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On additive maps of prime rings, II

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Abstract. Let R be a prime ring. The problem of describing the form of additive maps $f_1, \dots, f_n : R \to R$ satisfying $f_1(x)x^{n-1} + xf_2(x)x^{n-2} + \dots + x^{n-1}f_n(x) = 0$ for all $x \in R$ is discussed.

1. Introduction

Throughout, R will be a prime ring. The goal of this paper is to determine the form of the maps $f_1, \dots, f_n$ of R satisfying the condition

$$f_1(x)x^{n-1} + xf_2(x)x^{n-2} + \dots + x^{n-1}f_n(x) = 0 \tag{1}$$

for all $x \in R$. Let us first mention some results that have motivated this problem.

One should certainly start with an old, well-known theorem of Posner [14] stating that the existence of a nonzero derivation d of R such that $d(x)x - xd(x) = [d(x), x]$ is central for every $x \in R$ implies that R is commutative. This result has been generalized in several ways. In particular, Vukman showed that under certain restrictions concerning $\mathrm{char}(R)$ the same conclusion holds under a milder assumption $[[d(x), x], x] = 0$, $x \in R$, [16] and more generally, $[[[d(x), x], x], x] = 0$, $x \in R$ [17]. Subsequently, Lanski [13] generalized these results by showing that the same is true if $[d(x), x]_n = 0$, $x \in R$, where n is any positive integer. Here, $[y, x]_n$ is defined by $[y, x]_1 = [y, x]$ and $[y, x]_n = [[y, x]_{n-1}, x]$.

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In [4], the first named author described the structure of any additive map f of R satisfying $[f(x), x] = 0$, $x \in R$. It turns out that f must be of the form $f(x) = \lambda x + \zeta(x)$ where λ is an element of C, the extended centroid of R, and $\zeta : R \longrightarrow C$ is an additive map. In a series of papers it has been shown that, under certain restrictions concerning $\mathrm{char}(R)$, the same conclusion holds when $[f(x), x^2] = 0$, $x \in R$ [9], $[f(x), x]_n = 0$, $x \in R$ [8] (the special case, when $n = 2$, was previously proved in [5]), $[\dots[[f(x), x^{n_1}], x^{n_2}], \dots, x^{n_k}] = 0$, $x \in R$ [2], and finally, the most general condition, $\lambda_1 f(x)x^{n-1} + \lambda_2 x f(x)x^{n-2} + \dots + \lambda_n x^{n-1} f(x) = 0$, $x \in R$, where $\lambda_i \in C$ are not all zero [1].

We close the list of papers connected with the relation (1) by [6], where, among other results concerning additive maps, a characterization of additive maps f_1, f_2 of R satisfying $f_1(x)x + x f_2(x) = 0$, $x \in R$, is given.

Of course, the condition (1) is more general than all the conditions just mentioned. Assuming that $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) \geq n$, we will obtain a complete description of additive maps f_i of R satisfying (1) under the additional assumption that R has a nontrivial center (Theorem 2.2). Using the existence of a nonzero central element, we will reduce (1) to the situation where an additive map F satisfies $[F(x), x]_{n-1} = 0$, $x \in R$. It is natural to conjecture that the assumption concerning the center is redundant. However, it is certainly not obvious how to avoid it. Without this assumption we shall be able to obtain the structure of the maps f_i in the special case where they all are so-called generalized derivations (Theorem 3.3). A generalized derivation is an additive map $f : R \longrightarrow R$ such that there exists a derivation δ satisfying $f(xy) = f(x)y + x\delta(y)$, $x, y \in R$ [12]. For instance, the maps of the form $x \mapsto ax + xb$ with $a, b \in R$ fixed elements, are generalized derivations; they are called inner generalized derivations. It seems that it is somehow more natural to consider (1) in the case when the f_i's are generalized derivations rather than when they are ordinary derivations (compare Theorem 3.3 and Corollary 3.4).

Let us sketch briefly the idea of the proof of Theorem 3.3. It has turned out [12] that the so-called functional identities [7] can be used efficiently in the consideration of generalized derivations. Using this we will be able to prove Theorem 3.3 for the case when R is a non PI ring. A prime PI ring, however, always has a nontrivial center, so that Theorem 2.2 can be applied to obtain the desired conclusion.

Let us fix the notation. As already mentioned, R will be a prime ring. By Z, C and RC we denote the center, the extended centroid and the central closure of R, respectively. By $Q_r(RC)$ we denote the right Martindale ring of quotients of the central closure of R (the reason why we have to deal with this ring is that it appears in the study of functional identities [7]). For a full account of prime rings and their quotient rings we refer the reader to [3].

