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On the compactification of generalized ordered spaces

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Abstract. In this paper, we prove that every non empty generalized ordered space X has an ordered compactification with the same dimension and we compare the dimension of X with the dimensions of its other ordered compactifications.

In a previous paper (2), we proved that, for every ordered space (ordered set with its order topology) and more generaly, for every generalized ordered space (ordered set with a topology finer than the order topology and moreover generated by a set of left or right unlimited intervals and noted as in (5) G.O. space), the different definitions of topological dimension – the small inductive dimension, the large inductive dimension, the covering dimension, and the nonstandard definition or thickness (3) – coincide. More precisely, we established that, for every non empty G.O. space X, if $\tau(X)$ denotes the common value of these dimensions, we have:

(i) $\tau(X) = 0$ if and only if X is totally disconnected,

(ii) $\tau(X) = 1$ if and only if X is not totally disconnected.

In this paper, we prove that every non empty G.O. space X has an ordered compactification $\widehat{X}$ with the same dimension and that, if X is a not almost-compact (4), every ordered compactification of X is a quotient of $\widehat{X}$. Lastly, we compare the dimension of X with the dimensions of its other ordered compactifications.

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Preliminaries

Let X be an ordered set.

We will call on the one hand *interval* of X any non empty subset I of X such that:

$$\forall x \in I \quad \forall y \in I \quad \forall z \in X, \quad x \leq z \leq y \Longrightarrow z \in I.$$

We will call on the other hand *meatus* (méat in French) of X any pair (I, J) of complementary intervals such that $I < J$ (i.e. such that: $\forall x \in I \; \forall y \in J$, $x < y$).

Among the meatuses, we will distinguish:

– the improper meatuses: these are the meatuses (ϕ, X) and (X, ϕ),

– the proper meatuses and among these:

(a) the *gaps*: these are the meatuses (I, J) where I and J are not empty, I having no last element and J no first element.

(b) the *holes*: these are the meatuses (I, J) where I has a last element and J has a first element.

(c) the *left faults*: these are the meatuses (I, J) where I is not empty but without last element and J has a first element.

(d) the *right faults*: these are the meatuses (I, J) where I has a last element and J is not empty but without first element.

We will say that a meatus (I, J) of a G.O. space X is *open* if and only if I and J are both open sets in the topology on X. Thus, improper meatuses, gaps and holes are always open meatuses, but no fault is open for the order topology.

Moreover, we recall (2) that:

(i) a G.O. space is connected if and only if its only open meatuses are its improper meatuses,

(ii) a non empty ordered space is compact Hausdorff if and only if it has a first and a last element and has no gaps.

In the sequel, we will call *ordered compactification* of a G.O. space X a pair (Y, j) consisting of a compact Hausdorff ordered space Y (a compact Hausdorff G.O. space is necessarily an ordered space) and an isomorphism, for the topology and the order, j of X onto a dense subset of Y.

1. First ordered compactification theorem

1.1. Theorem. *Let X be a non empty G.O. space. There exists then an ordered compactification $\widehat{X}$ of X such that $\tau(X) = \tau(\widehat{X})$.*

PROOF.

(1) Notations: Let N be the set of all open meatuses of X.
For every $s = (I, J)$ of N, we will say that s^- exists if and only if I is not empty and without last element, and that s^+ exists if and only if J is not empty and without first element. If not, all what is said about s^- and s^+ will be considered as of no account.
We put then $N^- = \{s^- : s \in N\}$, $N^+ = \{s^+ : s \in N\}$, $\widehat{X} = X \cup N^- \cup N^+$ and we denote by j the canonical injection of X into $\widehat{X}$.

(2) Definition of an order on $\widehat{X}$:
We define an order on $\widehat{X}$ putting the elements of X in their order in X and deciding that:
- (i) for every gap $s = (I, J)$, we have $I < s^- < s^+ < J$,
- (ii) for every left fault (resp. right fault), we have $I < s^- < J$ (resp. $I < s^+ < J$),
- (iii) for every open meatuses $s = (I, J)$ and $t = (U, V)$ such that $s \leq t$ (i.e. $I \subset U$ or equivalently $V \subset J$), we have $s^- < s^+ < t^- < t^+$.

