] G. ECKSTEIN, Exact Controllability and Spectrum Assignment, Topics in Modern Operator Theory (Apostol C. et al, ed.), *Birkhauser*, 1981.

[3] L. GOHBERG and S. GOLDBERG, Basic Operator Theory, *Birkhauser*, 1981.

[4] M. R. A. MOUBARAK, Controllability for Infinite Dimensional Systems, Ph.D. Thesis, *Birmingham Univ.*, *England*, 1986.

[5] M. R. A. MOUBARAK, Controllability for Infinite Dimensional Linear singular Systems, *Bull. FAC. SCI. Assiut Univ.* **23** (I-C) (1994), 15–24.

[6] W. M. WONHAM, On Pole Assignment in Multi-Input Controllable Linear systems, *J. IEEE Trans., Automatic Control,* V. AC-12 no. 6 5 (1976), 660–665.

MOHAMED RABIE ABDALLAH MOUBARAK
MATHEMATICS DEPARTMENT
FACULTY OF SCIENCE
EL–MINIA UNIVERSITY, EL–MINIA
EGYPT

(Received May 13, 1996; revised May 28, 1997)