2. Additive maps

For convenience we state the following simple lemma; the lemma is obvious for $j = n-1$ and can be proved easily by proceeding by induction on j.

Lemma 2.1. *For every integer $n \geq 2$ and for every $j = 1, 2, \dots, n-1$ we have*

$$\sum_{k=j}^{n-1} (-1)^k \binom{n-1}{k} = (-1)^j \binom{n-2}{j-1}.$$

The goal of this section is to prove

Theorem 2.2. *Let n be a positive integer and R be a prime ring such that $\operatorname{char}(R) = 0$ or $\operatorname{char}(R) \geq n$. Let $f_1, \dots, f_n : R \to R$ be additive maps satisfying (1) for all $x \in R$. If the center Z of R is nonzero, then there exist elements $b_1, b_2, \dots, b_{n-1} \in RC + C$ and additive maps $\zeta_1, \dots, \zeta_n : R \to C$, such that*

$$\begin{aligned} f_1(x) &= xb_1 + \zeta_1(x) \\ f_k(x) &= -b_{k-1}x + xb_k + \zeta_k(x), \quad k = 2, \dots, n-1 \\ f_n(x) &= -b_{n-1}x + \zeta_n(x) \end{aligned}$$

for all $x \in R$. Moreover, $\zeta_1 + \dots + \zeta_n = 0$.

Proof. The proof will be by induction on n. There is nothing to prove when $n = 1$, so let $n > 1$. It is also convenient to assume that $n > 2$; for when $n = 2$ even a stronger result has already been proved [6, Corollary 4.9], and on the other hand, the necessary modifications to the proof below for $n = 2$ are obvious.

Let $c \in Z$ be a nonzero element. Define $a_i = f_i(c)$ for $i = 1, \dots, n$. By (1) we have $(a_1 + \ldots + a_n)c^{n-1} = 0$ and therefore

$$a_1 + \dots + a_n = 0. \tag{2}$$

Using the assumption on $\mathrm{char}(R)$ we see that a linearization of (1) yields

$$\begin{aligned}(3)\qquad &\sum_{k=2}^{n}\sum_{j=1}^{k-1} x^{j-1}yx^{k-j-1}f_k(x)x^{n-k} + \sum_{k=1}^{n} x^{k-1}f_k(y)x^{n-k}\\ &\qquad + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} x^{k-1}f_k(x)x^{j-1}yx^{n-k-j} = 0\end{aligned}$$

for all $x, y \in R$. Taking $y = c$ we obtain

$$\begin{aligned}&\sum_{k=2}^{n} c(k-1)x^{k-2}f_k(x)x^{n-k} + \sum_{k=1}^{n} x^{k-1}a_k x^{n-k}\\ &\qquad + \sum_{k=1}^{n-1}(n-k)cx^{k-1}f_k(x)x^{n-k-1} = 0.\end{aligned}$$

Set $F_i = cf_i$ and note that the last relation can be written in the form

$$\sum_{k=1}^{n-1} x^{k-1}(kF_{k+1}(x) + a_k x + (n-k)F_k(x))x^{n-k-1} + x^{n-1}a_n \ = \ 0.$$

Setting

$$\begin{aligned}G_k(x) &= a_k x + (n-k)F_k(x) + kF_{k+1}(x), \quad k = 1, \dots, n-2\\ G_{n-1}(x) &= a_{n-1}x + F_{n-1}(x) + (n-1)F_n(x) + xa_n,\end{aligned}$$

we thus have $\sum_{k=1}^{n-1} x^{k-1}G_k(x)x^{n-k-1} = 0$. Using the induction hypothesis we see that there exist elements $d_1, \dots, d_{n-2} \in RC + C$ and additive maps $\mu_1, \dots, \mu_{n-1} : R \to C$, such that $\mu_1 + \cdots + \mu_{n-1} = 0$ and

$$(4)\qquad G_1(x) = a_1 x + (n-1)F_1(x) + F_2(x) = xd_1 + \mu_1(x)$$

$$\begin{aligned}(5)\qquad G_k(x) &= a_k x + (n-k)F_k(x) + kF_{k+1}(x)\\ &= -d_{k-1}x + xd_k + \mu_k(x) \quad k = 2, \dots, n-2\end{aligned}$$

$$\begin{aligned}(6)\qquad G_{n-1}(x) &= a_{n-1}x + F_{n-1}(x) + (n-1)F_n(x) + xa_n\\ &= -d_{n-2}x + \mu_{n-1}(x)\end{aligned}$$

for all $x \in R$. Taking $x = c$ in these relations, denoting $\alpha_k = c^{-1}\mu_k(c) \in C$, $k = 1, \dots, n-1$, and using $F_k(c) = ca_k$, we obtain

$$\begin{aligned} na_1 + a_2 &= d_1 + \alpha_1 \\ (n-k+1)a_k + ka_{k+1} &= -d_{k-1} + d_k + \alpha_k, \quad k = 2, \dots, n-2 \\ 2a_{n-1} + na_n &= -d_{n-2} + \alpha_{n-1}. \end{aligned}$$