It follows from this definition that:
- (a) If X has a first (resp. last) element, this first (resp. last) element is also the first (resp. last) element of $\widehat{X}$.
If X has not a first (resp. last) element, then the element s^+ (resp. t^-) associated to meatus $s = (\phi, X)$ (resp. $t = (X, \phi)$) is the first (resp. last) element of $\widehat{X}$.
- (b) for every open meatus $s = (I, J), s^-$ and s^+ are such that:

$$s^- = \sup_{\widehat{X}} I, \quad s^+ = \inf_{\widehat{X}} J,$$
$$I = \,]\rightarrow, s^-[\, \cap X = \,]\rightarrow, s^-] \cap X = \,]\rightarrow, s^+] \cap X$$
$$= \,]\rightarrow, s^+[\, \cap X,$$
$$J = [s^-, \rightarrow[\, \cap X = \,]s^-, \rightarrow[\, \cap X = \,]s^+, \rightarrow] \cap X$$
$$= [s^+, \rightarrow[\, \cap X.$$

In the sequel, we will also denote by $\widehat{X}$ the ordered space obtained in this way.

(3) It follows immediatly from (2) that j is a (strictly) increasing continuous mapping of X into $\widehat{X}$ and that $j(X)$ is a dense subset of $\widehat{X}$.

(4) $\widehat{X}$ is compact:
It suffices to prove that $\widehat{X}$ has no gaps. Let us suppose $\widehat{X}$ has a gap $s = (S, T)$. Let us put $U = S \cap X$ and $V = T \cap X$; U and V are then two open complementary intervals such that $U < V$, so that $t = (U, V)$ is an open meatus of X. For this meatus, t^- and t^+ do not exist. Indeed, let us suppose t^- exists (we reason in the same way if t^+ exists). Then t^- would be the last element of U and consequently s would not be a gap. It follows from this that t is neither a fault (on the left or on the right), nor a gap. Consequently, t is an improper meatus or a hole, which is impossible.

Indeed: If $t = (\phi, X)$ or $t = (X, \phi)$, we would have $s = (\phi, \widehat{X})$ or $(\widehat{X}, \phi)$, which it is not.
If t was a hole, we would have $U = \,]\rightarrow, \alpha]$ and $V = [\beta, \rightarrow[$ with $\alpha < \beta$ and consequently $\alpha \in S$, $S \leq \alpha$ and $\beta \in T$, $\beta \leq T$. The meatus $s = (S, T)$ would be a hole, which it is not. Finally, $\widehat{X}$ has no gaps.

(5) $\tau(X) = \tau(\widehat{X})$:
Since X is homeomorphic to a subspace of $\widehat{X}$, we have $\tau(X) \leq \tau(\widehat{X})$. Moreover, if X is not totally disconnected, $\widehat{X}$ is a fortiori not totally disconnected, so that $\tau(X) = \tau(\widehat{X}) = 1$.
Let us prove lastly that, if X is totally disconnected, $\widehat{X}$ is also totally disconnected, which will imply that $\tau(X) = \tau(\widehat{X}) = 0$.
Let $I = [x, y]$ (with $x < y$) be an interval of $\widehat{X}$.
If I is finite, I is not connected.
If I is infinite, $J = I \cap X$ is an infinite interval of X. Consequently, since X is totally disconnected, J is not a connected subset of X, so that there exists an open meatus $s = (U, V)$ of X such that $U \cap J \neq \emptyset$ and $V \cap J \neq \emptyset$. This meatus defines a hole in I. Indeed:

- if s is a hole in X, we have $U = \,]\rightarrow, a]$ and $V = [b, \rightarrow[$ with $a < b$ and then (a, b) defines a hole in I.

– if s is a left fault in X, we have $V = [b, \rightarrow[$ and then (s^-, b) defines a hole in I.

– if s is a right fault in X, we have $U =]\rightarrow, a]$ and then (a, s^+) defines a hole in I.

– if s is a gap, (s^-, s^+) defines a hole in I.

It follows then that I is not connected.

The only connected intervals of $\widehat{X}$ are therefore the one-point sets, whence the result.

1.2. *Remarks.* (1) If X is connected and without first and last element, $\widehat{X}$ is the ordered space obtained by the addition of a first and a last element.

(2) If X is almost-compact (see (4)), $\widehat{X}$ is the only compactification of X, so that, in particular, $\widehat{X} = \beta(X)$.

1.3. *Examples.*

(1) Case where X is the rational G.O. space $\mathbb{Q}$.
The meatuses of $\mathbb{Q}$ are on one hand the improper meatuses $(\phi, \mathbb{Q})$ and $(\mathbb{Q}, \phi)$, on the other hand the faults associated to each rational and lastly the gaps associated to each irrational. Among these meatuses, only are open the two improper meatuses and the gaps. Consequently, from 1.1, $\widehat{\mathbb{Q}}$ consists of a first and a last element and, for every irrational x, of a pair (x^-, x^+) with $x^- < x^+$. Moreover, since $\mathbb{Q}$ is totally disconnected, we have $\tau(\mathbb{Q}) = \tau(\widehat{\mathbb{Q}}) = 0$.

