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Relaxed solutions for stochastic evolution equations on Hilbert space with polynomial nonlinearities

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Abstract. In this paper we introduce a new concept of generalized solutions or relaxed solutions for stochastic evolution equations on Hilbert space along the line of concept recently introduced for deterministic evolution equations on Banach spaces (see [1, 6]). We present here a result on the question of existence of generalized or measure valued solutions for stochastic semilinear evolution equations on Hilbert space. The result is sufficiently general to admit drift and diffusion parameters having polynomial growth without requiring Hilbert–Schmidt property for the later. As a corollary, an existence result of generalized solutions for forward Kolmogorov equation is obtained. Our main result is illustrated by three different examples one of which arises from structural mechanics.

1. Motivation

For motivation let us consider the deterministic evolution equation

$$\begin{aligned} &\dot{x} = Ax + F(x), \quad t \geq 0 \\ &x(0) = x_0 \end{aligned} \tag{1.1}$$

in a Banach space E where A is the infinitesimal generator of a C_0-semigroup, $T(t), t \geq 0$, on E and $F : E \longrightarrow E$ is a continuous map. It is well known that if E is finite dimensional, mere continuity of F is good enough to prove the existence of local solutions with possibly finite blow up time. If E is an infinite dimensional Banach space mere continuity no longer guarantees existence of even local solutions unless the semigroup $T(t)$,

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$t > 0$ is compact. For example, see Theorem 5.3.6 [2]. Generalizing the concept of solutions beyond the so called mild solutions it is possible to prove the existence of (generalized) solutions without requiring either of the hypothesis: the Lipschitz property of F and the compactness of the semigroup. The same comment applies to the stochastic system,

$$\begin{aligned} &dx = Ax\,dt + F(x)dt + \sigma(x)dW, \quad t \geq 0 \\ &x(0) = x_0, \end{aligned} \tag{1.2}$$

where W is a Wiener process defined on a suitable probability space and σ is a suitable operator valued function not necessarily Hilbert–Schmidt. For example, if F is merely continuous and satisfies the polynomial growth condition:

$$\|F(x)\|_E \leq K(1 + \|x\|_E^p), \quad p \geq 1, \tag{1.3}$$

standard results on stochastic differential equations in infinite dimensional Hilbert spaces [5, 10] can not be applied unless some additional assumptions such as dissipativity are used. The usual notions of mild and martingale solutions do not apply. However the notion of generalized solutions introduced in Section 3 does apply. There is an interesting similarity between the notions of martingale [14, 15] and generalized solutions as discussed later following Corollary 3.4. For simplicity of presentation we have considered both F and σ independent of time. However the results given here can be easily extended to the time varying case without any difficulty.

The rest of the paper is organized as follows. In Section 2 we recall some important results from analysis sufficient to serve our needs. In Section 3 we introduce the new concept of generalized (or measure valued) solutions and present some results on existence of generalized solutions and their regularity properties. In Section 4, we introduce the notion of a path process corresponding to a generalized solution and discuss its relationship with the standard notions of solutions. Further we present a brief discussion relating the notions of generalized, weak and mild solutions. In the final section we give three illustrating examples, the last one arising from structural mechanics.

2. Introduction

Recently we generalized the concept of relaxed solutions or equivalently the measure valued solutions of Fattorini and proved the existence of generalized solutions for the system (1.1) under much milder hypothesis on F admitting polynomial growth. In this paper we extend this result to stochastic systems given by equation (1.2). For this purpose we need the characterization of the dual of the Banach space $L_1(I, X)$ where $I \equiv [0, T]$ is a finite interval of the real line and X is a Banach space. Let X^* denote the dual of X, and $\langle \cdot \rangle$ the duality pairing of X^* and X. An X^*-valued function h is X-weakly measurable or equivalently w^*-measurable if and only if $\langle h(\cdot), x\rangle$ is an ordinary measurable function for each $x \in X$. Let $L^w_\infty(I, X^*)$ denote the class of all w^*-measurable functions h for each of which there exists a finite number $\alpha > 0$ such that for every $x \in X$

$$|\langle h(t), x\rangle| \leq \alpha \|x\|_X, \text{ for almost all (possibly depending on } x) \ t \in I.$$