Using induction on k we see that

$$d_k = n(a_1 + \cdots + a_k) + ka_{k+1} - (\alpha_1 + \cdots + \alpha_k), \quad k = 1, \dots, n-2. \tag{7}$$

Denote

$$F(x) = F_1(x) - xa_1 \tag{8}$$

and rewrite (4) as

$$F_2(x) = -(n-1)F(x) + [x, a_1] + xa_2 + (\mu_1(x) - \alpha_1 x). \tag{9}$$

Next we shall prove by induction that

$$\begin{aligned} F_{k+1}(x) = (-1)^k \binom{n-1}{k} F(x) + \sum_{j=1}^{k} [x, a_j] + xa_{k+1} \\ + \binom{n-1}{k} \sum_{j=1}^{k} (-1)^{j+k} \frac{1}{j\binom{n-1}{j}} (\mu_j(x) - \alpha_j x) \end{aligned} \tag{10}$$

holds for $k = 1, \dots, n-2$ and all $x \in R$. In the case $k = 1$ this is just the relation (9). Assume now that (10) holds true for some $k < n-2$.

By (5) we have

$$(k+1)F_{k+2}(x) = -a_{k+1}x - (n-k-1)F_{k+1}(x) - d_k x + xd_{k+1} + \mu_{k+1}(x).$$

Using the induction hypothesis and the fact that $(n-k-1)\binom{n-1}{k} = (k+1)\binom{n-1}{k+1}$ we obtain

$$\begin{aligned} (k+1)F_{k+2}(x) = -a_{k+1}x - (-1)^k(k+1)\binom{n-1}{k+1} F(x) \\ - (n-k-1) \sum_{j=1}^{k} [x, a_j] - (n-k-1)xa_{k+1} \end{aligned}$$

$$
\begin{aligned}
&- (k+1)\binom{n-1}{k+1}\sum_{j=1}^{k}(-1)^{j+k}\frac{1}{j\binom{n-1}{j}}(\mu_j(x)-\alpha_j x)\\
&- d_k x + x d_{k+1} + \mu_{k+1}(x).
\end{aligned}
$$

From (7) it follows

$$
-d_k x + x d_{k+1} = n\sum_{j=1}^{k}[x,a_j] + n x a_{k+1} - k a_{k+1} x + (k+1) x a_{k+2} - \alpha_{k+1} x.
$$

Whence we get

$$
\begin{aligned}
(k+1)F_{k+2}(x) = {}& (-1)^{k+1}(k+1)\binom{n-1}{k+1}F(x)\\
&+ (k+1)\sum_{j=1}^{k+1}[x,a_j] + (k+1) x a_{k+2}\\
&+ (k+1)\binom{n-1}{k+1}\sum_{j=1}^{k+1}(-1)^{j+k+1}\frac{1}{j\binom{n-1}{j}}(\mu_j(x)-\alpha_j x).
\end{aligned}
$$

The relation (10) is now proved. In the case $k = n-2$ we have

$$
\begin{aligned}
F_{n-1}(x) = {}& (-1)^n (n-1)F(x) + \sum_{j=1}^{n-2}[x,a_j] + x a_{n-1}\\
&+ (n-1)\sum_{j=1}^{n-2}(-1)^{j+n}\frac{1}{j\binom{n-1}{j}}(\mu_j(x)-\alpha_j x).
\end{aligned}
$$

This, together with (6) and (7) gives us

$$
\begin{aligned}
(n-1)F_n(x) &= -(a_{n-1}+d_{n-2})x - F_{n-1}(x) - x a_n + \mu_{n-1}(x)\\
&= -(n(a_1+\cdots+a_{n-2}) + (n-1)a_{n-1}\\
&\quad - (\alpha_1+\cdots+\alpha_{n-2}))x - x a_n\\
&\quad + \mu_{n-1}(x) - (-1)^n(n-1)F(x) - \sum_{j=1}^{n-2}[x,a_j] - x a_{n-1}\\
&\quad - (n-1)\sum_{j=1}^{n-2}(-1)^{j+n}\frac{1}{j\binom{n-1}{j}}(\mu_j(x)-\alpha_j x).
\end{aligned}
$$

