

A new kind of infinite dimensional spaces

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Abstract. In this paper we introduce a kind of infinite dimensional spaces, called finite dimensional separated, related to a dimension defined by G. STEINKE in [12].

This new kind of infinite dimensional spaces satisfies some properties such as the subspace theorem, a product theorem and an addition theorem. Likewise, we shall obtain characterizations of this property for regular spaces, compactifications of the space that preserve finite dimensional separateness and relations with some transfinite dimensions and another kind of infinite dimensionality (w.i.d. spaces, C-spaces). We also define a related property which has a local theorem.

1. Introduction

A very important part of Dimension Theory is what is called Infinite Dimensional Theory which is devoted to study spaces like countable dimensional, strongly countable dimensional, weakly infinite dimensional (in the senses of ALEXANDROFF and SMIRNOV), C-spaces and so on.

In this paper we introduce a kind of infinite dimensional spaces, called finite dimensional separated, related to the dimension t defined by G. STEINKE in [12].

This new kind of infinite dimensional spaces satisfies some properties such as the subspace theorem, a product theorem and an addition theorem. Likewise, we shall obtain characterizations of this property for regular spaces, compactifications of the space that preserve the finite dimensional separateness and relations with some transfinite dimensions and another kind of infinite dimensionality (w.i.d. spaces, C-spaces).

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It is important to note that some of the properties of our new kind of infinite dimensionality are not satisfied by the classical infinite dimensionalities cited above.

All the topological spaces considered in this paper are supposed to be T_1 .

This definition is quoted from [12], Definition 1.1.

Definition 1.1. A subset T of a nonempty topological space X is called a (closed) separating set of X if (T is closed and) $X \setminus T$ is not connected.

The separation dimension of a space X , denoted by $t(X)$ is defined inductively as follow: $t(\emptyset) = -1$; $t(X) = 0$ if $|X| = 1$. Let $|X| \geq 2$ and $n \in \mathbb{N} \cup \{0\}$; if for any subset M with $|M| \geq 2$ a separating set T of M exists with $t(T) \leq n - 1$ then we say $t(X) \leq n$. If $t(X) \leq n$ is true and $t(X) \leq n - 1$ is false then $t(X) = n$. If $t(X) \leq n$ is false for all $n \in \mathbb{N}$ then $t(X) = \infty$.

Definition 1.2. A space X is finite-dimensional separated (briefly f.d.s.) if, for any subset M of X with $\text{card}(M) \geq 2$, there exists a separating set T of M with $t(X) < \infty$.

Remark 1.3. If X is a topological space with $t(X) < \infty$ then X is f.d.s.

Remark 1.4. For a topological space X , note that $t(X) = 0$ if and only if X is totally disconnected (see [12, Corollary 2.1]) and there are totally disconnected spaces which are not countable dimensional (c.d.) (see [10], Lemma), then the classes f.d.s. and c.d. are not equal.

For regular spaces the finite-dimensional separateness can be described in an easier way. To do this we need the following lemma.

Lemma 1.5. *In a regular space X with $\text{card}(X) \geq 2$, for any separating set T of X there exists a closed separating set contained in T .*

PROOF. Let T be a separating set of X . Since $Y = X \setminus T$ is not connected, there exist two nonempty disjoint open sets in the subspace Y , A and B , such that $X = T \cup A \cup B$. Let then U and V two open subsets in the space X such that $A = U \cap Y$ and $B = V \cap Y$. Consider the open set $W = U \cap V$. If $W = \emptyset$, we take $S = \emptyset$. Otherwise, there exists, because X is regular, a nonempty open subset Z such that $\bar{Z} \subset W$. Let then

$S = \text{Fr}(Z)$. It can be easily seen that $S \subset T$ and S is a closed separating set of X . \square

Now we can prove the following characterization of the property f.d.s. for regular spaces.

Proposition 1.6. *For a regular space X are equivalent:*

1. X is f.d.s.;
2. for any subset M of X with $\text{card}(M) \geq 2$ there exists a closed separating set T of M with $t(T) < \infty$;
3. for any closed subset M of X with $\text{card}(M) \geq 2$ there exists a separating set T of M with $t(T) < \infty$;
4. for any closed subset M of X with $\text{card}(M) \geq 2$ there exists a closed separating set T of M with $t(T) < \infty$.

