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On Nemytskii operator

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Abstract. J. MATKOWSKI has proved that every function h, which generates the Nemytskii operator mapping the Banach space of Lipschitzian functions into itself must be Jensen function with respect to the second variable (see [2], [3], [4]). This paper is devoted to Nemytskii operator defined on a space of polynomials with values in a space of functions of class C^1.

We will start with some definitions and notations. Let $f : \langle a, b\rangle \to \mathbb{R}$, and $x_1, \dots, x_p \in \langle a, b\rangle$ be distinct points. The *divided difference* $[x_1, \dots, x_p; f]$ of f at points $x_1, \dots, x_p$ is defined by recurrence

$$[x_1, f] = f(x_1),$$

$$[x_1, \dots, x_p; f] = \frac{[x_2, \dots, x_p; f] - [x_1, \dots, x_{p-1}; f]}{x_p - x_1}, \quad p \geq 2$$

(cf. [1]). Let us consider

$$\mathrm{lip}^2(\langle a, b\rangle) := \left\{ f : \langle a, b\rangle \to \mathbb{R} : \sup_{\substack{t_1, t_2, t_3 \in \langle a, b\rangle \\ t_i \neq t_j,\ j \neq i}} |[t_1, t_2, t_3; f]| < \infty \right\},$$

and denote as $P^2(\langle a, b\rangle)$ the set of all restrictions of polynomials of degree at most 2 to the interval $\langle a, b\rangle$.

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Then $P^2(\langle a,b\rangle)$ is a subspace of a real Banach space $\mathrm{lip}^2(\langle a,b\rangle)$ with the norm

$$\|f\| := |f(a)| + |f(b)| + \sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle \\ t_i\neq t_j,\ j\neq i}} |[t_1,t_2,t_3;f]|.$$

Every function $h : \langle a,b\rangle \times \mathbb{R} \to \mathbb{R}$ generates the Nemytskii operator defined on the space $P^2(\langle a,b\rangle)$ by formula

$$N\varphi(x) := h(x,\varphi(x)), \quad \varphi \in P^2(\langle a,b\rangle),\ x \in \langle a,b\rangle.$$

Lemma. *There exists a constant $M > 0$ such that $\|A\varphi\| \le M\|A\| \cdot \|\varphi\|$ for all $A,\varphi \in \mathrm{lip}^2(\langle a,b\rangle)$.*

Proof. Indeed, let us fix $x,y,z \in \langle a,b\rangle$, distinct. Then

$$|[x,y,z;A\varphi]| \le |A(y)| \cdot \|\varphi\| + \|A\| \cdot |\varphi(z)| + \left|\frac{A(x)-A(y)}{x-y}\right| \cdot \left|\frac{\varphi(x)-\varphi(z)}{x-z}\right|.$$

It is easy to check, that

$$|\varphi(z)| \le (2+|a–b|^2)\|\varphi\|, \quad \left|\frac{\varphi(z)-\varphi(x)}{z-x}\right| \le \left(2|a-b| + \frac{1}{|a-b|}\right)\|\varphi\|,$$

(and analogous formulas hold for A). Thus

$$|[x,y,z;A\varphi]| \le 2(2+|a-b|^2)\|A\|\cdot\|\varphi\| + \left(2|a-b| + \frac{1}{|a-b|}\right)^2 \|A\|\cdot\|\varphi\|$$

for all $x,y,z \in \langle a,b\rangle$, distinct and

$$\|A\varphi\| \le \left(10 + 6|a-b|^2 + \frac{1}{|a–b|^2}\right)\|A\|\cdot\|\varphi\|.$$