Since $\alpha_1 + \cdots + \alpha_{n-1} = 0$ and $a_1 + \cdots + a_{n-1} + a_n = 0$, we obtain

$$\text{(11)} \quad F_n(x) = a_n x + (-1)^{n+1} F(x) + \sum_{j=1}^{n-1} (-1)^{j+n+1} \frac{1}{j\binom{n-1}{j}} (\mu_j(x) - \alpha_j x).$$

Next we intend to prove that

$$\text{(12)} \qquad [F(x), x]_{n-1} = 0$$

for all $x \in R$. We have

$$[F(x), x]_{n-1} = [\dots[F(x), x], \dots, x] = F(x)x^{n-1}$$
$$+ \sum_{k=1}^{n-2} x^k \left((-1)^k \binom{n-1}{k} F(x) \right) x^{n-k-1} + (-1)^{n-1} x^{n-1} F(x).$$

By (10) it follows that

$$\begin{aligned}[F(x), x]_{n-1} = F(x)x^{n-1} &+ \sum_{k=1}^{n-2} x^k F_{k+1}(x) x^{n-k-1} + (-1)^{n-1} x^{n-1} F(x) \\ &+ \sum_{k=1}^{n-2} \sum_{j=1}^{k} x^k [a_j, x] x^{n-k-1} - \sum_{k=1}^{n-2} x^{k+1} a_{k+1} x^{n-k-1} \\ &+ \sum_{k=1}^{n-2} \sum_{j=1}^{k} \binom{n-1}{k} (-1)^{j+k} \frac{1}{j\binom{n-1}{j}} (\alpha_j x - \mu_j(x)) x^{n-1}.\end{aligned}$$

Since $F_j = cf_j$ it follows from (1) that the sum of the first three terms on the right side of this identity is equal to

$$(F(x) - F_1(x))x^{n-1} - x^{n-1}(F_n(x) - (-1)^{n-1}F(x)).$$

According to (8) and (11), this is further equal to

$$-xa_1x^{n-1} - x^{n-1}a_n x + \sum_{j=1}^{n-1} (-1)^{j+n-1} \frac{1}{j\binom{n-1}{j}} (\alpha_j x - \mu_j(x)) x^{n-1}.$$

Therefore, we have

$$
\begin{aligned}
(13)\quad [F(x),x]_{n-1} &= \sum_{k=1}^{n-2}\sum_{j=1}^{k} x^k[a_j,x]x^{n-k-1} - \sum_{k=0}^{n-2} x^{k+1}a_{k+1}x^{n-k-1} \\
&\quad - x^{n-1}a_n x + \sum_{k=1}^{n-1}\sum_{j=1}^{k}\binom{n-1}{k}(-1)^{j+k}\frac{1}{j\binom{n-1}{j}}(\alpha_j x - \mu_j(x))x^{n-1}.
\end{aligned}
$$

The last term in (13) is equal to

$$
\begin{aligned}
&\left(\sum_{k=1}^{n-1}\sum_{j=1}^{k}\binom{n-1}{k}(-1)^{j+k}\frac{1}{j\binom{n-1}{j}}\alpha_j\right)x^n \\
&- \left(\sum_{k=1}^{n-1}\sum_{j=1}^{k}\binom{n-1}{k}(-1)^{j+k}\frac{1}{j\binom{n-1}{j}}\mu_j(x)\right)x^{n-1}.
\end{aligned}
$$

First changing the order of summation and then applying Lemma 2.1 we see that

$$
\begin{aligned}
\sum_{k=1}^{n-1}\sum_{j=1}^{k}\binom{n-1}{k}(-1)^{j+k}\frac{1}{j\binom{n-1}{j}}\alpha_j &= \sum_{j=1}^{n-1}(-1)^j\frac{1}{j\binom{n-1}{j}}\alpha_j\sum_{k=j}^{n-1}(-1)^k\binom{n-1}{k} \\
&= \sum_{j=1}^{n-1}\frac{\binom{n-2}{j-1}}{j\binom{n-1}{j}}\alpha_j = \frac{x^n}{n-1}\sum_{j=1}^{n-1}\alpha_j.
\end{aligned}
$$