PROOF. The only implication we have to show is $4 \Rightarrow 1$. Let M a subset of X with $\text{card}(M) \geq 2$ then \overline{M} is a closed subset of X with $\text{card}(\overline{M}) \geq 2$ and there exists a closed separating set T of \overline{M} with $t(T) < \infty$. As $M \setminus T \subset \overline{M} \setminus T \subset \overline{M \setminus T}$ and $\overline{M} \setminus T$ is not connected, then $M \setminus T = M \setminus (M \cap T)$ is not connected and $M \cap T$ is a separating set of M with $t(M \cap T) \leq t(T) < \infty$. \square

For locally compact metrizable spaces we have the following characterization.

Proposition 1.7. *Let X be a locally compact metrizable space, then are equivalent:*

1. X is f.d.s.;
2. for all closed subset M of X with $\text{card}(M) \geq 2$ there exists a closed separating set T of M such that $\dim(T) < \infty$ ($\text{ind}(T) < \infty$, $\text{Ind}(T) < \infty$).

PROOF. Note that for any locally compact metrizable space T , $t(T) = \dim(T) = \text{ind}(T) = \text{Ind}(T)$ (see [12, Theorem 4.7]). \square

2. Classical theorems

In this section we develop some theorems such as the subspace or the product theorems for f.d.s. spaces.

Theorem 2.1 (The subspace theorem). *If A is a subspace of a f.d.s. space X then A is f.d.s.*

PROOF. Let M be a subset of A with $\text{card}(M) \geq 2$, then there exists a separating set T of M (in X) with $t(T) < \infty$. Thus T is a separating set of M in A with $t(T) < \infty$. \square

The following theorem shows a distinguishing property of f.d.s. spaces.

Theorem 2.2. *Let $\{X_i : i \in I\}$ be the family of connected components of the space X , then:*

$$X \text{ is f.d.s. if and only if } \forall i \in I, X_i \text{ is f.d.s.}$$

PROOF. Suppose that X is f.d.s. For each $i \in I$, $X_i \subset X$ and then X_i is f.d.s.

For the converse, suppose that X_i is f.d.s., $\forall i \in I$. Let M be a subspace of X with $\text{card}(M) \geq 2$. If M is not connected then $T = \emptyset$ separates M and $t(T) = -1$. If M is connected there exists $i \in I$ such that $M \subset X_i$ and exists a separating set T of M in X_i with $t(T) < \infty$. Thus T is a separating set of M in X with $t(T) < \infty$. \square

Corollary 2.3. *If $X = \bigoplus_{i \in I} X_i$ then:*

$$X \text{ is f.d.s. if and only if } \forall i \in I, X_i \text{ is f.d.s.}$$

Proposition 2.4. *Let X, Y be two nonempty topological spaces and let $f : X \rightarrow Y$ be a continuous mapping with $t(f) = \sup\{t(f^{-1}(y)) : y \in Y\} < \infty$ (see [12, 2.1]). If Y is f.d.s. then X is f.d.s.*

PROOF. Let M be a subspace of X with $\text{card}(M) \geq 2$, then $f(M) \subset Y$. If $\text{card}(f(M)) = 1$, there exists $y \in Y$ such that $M \subset f^{-1}(y)$ and $T = M \setminus \{a, b\}$, with $a, b \in M$ $a \neq b$, is a separating set of M with $t(T) \leq t(f^{-1}(y)) \leq t(f) < \infty$, (see [12, Theorem 1.1]). If $\text{card}(f(M)) \geq 2$, there exists a separating set K of $f(M)$ in Y with $t(K) < \infty$. If we put $T = f^{-1}(K)$, we have, since f is continuous, $t(T) \leq t(K) + t(f)$ (see [12, 2.2]), with implies, because $f(M \setminus T) = f(M) \setminus K$ is not connected that T is a separating set of M . \square

Corollary 2.5 (Product theorem). *Let X be a topological space with $t(X) < \infty$ and let Y be a f.d.s. space. Then $X \times Y$ is f.d.s.*

PROOF. Let $p_2 : X \times Y \rightarrow Y$ the continuous mapping $p_2(x, y) = y$, $\forall (x, y) \in X \times Y$. Then $t(p_2) = \text{Sup}\{t(p_2^{-1}(y)) : y \in Y\} = \text{Sup}\{t(X \times \{y\}) : y \in Y\}$. Since $X \times \{y\}$ is homeomorphic to X for all $y \in Y$ and $t(X \times \{y\}) = t(X) \forall y \in Y$, we have $t(p_2) = t(X) < \infty$. Using the previous theorem, $X \times Y$ is f.d.s. \square

Remark 2.6. It is not known if the product of two w.i.d. compacta is w.i.d. However, for f.d.s. spaces, the following example shows that the product of two f.d.s. spaces is not necessarily f.d.s. In fact, following ENGELKING (see [4, p. 260]) one can get a metric continuum X such that each closed set in X can be separated by a finite-dimensional compactum (i.e. X is f.d.s.), but the square $X \times X$ can not be separated by any finite-dimensional set (i.e. the square fails to be f.d.s.).