(2) Case where X is the real G.O. space $\mathbb{R}$.
Since $\mathbb{R}$ is a connected space, we have, from 1.2 1), $\widehat{\mathbb{R}} = \overline{\mathbb{R}}$ $(= \mathbb{R} \cup \{-\infty, +\infty\})$ and therefore $\tau(\mathbb{R}) = \tau(\widehat{\mathbb{R}}) = 1$.

(3) Case where X is the Sorgenfrey's G.O. space.
Sorgenfrey's G.O. space is the G.O. space, denoted by $\mathbf{E}$, consisting of the set $\mathbb{R}$ with the topology generated by all intervals $]\leftarrow, a[$, $]\leftarrow, a]$ and $]a, \rightarrow[$. Let us note that $\mathbf{E}$ has no holes and no gaps and the only open faults of $\mathbf{E}$ are the right faults. Consequently, from 1.1, $\widehat{\mathbf{E}}$ consists of a first and a last element and, for every real number x, of a pair (x, x^+) with $x < x^+$. Moreover, since $\mathbf{E}$ is totally disconnected (see for example (6)), we have $\tau(\mathbf{E}) = \tau(\widehat{\mathbf{E}}) = 0$.

(4) Case where X is the subspace $[0, 1[\cup \{2\}$ of the real G.O. space $\mathbb{R}$.

Since $s = ([0,1[, \{2\})$ is an open left fault of X, we have, from 1.1, $\widehat{X} = X \cup \{s^-\}$. Let us note $\widehat{X}$ is homeomorphic to the compact subspace $[0,1] \cup \{2\}$ of $\mathbb{R}$. Moreover, since X is not totally disconnected, we have $\tau(X) = \tau(\widehat{X}) = 1$.

(5) Case where X is the subspace $(\,]0,1[\cap\mathbb{Q}) \cup (\,]2,+\infty[\,)$ of the real G.O. space $\mathbb{R}$.
In this case, $\widehat{X}$ consists of a first and a last element and, for every irrational x of $]0,1[$, of a pair (x^-, x^+) with $x^- < x^+$ and lastly of a pair (s^-, s^+) corresponding to the gap $s = (\,]0,1[\cap\mathbb{Q},]2,+\infty[\,)$.
Since X is not totally disconnected, we have $\tau(X) = \tau(\widehat{X}) = 1$.

(6) Case where X is the ordinal space $\Omega = [0, \omega_1[$.
Ω is an almost-compact ordered space. Consequently, from 1.2 2), we have

$$\widehat{\Omega} = \beta(\Omega) = \Omega \cup \{\omega_1\} (= [0, \omega_1]).$$

Since X is totally disconnected, we have $\tau(\Omega) = \tau(\widehat{\Omega}) = 0$.

(7) Case where X is the long G.O. space L_0 (connexification of Ω).
L_0 is an almost-compact ordered space. Consequently, we have $\widehat{L}_0 = \beta(L_0) = L$, where L denotes the connexification of Ω, called the long segment.
Since L_0 is a connected space, we have $\tau(L_0) = \tau(L) = 1$.

2. Other ordered compactifications of a non empty and non almost compact

2.1. The spaces $(\widehat{X,S})$.

(1) Notations: Let us denote by N the set of all open meatuses of X, L the set of all gaps of X, S a subset of L and put $D = N \setminus S$, $D^- = \{s^- : s \in D\}, D^+ = \{s^+ : s \in D\}$ and $(\widehat{X,S}) = X \cup S \cup D^- \cup D^+$.
Let us note that, if $S = \emptyset$, $(\widehat{X,\phi}) = X \cup N^- \cup N^+$ is the space $\widehat{X}$ introduced in 1.1.

(2) Definition of an order on $(\widehat{X,S})$:
We define an order on $(\widehat{X,S})$ putting the elements of X in their order in X and deciding that:

(i) for every $s = (I, J)$ of S, we have $I < s < J$,

(ii) for every $s = (I, J)$ of D, we have $I < s^- < s^+ < J$,

(iii) for every open meatuses $s = (I, J)$ and $t = (U, V)$ such that $s \leq t$ (i.e. such that $I \subset U$ or equivalently $V \subset J$), we have $s^- < s^+ < t^- < t^+$.