But then

$$
(14)\qquad \sum_{k=1}^{n-1}\sum_{j=1}^{k}\binom{n-1}{k}(-1)^{j+k}\frac{1}{j\binom{n-1}{j}}\alpha_j = 0
$$

for $\sum_{j=1}^{n-1}\alpha_j = 0$. Similarly,

$$
(15)\qquad \sum_{k=1}^{n-1}\sum_{j=1}^{k}\binom{n-1}{k}(-1)^{j+k}\frac{1}{j\binom{n-1}{j}}\mu_j(x) = 0.
$$

Therefore, (13) reduces to

$$[F(x), x]_{n-1} = \sum_{k=1}^{n-2} \sum_{j=1}^{k} x^k [a_j, x] x^{n-k-1} - \sum_{k=0}^{n-2} x^{k+1} a_{k+1} x^{n-k-1} - x^{n-1} a_n x.$$

However, noting that

$$\begin{aligned} \sum_{k=1}^{n-2} \sum_{j=1}^{k} x^k [a_j, x] x^{n-k-1} &= \sum_{k=1}^{n-2} x^k \sum_{j=1}^{k} a_j x^{n-k} - \sum_{k=2}^{n-1} x^k \sum_{j=1}^{k-1} a_j x^{n-k} \\ &= \sum_{k=1}^{n-2} x^k a_k \; x^{n-k} - x^{n-1} \sum_{j=1}^{n-2} a_j x \end{aligned}$$

and using $\sum_{j=1}^{n-2} a_j = -a_{n-1} - a_n$ we now see that (12) holds.

Applying the result of [8] it follows that there exist $\lambda \in C$ and an additive map $\xi : R \to C$ such that

$$F(x) = \lambda x + \xi(x),$$

$x \in R$. Thus, according to (8) we have

$$F_1(x) = x a_1 + \lambda x + \xi(x).$$

From (10) we obtain that

$$\begin{aligned} F_{k+1}(x) = (-1)^k \binom{n-1}{k} \lambda x + (-1)^k \binom{n-1}{k} \xi(x) + x \sum_{j=1}^{k+1} a_j - \sum_{j=1}^{k} a_j x \\ + \binom{n-1}{k} \sum_{j=1}^{k} (-1)^{j+k} \frac{1}{j \binom{n-1}{j}} (\mu_j(x) - \alpha_j x) \\ = x \sum_{j=1}^{k+1} a_j - \sum_{j=1}^{k} a_j x + \lambda_{k+1} x + \xi_{k+1}(x), \end{aligned}$$

$k = 1, \dots, n-2$, where

$$\lambda_{k+1} = \binom{n-1}{k}(-1)^k \left(\lambda - \sum_{j=1}^{k}(-1)^j \frac{1}{j\binom{n-1}{j}}\alpha_j\right)$$

$$\xi_{k+1}(x) = \binom{n-1}{k}(-1)^k \left(\xi(x) + \sum_{j=1}^{k}(-1)^j \frac{1}{j\binom{n-1}{j}}\mu_j(x)\right).$$

Similarly, (11) implies that

$$F_n(x) = a_n x + \lambda_n x + \xi_n(x),$$

where

$$\lambda_n = (-1)^{n+1} \left(\lambda - \sum_{j=1}^{n-1}(-1)^j \frac{1}{j\binom{n-1}{j}}\alpha_j\right)$$

$$\xi_n(x) = (-1)^{n+1} \left(\xi(x) + \sum_{j=1}^{n-1}(-1)^j \frac{1}{j\binom{n-1}{j}}\mu_j(x)\right).$$

Set $\lambda_1 = \lambda$ and $\xi_1 = \xi$ and compute

$$\sum_{k=1}^{n}\lambda_k = \lambda + \sum_{k=1}^{n-1}\binom{n-1}{k}(-1)^k \left(\lambda - \sum_{j=1}^{k}(-1)^j \frac{1}{j\binom{n-1}{j}}\alpha_j\right)$$

$$= \left(\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\right)\lambda - \sum_{k=1}^{n-1}\sum_{j=1}^{k}\binom{n-1}{k}(-1)^k(-1)^j \frac{1}{j\binom{n-1}{j}}\alpha_j.$$

The first term, of course, equals 0, and by (14) the second term is 0 as well. Therefore, $\sum_{k=1}^{n}\lambda_k = 0$. Similarly, using (15) we see that $\sum_{k=1}^{n}\xi_k(x) = 0$.