3. A sum theorem

The following property is a variant of the f.d.s. property in a way similar to the usual one to construct local dimensions. It will be useful in order to obtain an addition theorem for f.d.s. spaces and it is also interesting by itself.

Definition 3.1. A space X is point finite dimensional separated (briefly p.f.d.s.) if for every $x \in X$ and every open neighborhood U of x there exists an open neighborhood V of x such that $\bar{V} \subset U$ and $t(\text{Fr}(V)) < \infty$. Moreover, this condition implies that X is a regular space.

Clearly p.f.d.s. spaces fulfill the subspace theorem. We also have the following theorem that shows that p.f.d.s. is a local concept.

Proposition 3.2 (Local theorem). *Let X be a regular space. Then X is p.f.d.s. if and only if every point of X has a neighborhood which is p.f.d.s.*

PROOF. If X is p.f.d.s., then X is a neighborhood of each of its points which is p.f.d.s.

For the converse, let $x \in X$, W a p.f.d.s. open neighborhood of x and U an open neighborhood of x ; then $U \cap W$ is an open neighborhood of x

and, as X is a regular space, there exists an open neighborhood B of x such that $B \subset \overline{B} \subset U \cap W$. Thus B is an open neighborhood of x in W and by hypothesis there exists an open neighborhood of x in W (and so in X), V , such that $V \subset \overline{V}^W \subset B$ and $t(\text{Fr}_W(V)) < \infty$. Then $\overline{V} \subset \overline{B} \subset U \cap W$; in particular $\overline{V} \subset W \Rightarrow \overline{V}^W = \overline{V}$ and $\text{Fr}_W(V) = \text{Fr}(V)$. Hence V is an open neighborhood of x in X such that $\overline{V} \subset U$ and $t(\text{Fr}(V)) < \infty$. \square

Here is the relation between p.f.d.s. and f.d.s. spaces.

Proposition 3.3. *Let X be a space. If X is p.f.d.s. then X is f.d.s.*

PROOF. Let M be a subset of X with $\text{card}(M) \geq 2$ and let $x, y \in M$ be with $x \neq y$; there exists an open neighborhood U of x such that $y \notin U$ and there exists V an open neighborhood of x with $\overline{V} \subset U$ and $t(V) < \infty$. Thus $M \cap \text{Fr}(V)$ is a separating set of M since $M \setminus (M \cap \text{Fr}(V)) = M \setminus \text{Fr}(V) = (V \cap M) \cup (M \setminus \overline{V}) = (V^\circ \cap M) \cup (M \cap (X \setminus \overline{V}))$, and $t(M \cap \text{Fr}(V)) \leq t(\text{Fr}(V)) < \infty$. Hence X is f.d.s. \square

Now we have a kind of addition theorem mixing both properties.

Proposition 3.4. *Let the regular space X be the union of two sets A, B . If A is closed and f.d.s. and B is p.f.d.s. then X is f.d.s.*

PROOF. Let M be a subset of X with $\text{card}(M) \geq 2$ such that $M \setminus A \neq \emptyset$ and $M \cap A \neq \emptyset$ (otherwise $M \subset A$ or $M \subset B$ and both, A and B , are f.d.s. spaces and we can find a separating set T of M with $t(T) < \infty$). Choose $x \in M \setminus A$ and an open neighborhood U of x with $\overline{U} \subset X \setminus A \subset B$; for U there exists an open neighborhood V of x in B such that $\overline{V}^B \subset U$ and $t(\text{Fr}_B(V)) < \infty$. From $V \subset U$ and $\overline{V} \subset B$ we get that V is open, $\overline{V}^B = \overline{V}$ is closed and $\text{Fr}_B(V) = \text{Fr}(V)$; hence $T = (\overline{V} \setminus V) \cap M$ is a separating set of M since $M \setminus T = M \setminus (\overline{V} \setminus V) = (M \cap U^\circ) \cup (M \cap (X \setminus \overline{U}))$, with $t(T) \leq t(\text{Fr}(U)) < \infty$. Then X is f.d.s. space. \square

4. Compactifications

In this section we consider compactifications of a nonempty locally compact, Hausdorff space X and $X^* = X \cup \{a\}$ denotes the one-point compactification of X .