The space is furnished with the norm $\|h\|_{L^w_\infty(I,X^*)} = \alpha_h$ where α_h is the smallest number α for which the inequality is satisfied. As a consequence of Dunford–Pettis theorem (see [7, Theorem 2.1; 8, Theorem 6, p. 503]), the dual of $L_1(I, X)$ is isometrically isomorphic to $L^w_\infty(I, X^*)$. Based on this result Fattorini constructed several classes of relaxed controls and later in [6] relaxed trajectories. We also make critical use of this result.

Let Z denote any normal topological space and $BC(Z)$ the space of bounded continuous functions on Z with the topology of sup norm, and let $\Sigma_{\mathrm{rba}}(Z)$ denote the space of regular bounded finitely additive set functions on Φ_{c} with total variation norm where Φ_{c} denotes the algebra generated by the closed subsets of Z. With respect to these topologies, these are Banach spaces and the dual of $BC(Z)$ is $\Sigma_{\mathrm{rba}}(Z)$ (see [8, Theorem 2, p. 262]). Let $\Pi_{\mathrm{rba}}(Z) \subset \Sigma_{\mathrm{rba}}(Z)$ denote the class of regular finitely additive probability measures furnished with the relative topology. It follows from the characterization result discussed above that the dual of $L_1(I, BC(Z))$ is given by $L^w_\infty(I, \Sigma_{\mathrm{rba}}(Z))$ which is furnished with the weak star topology. For brevity we replace the phrase "L_1 weak convergence in L^w_∞" often used in [6, 7], by simply w^* (weak star) convergence.

Let H, E be two separable Hilbert spaces and $(\Omega, \mathcal{F}, \mathcal{F}_t \uparrow, P)$ a complete filtered probability space, $W(t)$, $t \in I$, is an E valued $\mathcal{F}_t$ adapted cylindrical Wiener process and $\sigma : H \mapsto \mathcal{L}(E, H)$ where $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators from X to Y. For the purpose of this paper we consider $\mathcal{F}_t \equiv \mathcal{F}_t^W \vee \sigma(x_0)$, where $\sigma(x_0)$ is the smallest σ-algebra with respect to which x_0 is measurable. Let $I \times \Omega$ be furnished with the predictable σ-field with reference to the filtration $\mathcal{F}_t$, $t \in I$. Let $M_\infty^w(I, L_2(\Omega, \Sigma_{\mathrm{rba}}(H))) \subset L_\infty^w(I, L_2(\Omega, \Sigma_{\mathrm{rba}}(H)))$ denote the vector space of $\Sigma_{\mathrm{rba}}(H)$ valued random processes $\{\lambda_t, t \in I\}$, which are $\mathcal{F}_t$-adapted and w^*-measurable in the sense that $t \longrightarrow \lambda_t(\phi)$ is $\mathcal{F}_t$ measurable for each $\phi \in BC(H)$ and have finite second moments. We furnish this space with the w^* topology as before. Clearly this is the dual of the Banach space $M_1(I, L_2(\Omega, BC(H))) \subset L_1(I, L_2(\Omega, BC(H)))$. Here we have chosen $X \equiv L_2(\Omega, BC(H))$ and $X^* \equiv L_2(\Omega, \Sigma_{\mathrm{rba}}(H))$.

3. Existence of generalized solutions

Recently a notion of generalized solutions which consist of regular finitely additive measure valued functions was introduced (see [6, 1, 11]) and existence of solutions for deterministic systems such as (1.1) was proved. Our objective here is to prove similar results for the stochastic system (1.2).