Recall that $f_k = c^{-1}F_k$. Now set $\zeta_k = c^{-1}\xi_k$ and

$$b_k = c^{-1}(a_1 + \dots + a_k + \lambda_1 + \dots + \lambda_k) \in RC + C$$

for $k = 1, \dots, n-1$, and note that $\zeta_1 + \cdots + \zeta_n = 0$, and

$$\begin{aligned} f_1(x) &= xb_1 + \zeta_1(x) \\ f_k(x) &= -b_{k-1}x + xb_k + \zeta_k(x), \quad k = 2, \dots, n-1 \\ f_n(x) &= -b_{n-1}x + \zeta_n(x), \end{aligned}$$

$x \in R$. The theorem is thereby proved. $\square$

3. Generalized derivations

The proof of the main theorem of this section rests heavily on the following result.

Proposition 3.1 [7, Proposition 8]**.** *Let R be a prime ring and suppose that*

$$\sum_{j=1}^{n} f_j(z)xa_j + \sum_{i=1}^{k} c_i z h_i(x) = 0 \quad \text{for all } x, z \in R,$$

where $a_j, c_i \in R$ and $f_j, h_i : R \to R$ are any maps. If the sets $\{a_1, \dots, a_n\}$ and $\{c_1, \dots, c_k\}$ are C-independent, then there exist $q_{ij} \in Q_r(RC)$, $i = 1, \dots, k$, $j = 1, \dots n$, such that

$$f_j(z) = -\sum_{i=1}^{k} c_i z q_{ij}, \qquad h_i(x) = \sum_{j=1}^{n} q_{ij} x a_j,$$

for all $x, z \in R$, $i = 1, \dots, k$, $j = 1, \dots, n$.

It is clear from the proof that Proposition 3.1 remains true when assuming that some a_j or c_i lies in the centroid of R instead of in R (that is to say, we can take a_j or c_i to be equal to 1 even when R does not contain a unit element).

Lemma 3.2. *Let R be a prime ring such that* $\mathrm{char}(R) = 0$ *or* $\mathrm{char}(R) > n$. *Suppose that $a_0, a_1, \dots, a_n \in Q_r(RC)$ satisfy*

$$a_0x^n + xa_1x^{n-1} + \cdots + x^{n-1}a_{n-1}x + x^na_n = 0 \tag{16}$$

for all $x \in R$. Then $a_0, \dots, a_n$ lie in C and $a_0 + a_1 + \cdots + a_n = 0$.

Proof. First of all, note that a complete linearization of (16) shows that (16) holds for every $x \in RC$ as well.

We consider

$$\phi(x) = a_0x^n + xa_1x^{n-1} + \cdots + x^{n-1}a_{n-1}x + x^na_n$$

as an element of $Q_r(RC)_C\langle X\rangle$ the copruduct of the C-algebra $Q_r(RC)$ and the free algebra $C\langle X\rangle$ over C (here, X is an (infinite) set – cf. [3, p. 212]). The desired conclusion can be expressed simply as $\phi = 0$. Assume, therefore, that $\phi \neq 0$. Then ϕ is a generalized polynomial identity (GPI) on RC. But then, by [3, Theorem 6.4.4], ϕ is also a GPI on $Q_r(RC)$. In particular, (16) holds for $x = 1$ showing that $a_0 + a_1 + \cdots + a_n = 0$.

We proceed by induction on n. When $n = 1$, since $a_0 = -a_1$, we have $a_0x = xa_0$, $x \in Q_r(RC)$, meaning that $a_0, a_1 \in C$. Now assume that the lemma is true for all positive integers smaller than n. Replacing x by $x+1$ in (16) we arrive at

$$\sum_{k=0}^{n-1} x^k\left((k+1)a_{k+1} + (n-k)a_k\right)x^{n-k-1} = 0$$

for all $x \in Q_r(RC)$. By the induction hypothesis it follows that $(k+1)a_{k+1} + (n-k)a_k \in C$ for $k = 0, 1, \ldots, n-1$. Therefore, the lemma will be proved by showing that $a_0 \in C$.

Let $e \in Q_r(RC)$ be any idempotent. As (16) holds for any $x \in Q_r(RC)$, it follows that $a_0e \in eQ_r(RC)$, which yields $(1-e)a_0e = 0$. Replacing the roles of e and $1-e$ we get $ea_0(1-e) = 0$, and the two relations show that a_0 commutes with any idempotent $e \in Q_r(RC)$.