Proposition 4.1. *For any locally compact, Hausdorff space X :*

X is f.d.s. if and only if X^ is f.d.s.*

PROOF. Suppose that X^* is f.d.s. then $X \subset X^*$ is also f.d.s. (see 2.1).

For the converse, suppose that X is a f.d.s. space and let M be a subset of X^* with $\text{card}(M) \geq 3$ (for $\text{card}(M) = 2$ the empty set separates M). There are two cases:

1. $a \notin M$, then M is a subset of X and there exists a separating set T of M such that $t(T) < \infty$;
2. $a \in M$, then $M \setminus \{a\} \subset X$ and there exists a separating set S of $M \setminus \{a\}$ in X such that $t(S) < \infty$. Thus $T = S \cup \{a\}$ is a separating set of M with $t(T) \leq t(S) + 1$ (see [12, 3.1]). Then X^* is f.d.s.

Corollary 4.2. *Let Y be a compactification of a nonempty, locally compact, Hausdorff space X such that $t(Y \setminus X) < \infty$. Then:*

X is f.d.s. if and only if Y is f.d.s.

This especially holds for X^ and the Freudenthal compactification (see [1, p. 266–282]).*

PROOF. It is well known (see [7, p. 12–13]) that the compactifications of X form a complete lattice with minimal element X^* , i.e. for any compactification Y of X there exists a continuous mapping $f : Y \rightarrow X^*$ with $f(x) = x$ for all $x \in X$ and $f(Y \setminus X) = \{a\}$. Then $t(f) = \text{Sup}\{t(f^{-1}(y)) : y \in X^*\} = \text{Sup}\{\text{Sup}\{t(x) : x \in X\}, t(Y \setminus X)\} < \infty$; since X f.d.s. then X^* is f.d.s., so Y f.d.s. (from 2.4).

For the converse, see 2.1. □

5. Examples

In this section we show some examples of f.d.s. spaces. We first recall the definition of trind from [3].

Definition 5.1. We define $\text{trind}(\emptyset) = -1$. Let X be a nonempty topological space. We say that $\text{trind } X \leq \alpha$, where α is an ordinal number or zero, if and only if for every point x in X and every open set V with $x \in V$, there exists an open subset U with $x \in U \subset V$, such that $\text{trind } \text{Fr } U < \alpha$.

We say that $\text{trind } X = \alpha$, if and only if $\text{trind } X \leq \alpha$ and it is not true that $\text{trind } X \leq \beta$ for every $\beta < \alpha$. If $\text{trind } X > \alpha$ for every ordinal number, we say that $\text{trind } X$ does not exist.

1. *Example 5.2.* Let X be a regular space with $\text{trind}(X) \leq \omega_0$, then X is a p.f.d.s. space (and so a f.d.s. space).

PROOF. Let $x \in X$ and U an open neighborhood of x , then there exists an open neighborhood V of x such that $\overline{V} \subset U$ and $\text{trind}(\text{Fr}(V)) < \infty$. Thus $t(\text{Fr}(V)) < \infty$ (see [12, Proposition 4.1]). \square

2. *Example 5.3.* The Henderson's spaces J^α , (see [6, Definition 2.2]) are f.d.s. spaces for every ordinal number α .

PROOF. We define, for every ordinal number α , J^α as follows:

- (a) $J^0 = \{0\}$, $p_0 = \{0\}$;
- (b) $J^1 = I = 3D[0, 1]$, $p_1 = \{0\}$;
- (c) $J^{\alpha+1} = J^\alpha \times I$, $p_{\alpha+1} = p_\alpha \times \{0\}$;
- (d) if α is a limit ordinal, for $\beta < \alpha$, let A_α^β be a half-open arc with $A_\alpha^\beta \cap J^\beta = p_\beta$, let $J^\alpha = (\bigoplus_{\beta < \alpha} A_\alpha^\beta \cup J^\beta)^*$, and let p_α the compactification point.

Let show that J^α is a f.d.s. space for every ordinal number α . We proceed inductively. For $\alpha = 0$ is trivial. Supposed for an ordinal number α that J^β is a f.d.s. space $\forall \beta < \alpha$, then there are two cases:

- (a) α is not a limit ordinal, then $J^\alpha = J^{\alpha-1} \times I$ where $J^{\alpha-1}$ is a f.d.s. space by the inductive hypothesis and $t(I) = 1$; using 2.5 J^α is a f.d.s. space;
 - (b) α is a limit ordinal, then J^α is the one-point compactification of the space $\bigoplus_{\beta < \alpha} A_\alpha^\beta \cup J^\beta$. For every ordinal number $\beta < \alpha$, J^β is closed in $A_\alpha^\beta \cup J^\beta$ and f.d.s. by the inductive hypothesis and easily A_α^β is p.f.d.s., so using 3.4, the space $A_\alpha^\beta \cup J^\beta$ is f.d.s. By 2.3, $\bigoplus_{\beta < \alpha} A_\alpha^\beta \cup J^\beta$ is a f.d.s. space and from 4.1, its one-point compactification J^α is also f.d.s. \square
3. *Example 5.4.* The Smirnov's spaces S_α (see [3, Example 2.2]) are f.d.s. spaces for every ordinal number α .