□

Theorem. *Let $h : \langle a,b\rangle \times \mathbb{R} \to \mathbb{R}$. Then the Nemytskii operator $N : P^2(\langle a,b\rangle) \to \mathrm{lip}^2(\langle a,b\rangle)$ defined by h satisfies the Lipschitz condition*

$$(1) \qquad \|N\varphi_1 - N\varphi_2\| \le L\|\varphi_1 - \varphi_2\|, \quad \varphi_1,\varphi_2 \in P^2(\langle a,b\rangle),$$

iff there exist functions $A,B \in \mathrm{lip}^2(\langle a,b\rangle)$ such that

$$h(x,y) = A(x)y + B(x), \quad x \in \langle a,b\rangle,\ y \in \mathbb{R}.$$

Proof. Suppose that h generates the Nemytskii operator N satisfying (1). For a fixed $y_0 \in \mathbb{R}$, the function $\varphi_0(x) = y_0$, $x \in \langle a, b\rangle$ belongs to $P^2(\langle a, b\rangle)$. Consequently

$$N\varphi_0 = h(\cdot, y_0) \in \operatorname{lip}^2(\langle a, b\rangle) \quad \text{for all } y_0 \in \mathbb{R}.$$

By the proof of Theorem 1 in ([1], p. 391) the function $h(\cdot\,, y_0)$ is of class C^1 in $\langle a, b\rangle$, in particular h is continuous with respect to the first variable for every $y_0 \in \mathbb{R}$. Let us fix $x, y, z \in \langle a, b\rangle$, distinct, $y_1, y_2, y_3, \bar{y}_1, \bar{y}_2, \bar{y}_3 \in \mathbb{R}$ and define finctions $\varphi_1, \varphi_2 : \langle a, b\rangle \to \mathbb{R}$,

$$\varphi_i(t) = A_i t^2 + B_i t + C_i, \qquad t \in \langle a, b\rangle,\ i = 1, 2,$$

where

$$A_1 = \frac{\frac{y_3 - y_2}{z - y} - \frac{y_2 - y_1}{y - x}}{z - x}, \qquad A_2 = \frac{\frac{\bar{y}_3 - \bar{y}_2}{z - y} - \frac{\bar{y}_2 - \bar{y}_1}{y - x}}{z - x},$$

$$B_1 = \frac{y_3 - y_1}{z - x} - (z + x)A_1, \qquad B_2 = \frac{\bar{y}_3 - \bar{y}_1}{z - x} - (z + x)A_2,$$

$$C_1 = A_1 zx + y_1 - \frac{y_3 - y_1}{z - x}x, \qquad C_2 = A_2 zx + \bar{y}_1 - \frac{\bar{y}_3 - \bar{y}_1}{z - x}x.$$

Then $\varphi_1(x) = y_1$, $\varphi_1(y) = y_2$, $\varphi_1(z) = y_3$, $\varphi_2(x) = \bar{y}_1$, $\varphi_2(y) = \bar{y}_2$, $\varphi_2(z) = \bar{y}_3$, $\varphi_1, \varphi_2 \in P^2(\langle a, b\rangle)$, and

$$\begin{aligned} \|\varphi_1 - \varphi_2\| &= |(A_1 - A_2)a^2 + (B_1 - B_2)a + C_1 - C_2| + |(A_1 - A_2)b^2 \\ &\quad + (B_1 - B_2)b + C_1 - C_2| + |A_1 - A_2| \\ &\le \left| \frac{\frac{y_3 - \bar{y}_3 - y_2 + \bar{y}_2}{z - y} - \frac{y_2 - \bar{y}_2 - y_1 + \bar{y}_1}{y - x}}{z - x} \right| (1 + 8\alpha^2) + \frac{|y_3 - y_1 - \bar{y}_3 + \bar{y}_1|}{|z - x|}(4\alpha) \\ &\quad + 2|y_1 - \bar{y}_1|, \end{aligned}$$

where $\alpha = \max\{|a|, |b|\}$.