Since we do not impose the standard assumptions such as the drift parameter having Lipschitz property with at most linear growth and the diffusion operator being Lipschitz and Hilbert–Schmidt, we expect the solutions to escape the original state space H at some point in time. Thus we may extend our state space through various topological compactification techniques (Alexandrov one point compactification, Stone–Cech compactification or Wallman compactification) so as to capture the supports of our measure solutions and that they may also posses the countable additivity property. Unless the original space is a locally compact Hausdorff space the Alexandrov compactification does not produce a compact Hausdorff space. It is well known that for any Tychonoff space $\mathcal{G}$, its Stone–Cech compactification denoted by $\beta\mathcal{G} \equiv \mathcal{G}^+$ is a compact Hausdorff space. In fact, for the purpose of this paper, $\mathcal{G} = H$ where H is a Hilbert space and hence a metric space with respect to its usual norm topology. Since every metric space is a Tychonoff space, H is a Tychonoff space. Hence H^+ is a compact Hausdorff space and consequently bounded continuous functions on H can be extended to continous functions on H^+. We write $H^+ = H \cup \triangle$ and consider $\triangle$ as the "dead Zone" or the "anihilator". In view of this we shall often use H^+ in place of H and hence the spaces $M_1(I, L_2(\Omega, BC(H^+)))$ with dual $M_\infty^w(I, L_2(\Omega, \Sigma_{\mathrm{rba}}(H^+))) \subset M_\infty^w(I, L_2(\Omega, \Pi_{\mathrm{rba}}(H^+)))$. Here $M_\infty^w(I, L_2(\Omega, \Pi_{\mathrm{rba}}(H^+)))$ is the set of probability measure valued processes, a subset of the vector space $M_\infty^w(I, L_2(\Omega, \Sigma_{\mathrm{rba}}(H^+)))$. Note that since H^+ is a compact Hausdorff space $\Sigma_{\mathrm{rba}}(H^+) = \Sigma_{\mathrm{rca}}(H^+)$. In view of the fact that the measure solutions (of stochastic evolution equations we consider) restricted to H are only finitely additive we prefer to use the notation $\Sigma_{\mathrm{rba}}(H^+)$ to emphasize this fact though they are countably additive on H^+.

Without further notice, throughout this paper we use $D\phi$ and $D^2\phi$ to denote the first and second Frechet derivatives of the function ϕ whenever they exist. We denote by Ψ the class of test functions as defined below

$$\begin{aligned}\Psi \equiv \{\phi \in BC(H) : D\phi, D^2\phi \text{ exist and are continuous}\\ \text{having bounded supports in } H \text{ and } \sup_{x\in H} \|D^2\phi(x)\|_{L_1(H)} < \infty\},\end{aligned}$$

where $L_1(H)$ denotes the space of nuclear operators in H. Define the operators $\mathcal{A}$ and $\mathcal{B}$ with domain given by

$$\mathcal{D}(\mathcal{A}) \equiv \{\phi \in \Psi : \mathcal{A}\phi \in BC(H^+)\}$$

where

$$\begin{aligned}\mathcal{A}\phi)(\xi) = (1/2)\,\mathrm{Tr}(\sigma^*(D^2\phi)\sigma)(\xi) + (A^*D\phi(\xi), \xi) + (F(\xi), D\phi(\xi))\\ \text{for} \quad \phi \in \mathcal{D}(\mathcal{A}),\ \mathcal{B}\phi(\xi) \equiv (\sigma^* D\phi)(\xi) \in E.\end{aligned} \tag{3.1}$$

Note that $\mathcal{D}(\mathcal{A}) \neq \emptyset$, for example, for $\psi \in \Psi$, the function ϕ given by $\phi(x) \equiv \psi(\lambda R(\lambda, A)x)$, belongs to $\mathcal{D}(\mathcal{A})$ for each $\lambda \in \rho(A)$, the resolvent set of A.

We consider the system

$$dx = Ax\,dt + F(x)dt + \sigma(x)dW, \quad x(0) = x_0, \tag{3.2}$$

and introduce a notion of generalized solutions which is applicable to stochastic systems with polynomial nonlinearities.