In the rest of the proof we, more or less, just repeat the arguments given in the proof of [13, Theorem 1]. By MARTINDALE's theorem [3, Theorem 6.1.6] we know that RC is a primitive ring with $H = \mathrm{soc}(RC) \neq 0$ and $eRCe$ is finite dimensional for any minimal idempotent e. If H contains a nontrivial idempotent, then it follows from [11, Corollary, p. 18 and Corollary, p. 9] that H is generated by its idempotents. Whence $[a_0, H] = 0$ showing that $a_0 \in C$. In the case when H contains no nontrivial idempotent, H is a finite dimensional division algebra over C. Then the a_i's lie in H for $H = RC = Q_r(RC)$. Of course, we may assume that H is noncommutative. Now it follows from [13, Lemma 2] that there exist a field F and an integer $n > 1$ such that H can be embedded into $M_n(F)$ and ϕ is a GPI on $M_n(F)$. But then, as has already been shown, a_0 commutes with every idempotent in $M_n(F)$ implying that $a_0 \in C$. The proof of the lemma is now complete. $\square$

Theorem 3.3. *Let n be a positive integer and R be a noncommutative prime ring such that* $\mathrm{char}(R) = 0$ *or* $\mathrm{char}(R) > n$. *Let $f_1, \dots, f_n : R \to R$ be generalized derivations satisfying* (1) *for all $x \in R$. Then there exist elements $b_1, b_2, \dots, b_{n-1} \in Q_r(RC)$ such that*

$$\begin{aligned} f_1(x) &= xb_1 \\ f_k(x) &= -b_{k-1}x + xb_k, \quad k = 2, \dots, n-1 \\ f_n(x) &= -b_{n-1}x \end{aligned}$$

for all $x \in R$.

Proof. As in the proof of Theorem 2.2 we see that (1) implies (3). Substituting y for x and zx for y in (3) we get

$$\begin{aligned} \sum_{k=2}^{n}\sum_{j=1}^{k-1} y^{j-1}zxy^{k-j-1}f_k(y)y^{n-k} + \sum_{k=1}^{n} y^{k-1}f_k(zx)y^{n-k} \\ + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} y^{k-1}f_k(y)y^{j-1}zxy^{n-k-j} = 0 \end{aligned} \tag{17}$$

for all $x, y \in R$. Changing the order of summation we see that

$$\begin{aligned} &\sum_{k=2}^{n}\sum_{j=1}^{k-1} y^{j-1}zxy^{k-j-1}f_k(y)y^{n-k} \\ &= \sum_{j=1}^{n-1} y^{j-1}z\left(x\sum_{k=j+1}^{n} y^{k-j-1}f_k(y)y^{n-k}\right), \end{aligned}$$

$$\sum_{k=1}^{n-1}\sum_{j=1}^{n-k} y^{k-1}f_k(y)y^{j-1}zxy^{n-k-j} = \sum_{j=1}^{n-1}\left(\sum_{k=1}^{n-j} y^{k-1}f_k(y)y^{n-k-j}z\right)xy^{j-1}.$$

Let δ_k be derivations satisfying $f_k(zx) = f_k(z)x + z\delta_k(x)$, $z, x \in R$, $k = 1, \dots, n$. Then we have

$$\begin{aligned} \sum_{k=1}^{n} y^{k-1}f_k(zx)y^{n-k} &= \sum_{k=1}^{n} y^{k-1}f_k(z)xy^{n-k} + \sum_{k=1}^{n} y^{k-1}z\delta_k(x)y^{n-k} \\ &= \sum_{j=1}^{n} y^{n-j}f_{n-j+1}(z)xy^{j-1} + \sum_{j=1}^{n} y^{j-1}z\delta_j(x)y^{n-j}. \end{aligned}$$

Thus, (17) can be written as

$$\sum_{j=1}^{n-1}\left(y^{n-j}f_{n-j+1}(z)+\sum_{k=1}^{n-j}y^{k-1}f_k(y)y^{n-k-j}z\right)xy^{j-1}+f_1(z)xy^{n-1}$$
$$+\sum_{j=1}^{n-1}y^{j-1}z\left(\delta_j(x)y^{n-j}+x\sum_{k=j+1}^{n}y^{k-j-1}f_k(y)y^{n-k}\right)+y^{n-1}z\delta_n(x)=0.$$

Set

$$F_j(z)=y^{n-j}f_{n-j+1}(z)+\sum_{k=1}^{n-j}y^{k-1}f_k(y)y^{n-k-j}z \tag{18}$$
$$H_i(x)=\delta_i(x)y^{n-i}+x\sum_{k=i+1}^{n}y^{k-i-1}f_k(y)y^{n-k}$$

for $i,j=1,\dots,n-1$, and

$$F_n(z)=f_1(z),\quad H_n(x)=\delta_n(x),$$

so that we have

$$\sum_{j=1}^{n}F_j(z)xy^{j-1}+\sum_{i=1}^{n}y^{i-1}zH_i(x)=0. \tag{19}$$

We now consider two different cases.