(3) It follows from this definition that:

(a) for every open meatus $s = (I, J), s^-$ and s^+, if they exist, are such that $s^- = \sup_{\widehat{(X,S)}} I$ and $s^+ = \inf_{\widehat{(X,S)}} J$.

(b) the canonical injection j of X into $\widehat{(X, S)}$ is a (strictly) increasing and continuous mapping such that $\overline{j(X)} = \widehat{(X, S)}$.

2.2. Theorem. *Every ordered compactification of a non empty and non almost compact G.O. space X is homeomorphic to a space $\widehat{(X, S)}$.*

Let us consider an ordered compactification Y of a non empty G.O. space X and put, for every $y \in Y \setminus X$, $U_y = \{x \in X : x < y\}$ and $V_y = \{x \in X : x > y\}$. Thus, we define, for every $y \in Y \setminus X$, an open meatus $s_y = (U_y, V_y)$. Let us note that $y \in \overline{U}_y \cup \overline{V}_y$ (with $\overline{A}$ the closure of A in Y). Let us put then:

$$T_1 = \{y \in Y \setminus X : y \in \overline{U}_y \cap \overline{V}_y\}$$
$$T_2 = \{y \in Y \setminus X : y \in \overline{U}_y \setminus \overline{V}_y\}$$
$$T_3 = \{y \in Y \setminus X : y \in \overline{V}_y \setminus \overline{U}_y\}.$$

We define thus a partition of $Y \setminus X$. Let $S = \{s_y : y \in T_1\}$ (note that $S \subset L), D = N \setminus S$ and h the mapping of Y into $\widehat{(X, S)} = X \cup S \cup D^- \cup D^+$ defined by:

$$h(y) = \begin{cases} y & \text{if } y \in X \\ s_y & \text{if } y \in T_1 \\ s_y^- & \text{if } y \in T_2 \\ s_y^+ & \text{if } y \in T_3. \end{cases}$$

This mapping h is, by construction and definition of the order on $\widehat{(X, S)}$, one to one and such that $h(X) = X$, $h(T_1) = S$, $h(T_2) \subset D^-$ and $h(T_3) \subset D^+$.

Moreover, since Y is compact Hausdorff, h is also onto.

Indeed, let $s = (I, J)$ be an element of D such that s^- exists (we reason in the same way if s^+ exists). Then s is an open meatus which does not belong to S and such that I is not empty and without last element. Since Y is compact Hausdorff, the closure $\overline{I}$ of I is a non empty compact Hausdorff interval of Y and therefore has a last element $y \notin I$. Let $s_y = (U_y, V_y)$. We have then $s_y = s$ and therefore, since $s \notin S, y \notin T_1$. Moreover, since $y \in \overline{I} \setminus I = \overline{U}_y \setminus U_y$, we have $y \in T_2$ which implies $h(y) = s^-$. The mapping h is consequently a bijection of Y onto $(\widehat{X,S})$.

Moreover, since any increasing mapping f of a totally ordered space T onto a totally ordered space Z is continuous (indeed the inverse image by f of each open interval of Z is an open interval of Y), the mapping h is a homeomorphism of Y onto $(\widehat{X,S})$.

2.3. Theorem. *Let X be a non empty and non almost compact G.O. space and L be the set of all gaps of X. If S and T are two subsets of L such that $S \subset T$, the space $(\widehat{X,T})$ is canonically a quotient of the space $(\widehat{X,S})$. In particular, every space $(\widehat{X,S})$ is canonically a quotient of the space $\widehat{X}$.*

Let f be the mapping of $(\widehat{X,S})$ into $(\widehat{X,T})$ obtained by identification of elements s^- and s^+ associated to gaps s belonging to $T \setminus S$ and without any other modification. By construction, this mapping is increasing and onto and therefore continuous. Since $(\widehat{X,S})$ and $(\widehat{X,T})$ are compact Hausdorff, this mapping is also closed. Let then $R(f)$ be the equivalence relation defined on $(\widehat{X,S})$ by "$(x,y) \in R(f) \Longleftrightarrow f(x) = f(y)$". The space $(\widehat{X,T})$ is then, from a classical result of topology ((1), I. 32) homeomorphic to the quotient space $(\widehat{X,S})/R(f)$.

Remark. In a pictorial way, we can say that a space $(\widehat{X,S})$ is obtained from the space $\widehat{X}$ by "suturing" the gaps of X belonging to S.

3. Characterization of the zero-dimensional spaces $(\widehat{X,S})$

In the sequel, we consider a non empty and non almost compact G.O. space X and we denote by L the set of all the gaps of X and by S a subset of L.