Definition 3.1. A measure valued random process

$$\mu \in M_\infty^w\left(I, L_2(\Omega, \Pi_{\mathrm{rba}}(H^+))\right) \subset M_\infty^w\left(I, L_2(\Omega, \Sigma_{\mathrm{rba}}(H^+))\right)$$

is said to be a generalized solution of equation (3.2) if for every $\phi \in \mathcal{D}(\mathcal{A})$ and $t \in I$, the following equality holds

$$\mu_t(\phi) = \phi(x_0) + \int_0^t \mu_s(\mathcal{A}\phi)ds + \int_0^t \langle \mu_s(\mathcal{B}\phi), dW(s)\rangle, \ P\text{-a.s.}$$

where

$$\mu_t(\psi) \equiv \int_{H^+} \psi(\xi)\mu_t(d\xi), \quad t \in I.$$

Our first existence result is given in the following theorem.

Theorem 3.2. *Let A be the generator of a C_0-semigroup in H and $F : H \longrightarrow H$ is continuous, and bounded in x on bounded subsets of H, and $\sigma : H \longrightarrow \mathcal{L}(E, H)$ is continuous and bounded on bounded subsets of H satisfying the following approximation property:*

(ai)*: there exists a sequence $\{F_n, \sigma_n\}$ such that $F_n(x) \in D(A)$, $\sigma_n(x) \in \mathcal{L}(E, D(A))$ for each $x \in H$, and*

$$\begin{aligned} &F_n(x) \longrightarrow F(x) \ \text{ in } H \text{ uniformly on compact subsets of } H,\\ &\sigma_n^*(x) \longrightarrow \sigma^*(x) \text{ strongly in } \mathcal{L}(H, E) \text{ uniformly}\\ &\qquad\qquad\qquad\quad \text{on compact subsets of } H. \end{aligned}$$

(aii)*: there exists a pair of sequences $\{\alpha_n, \beta_n > 0\}$ possibly α_n, $\beta_n \to \infty$ as $n \to \infty$, such that*

$$\|F_n(x) - F_n(y)\| \le \alpha_n \|x - y\|; \quad \|F_n(x)\| \le \alpha_n(1 + \|x\|),$$
$$\|\sigma_n(x) - \sigma_n(y)\|_{\mathcal{L}_2(E,H)} \le \beta_n \|x - y\|; \quad \|\sigma_n(x)\|_{\mathcal{L}_2(E,H)} \le \beta_n(1 + \|x\|)$$

for all $x, y \in H$.

Then for every x_0 for which $P\{\omega \in \Omega : |x_0|_H < \infty\} = 1$ the evolution equation (3.2) has at least one generalized solution $\lambda^0 \in M_\infty^w(I, L_2(\Omega, \Sigma_{\mathrm{rba}}(H^+)))$ in the sense of Definition 3.1. Further, $\lambda^0 \in M_\infty^w(I, L_2(\Omega, \Pi_{\mathrm{rba}}(H^+)))$ and it is P-a.s w^* continuous.

Proof. Since $D(A)$ is dense in H and $x_0 \in H$, a.s (almost surely), there exists a sequence $\{x_{0,n}\} \in D(A)$ such that $x_{0,n} \xrightarrow{s} x_0$ a.s. Consider the Cauchy problem:

$$\begin{aligned} dx &= A_n x dt + F_n(x)dt + \sigma_n(x)dW(t),\\ x(0) &= x_{0,n}, \end{aligned} \tag{3.3}$$

where $A_n = nAR(n, A)$, $n \in \rho(A)$, is the Yosida approximation of A. Since for each $n \in N$ and $x \in H$, $F_n(x) \in D(A)$ and $\sigma_n(x) : E \mapsto D(A)$, it follows from assumption (aii) that equation (3.3) has a unique strong solution $x_n = \{x_n(t), t \in I\}$ which is $\mathcal{F}_t$- adapted, continuous P-a.s., and for each $n \in N$

$$\sup\{E\|x_n(t)\|_H^2, t \in I\} < \infty,$$

and for almost all $t \in I$, $x_n(t) \in D(A)$. Now let $\phi \in \mathcal{D}(\mathcal{A})$ with $D\phi$ and $D^2\phi$ having compact supports in H. Since x_n is a strong solution it follows from Ito's formula that for each $t \in I$,