On account of (1) and the definition of the norm we have

$$\begin{aligned}
&\Big|\frac{h(z,y_3)-h(z,\bar{y}_3)-h(y,y_2)+h(y,\bar{y}_2)}{z-y}\\
&\quad-\frac{h(y,y_2)-h(y,\bar{y}_2)-h(x,y_1)+h(x,\bar{y}_1)}{y-x}\Big|\\
&\le L\Big\{(1+8\alpha^2)\Big|\frac{y_3-\bar{y}_3-y_2+\bar{y}_2}{z-y}-\frac{y_2-\bar{y}_2-y_1+\bar{y}_1}{y-x}\Big|\\
&\quad+4\alpha|y_3-y_1-\bar{y}_3+\bar{y}_1|+2|y_1-\bar{y}_1|\,|z-x|\Big\}
\end{aligned}$$

for all $x,y,z\in\langle a,b\rangle$, and $y_1,y_2,y_3,\bar{y}_1,\bar{y}_2,\bar{y}_3\in\mathbb{R}$. Letting $z\to x$ it follows from the continuity of $h(\cdot,y)$ that

$$\begin{aligned}
&\frac{1}{|x-y|}\,|h(x,y_3)-h(x,\bar{y}_3)-h(x,y_1)+h(x,\bar{y}_1)|\\
&\le L\left\{(1+8\alpha^2)\left|\frac{y_3-\bar{y}_3-y_1+\bar{y}_1}{x-y}\right|+4\alpha|y_3-y_1-\bar{y}_3+\bar{y}_1|\right\}
\end{aligned}$$

for all $x,y\in\langle a,b\rangle$, and $y_1,y_3,\bar{y}_1,\bar{y}_3\in\mathbb{R}$. Multiplying the both sides of this inequality by $|x-y|$ and passing $y\to x$ we get

$$(2)\quad |h(x,y_3)-h(x,\bar{y}_3)-h(x,y_1)+h(x,\bar{y}_1)|\le L(1+8\alpha^2)|y_3-\bar{y}_3-y_1+\bar{y}_1|$$

for all $x\in\langle a,b\rangle$, and $y_1,y_3,\bar{y}_1,\bar{y}_3\in\mathbb{R}$.

Putting $\bar{y}_3=\bar{y}_1=0$ in (2), we have

$$|h(x,y_3)-h(x,y_1)|\le L(1+8\alpha^2)|y_3-y_1|,\quad x\in\langle a,b\rangle,\ y_1,y_3\in\mathbb{R}.$$

Thus for all $x\in\langle a,b\rangle$, the function $h(x,\cdot)$ is continuous.

Now, let us fix $u,v\in\mathbb{R}$. Putting $y_3:=u$, $\bar{y}_3=y_1:=\frac{u+v}{2}$, $\bar{y}_1:=v$ in (2) we get

$$h(x,u)-2h\left(x,\frac{u+v}{2}\right)+h(x,v)=0\quad\text{for } x\in\langle a,b\rangle.$$

This means that for every $x\in\langle a,b\rangle$ the function $y\to h(x,y)$ is a continuous solution of Jensen equation. According to Theorem 2, p. 316 in [1], there exist unique functions $A,B:\langle a,b\rangle\to\mathbb{R}$, such that

$$h(x,y)=A(x)+B(x)y\quad\text{for all } x\in\langle a,b\rangle,\ y\in\mathbb{R}.$$

It is obvious that $A \in \mathrm{lip}^2(\langle a, b\rangle)$. Since

$$B(x) = h(x, 1) - h(x, 0), \quad x \in \langle a, b\rangle$$

we also have $B \in \mathrm{lip}^2(\langle a, b\rangle)$, which ends the first part of the proof.

Conversely, suppose that $h : \langle a, b\rangle \times \mathbb{R} \to \mathbb{R}$ is given by formula

$$h(x, y) = A(x) + B(x)y \quad \text{for all } x \in \langle a, b\rangle,\ y \in \mathbb{R},$$

where $A, B \in \mathrm{lip}^2(\langle a, b\rangle)$. By the lemma, the function h generates Nemytskii operator

$$N : P^2(\langle a, b\rangle) \to \mathrm{lip}^2(\langle a, b\rangle),$$

and

$$|N\varphi_1 - N\varphi_2\| \leq M\|B\| \cdot \|\varphi_1 - \varphi_2\| \quad \text{for } \varphi_1, \varphi_2 \in P^2(\langle a, b\rangle)$$

for some constant $M > 0$. □

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