PROOF. The Smirnov's spaces S_α , with α an ordinal number, are defined by transfinite induction:

- (a) $S_0 = \{0\}$ is a one-point space;
- (b) $S_{\alpha+1} = S_\alpha \times I$;
- (c) if α is a limit ordinal, $S_\alpha = (\bigoplus_{\beta < \alpha} S_\beta)^*$.

In a similar way to last example, this spaces are f.d.s. spaces. \square

6. Relation with other dimension functions

This definition is quoted from [2, Definition 1.2].

Definition 6.1. A continuous mapping $f : X \rightarrow Y$ of a space X into another space Y is called light if for every point $y \in Y$, $f^{-1}(y)$ is totally disconnected. In particular a light mapping f satisfies $t(f) = 0$.

The following definition is quoted from [2, 2.1].

Definition 6.2. Let X be a topological space. We define the light dimension of the space, denoted by $L\text{-dim } X$, as follows: $L\text{-dim}(\emptyset) = -1$. Let α be an ordinal number or zero, $L\text{-dim } X \leq \alpha$ if and only if there exists a light mapping $f : X \rightarrow J^\alpha$. If the last condition is not satisfied for any α ordinal number or zero, we say that $L\text{-dim}(X)$ does not exist.

Remark 6.3. Note that, if $L\text{-dim}(X)$ exists, using that J^α is f.d.s. and 2.4, it is clear that X is f.d.s.

We also quote the definition of countable dimensionality from [3].

Definition 6.4. A space X is called countable dimensional (c.d.) if it can be written as a countable union of finite-dimensional subspaces.

Proposition 6.5. *Let $f : X \rightarrow Y$ be a continuous mapping from a compact metrizable space X onto a Hausdorff c.d. space Y with $t(f) < \infty$. Then X is c.d. In particular if X is a compact metrizable space such that $L\text{-dim}(X)$ exists, then X is c.d.*

PROOF. For every $y \in Y$, $f^{-1}(y)$ is a compact metrizable space, so $\dim(f^{-1}(y)) = t(f^{-1}(y)) \leq t(f) < \infty$ (see [12, Theorem 4.7]) where \dim is the covering dimension (see [9, Definition I.4]). Then f is a closed continuous mapping from a metrizable space X onto a c.d. space Y such that $\dim f^{-1}(y) < \infty$ for every $y \in Y$ and so X is c.d. (see Theorem 7.7 of [5]).

Finally, if X is a compact metric space with light dimension, using that J^α is compact metric and c.d. (see [6]), all its subspaces are c.d. (see [5, Proposition 2.2]) and so we can obtain a light mapping from X onto a c.d. subspace of J^α . \square

The following result gives a relation between our new concept of f.d.s. spaces and the classical one of C-spaces.

Theorem 6.6. *If X is a compact metric f.d.s. space, then X is a C-space (see [5, 8.15]) and hence is weakly infinite-dimensional (w.i.d.) (see [5, 8.1 and 8.17]).*

PROOF. Suppose that X is a compact metric space which is not a C-space. Then there exists a non-trivial metric continuum $K \subset X$ without any compact subsets of positive finite dimension (see [8, Theorem 3.6]). By a theorem of TUMARKIN (see [13], also see footnote 3 in p. 162 of [9], K contains an infinite-dimensional Cantor manifold (that is, a compact space F such that $\dim(F) = \infty$ and for any subset A of F with $\dim(A) < \infty$, $F \setminus A$ is connected) and then X is not f.d.s. \square

Remark 6.7. Recall from Example 8.19 of [5] that there is a compact metric C-space that has a subspace that is not C-space. However it is f.d.s., so compactness is essential in the above theorem. From [10], Comment C, we can also conclude that f.d.s. does not imply w.i.d. if compactness is suppressed.

Remark 6.8. If X is the space of remark 2.6, $X \times X$ is a compact metric C-space (see Theorem 3 of [11]) (and hence w.i.d.) that is not f.d.s., so Theorem 6.6 cannot be reversed even for compact metric spaces.

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