Case 1. The set $\{1,y,y^2,\dots,y^{n-1}\}$ is C-dependent for every $y\in R$.

In other words, R is algebraic of degree $n-1$ over C. In particular, R is a PI ring [10, Lemma 6.2.3]. But then Z is nontrivial [15], and so the f_i's take the form described in Theorem 2.2. Moreover, since R is noncommutative, using the fact that f_i is a generalized derivation it is very easy to see that $\zeta_i=0$, $i=1,\dots,n$ [12, Lemma 3]. The proof is thus complete in this case.

Case 2. There exists an element $y\in R$ such that $\{1,y,y^2,\dots,y^{n-1}\}$ is a C-independent set.

According to Proposition 3.1, it follows from (19) that there exist $q_{ij} \in Q_r(RC)$, $i, j = 1, \dots, n$, such that

$$F_j(z) = -\sum_{i=1}^{n} y^{i-1} z q_{ij}, \qquad H_i(x) = \sum_{j=1}^{n} q_{ij} x y^{j-1}$$

for all $x, z \in R$, $i, j = 1, \dots, n$. Let $b_j = \sum_{k=1}^{n-j} y^{k-1} f_k(y) y^{n-k-j}$, $j = 1, \dots, n-1$ and $b_n = 0$. By (18) we have

$$y^{n-j} f_{n-j+1}(z) + b_j z - F_j(z) = 0$$

for $j = 1, \dots, n$ and all $z \in R$. Replacing z by zx, we obtain

$$y^{n-j} f_{n-j+1}(z)x + y^{n-j} z \delta_{n-j+1}(x) + b_j zx - F_j(zx) = 0.$$

Since $b_j z = -y^{n-j} f_{n-j+1}(z) + F_j(z)$ it follows that

$$y^{n-j} z \delta_{n-j+1}(x) - F_j(zx) + F_j(z)x = 0.$$

Consequently,

$$y^{n-j} z \delta_{n-j+1}(x) + \sum_{i=1}^{n} y^{i-1} z [x, q_{ij}] = 0.$$

Since the elements $1, y, \dots, y^{n-1}$ are C-independent, a simple extension (see, e.g., [7, Lemma 1]) of a well-known result of Martindale implies that $[x, q_{ij}] = 0$ for $i \neq n-j+1$ and $\delta_{n-j+1}(x) + [x, q_{n-j+1,j}] = 0$. This holds for $j = 1, \dots, n$ and all $x \in R$. This means that the derivations δ_j are of the form $\delta_j(x) = [q_j, x]$, where $q_j = q_{j,n-j+1}$, $j = 1, \dots, n$. Next, noting that $(f_j - \delta_j)(xy) = (f_j - \delta_j)(x)y$, $x, y \in R$, we see that $(f_j - \delta_j)(x) = p_j x$ for some $p_j \in Q_r(R)$ [3, Proposition 2.2.1 (iv)]. Whence

$$f_j(x) = a_j x + x d_j$$

for $j = 1, \dots, n$, where $a_j = p_j + q_j \in Q_r(RC)$, $d_j = -q_j \in Q_r(RC)$. Therefore, the initial relation (1) can be written in the form

$$a_1 x^n + x(d_1 + a_2)x^{n-1} + \dots + x^{n-1}(d_{n-1} + a_n)x + x^n d_n = 0.$$

Set $\lambda_0 = a_1$, $\lambda_1 = d_1 + a_2, \dots, \lambda_{n-1} = d_{n-1} + a_n$ and $\lambda_n = d_n$, and note that all the λ_i's lie in C and that their sum is zero by Lemma 3.2. This shows that the f_k's are of the desired form with $b_k = \lambda_0 + \dots + \lambda_{k-1} + d_k$, $k = 1, \dots, n-1$. The proof is complete. $\square$

Corollary 3.4. *Let n be a positive integer and R be a noncommutative prime ring such that* $\mathrm{char}(R) = 0$ *or* $\mathrm{char}(R) > n$. *Let $f_1, \dots, f_n : R \to R$ be derivations satisfying* (1) *for all $x \in R$. Then $f_1 = f_2 = \dots = f_n = 0$.*

Proof. By Theorem 3.3 we have $f_1(x) = xb_1$ for some $b_1 \in Q_r(RC)$. Therefore, $f_1(xy) = xyb_1 = xf_1(y)$. However, since f_1 is a derivation it follows that $f_1(x)y = 0$, $x, y \in R$; but then $f_1 = 0$ and $b_1 = 0$. Similarly we see that $f_2 = \dots = f_n = 0$. $\square$

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