Definition 3.1. We will say that a subset U of X is *S-connected* if and only if U is an interval such that any open meatus $s = (I, J)$ of X meeting U (i.e. such that $s|_U = (I \cap U, J \cap U)$ is a proper meatus of U) belongs to S.

It follows from this definition that:

(a) a subset is ϕ-connected if and only if it is connected.

(b) if $S \subset T \subset L$, any S-connected subset of X is T-connected which implies that any connected subset is S-connected.

Definition 3.2. We will call *S-connected component* of a point x of X the largest S-connected subset containing x (such a subset exists, it is the union of all S-connected subsets containing x) and we will say that X is *totally S-disconnected* if and only if the S-connected components are one-point sets.

It follows from this definition and the previous remark that, if $S \subset T \subset L$, any totally T-disconnected space is totally S-disconnected and therefore in particular totally disconnected.

Proposition 3.3. $\tau(\widehat{X,S})) = 0$ *if and only if X is totally S-disconnected.*

(a) Let us suppose X is not totally S-disconnected. There exists then an S-connected subset U of X which is not a one-point set. Since the closure of U in $(\widehat{X,S})$ is then a connected subset which is not a one-point set, the space $(\widehat{X,S})$ is not totally disconnected which implies $\tau(\widehat{X,S})) = 1$.

(b) Let us suppose X be totally S-disconnected and prove then $(\widehat{X,S})$ is totally disconnected which will imply $\tau((\widehat{X,S})) = 0$.
Let $I = [x, y]$ (with $x < y$) be an interval of $(\widehat{X,S})$.
If I is finite, I is not connected.
If I is infinite, $J = I \cap X$ is also an infinite interval.
Therefore, since X is totally S-disconnected, there exists an open meatus $s = (U, V)$ of X which meets J and does not belong to S.

That meatus defines a hole in I. Indeed:

(i) if s is a hole in X, we have $U = \,]\leftarrow, a]$ and $V = [b, \rightarrow[$ with $a < b$ and then (a, b) defines a hole in I.

(ii) if s is a left fault, we have $V = [b, \rightarrow[$ and then (s^-, b) defines a hole in I.

(iii) if s is a right fault, we have $U = \,]\leftarrow, a]$ and then (a, s^+) defines a hole in I.

(iv) if $s \in L \setminus S$, (s^-, s^+) defines a hole in I.

It follows then from 1.1 that I is not connected.

The only connected intervals of $(\widehat{X,S})$ are therefore the one-point sets, whence the result.

Remark. In the special case where $S = \emptyset$, we find again the result proved in 1.1.

4. Comparison between the dimensions of a G.O. space and its compactifications

4.1. Theorem 4. *Let X be a non empty and non almost compact G.O. space and L be the set of all the gaps of X. Then, for any $S \subset T \subset L$, we have:*

$$0 \leq \tau(X) = \tau(\widehat{X}) \leq \tau(\widehat{(X,S)}) \leq \tau(\widehat{(X,T)}) \leq \tau(\widehat{(X,L)}) \leq 1.$$

We already proved that, for every totally ordered compactification Y of X, we have $0 \leq \tau(X) \leq \tau(Y) \leq 1$ and that we have $\tau(X) = \tau(\widehat{X})$. It suffices therefore to prove that $\tau(\widehat{(X,S)}) \leq \tau(\widehat{(X,T)})$.

That is immediate if $\tau(\widehat{(X,S)}) = 0$ or $\tau(\widehat{(X,T)}) = 1$.

Let us suppose then that $\tau(\widehat{(X,T)}) = 0$. It follows, from 3.3, that X is totally T-disconnected which implies, since $S \subset T$, that X is totally S-disconnected and therefore such that $\tau(\widehat{(X,S)}) = 0$.

Conversely, if $\tau(\widehat{(X,S)}) = 1$, X is not totally D-disconnected are therefore not totally T-disconnected which implies that $\tau(\widehat{(X,T)}) = 1$.

4.2. *Remark.* It might happen that the previous inequalities are strict. Let us suppose, for example, that $X = \mathbb{Q}$. On one hand, we have, from 1.3, $\tau(\mathbb{Q}) = \tau(\widehat{\mathbb{Q}}) = 0$. On an other hand, $\widehat{(\mathbb{Q},L)}$, space obtained by "suturing" all the gaps of $\mathbb{Q}$ associated to each irrational, is homeomorphic to $\overline{\mathbb{R}}$, so that $\tau(\widehat{(\mathbb{Q},L)}) = 1$.

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