$$\phi(x_n(t)) = \phi(x_{0,n}) + \int_0^t (\mathcal{A}_n\phi)(x_n(s))ds + \int_0^t \langle (\mathcal{B}_n\phi)(x_n(s)), dW(s)\rangle, \tag{3.4}$$

where the angle bracket denotes the scalar product in E and

$$(\mathcal{A}_n\phi)(\xi) = (1/2)\,\mathrm{Tr}((\sigma_n^*(D^2\phi)\sigma_n)(\xi)) + (A_n^* D\phi(\xi), \xi) + (F_n(\xi), D\phi(\xi))$$

$$\text{for } \phi \in \mathcal{D}(\mathcal{A}) \qquad (\mathcal{B}_n\phi)(\xi) \equiv (\sigma_n^* D\phi)(\xi) \in E.$$

Letting $\delta_e(d\xi)$ denote the Dirac measure concentrated at the point $e \in H$, and defining $\lambda_t^n(d\xi) \equiv \delta_{x_n(t)}(d\xi)$, $t \in I$, $\lambda_0^n(d\xi) \equiv \delta_{x_{0,n}}(d\xi)$, and using the notation of Definition 3.1 we can rewrite (3.4) as

$$\lambda_t^n(\phi) = \lambda_0^n(\phi) + \int_0^t \lambda_s^n(\mathcal{A}_n\phi)ds + \int_0^t \langle \lambda_s^n(\mathcal{B}_n\phi), dW(s)\rangle, t \in I. \tag{3.5}$$

For each integer $n, \lambda^n \in M_\infty^w(I, L_2(\Omega, \Pi_{\mathrm{rba}}(H)))$ and hence the set $\{\lambda^n\}$ is contained in

$$M_\infty^w\left(I, L_2(\Omega, \Pi_{\mathrm{rba}}(H^+))\right) \subset M_\infty^w(I, L_2\left(\Omega, \Sigma_{\mathrm{rba}}(H^+))\right).$$

Indeed the functional ℓ_n, given by

$$\ell_n(\psi) \equiv E\int_{I\times H^+} \psi(t,\xi)\lambda_t^n(d\xi)dt \equiv \int_{I\times\Omega\times H^+} \psi(t,\omega,\xi)\,\lambda_{t,\omega}^n(d\xi)\,dPdt,$$

is well defined for each $\mathcal{F}_t$ adapted $\psi \in M_1(I, L_2(\Omega, BC(H^+)))$ and

$$|\ell_n(\psi)| \leq \|\psi\|_{M_1(I,L_2(\Omega,BC(H^+)))}, \quad \text{for all } n \in N.$$

Thus the sequence $\{\ell_n\}$ is contained in a bounded subset of $(M_1(I, L_2(\Omega, BC(H^+))))^*$ and by the characterization of the dual space the corresponding sequence of measures $\{\lambda^n\}$ is confined in a bounded subset of $M_\infty^w(I, L_2(\Omega, \Sigma_{\text{rba}}(H^+)))$. Hence by Alaoglu's theorem, there exists a generalized subsequence (subnet) of the sequence (net) $\{\lambda^n\}$, relabeled as $\{\lambda^n\}$, and a $\lambda^0 \in M_\infty^w(I, L_2(\Omega, \Sigma_{\text{rba}}(H^+)))$, so that $\lambda^n \xrightarrow{w^*} \lambda^0$. We show that λ^0 is a generalized solution of equation (3.2) in the sense of Definition 3.1. Define

$$\psi_{1,n}(\xi) \equiv (1/2)\,\text{Tr}(\sigma_n^*(D^2\phi)\sigma_n)(\xi)$$
$$\psi_1(\xi) \equiv (1/2)\,\text{Tr}(\sigma^*(D^2\phi)\sigma)(\xi).$$

Since $\sigma_n^*(x) \longrightarrow \sigma^*(x)$ strongly in $\mathcal{L}(H, E)$ uniformly on compact subsets of H and $D^2\phi$ has compact support, and for each $\phi \in D(\mathcal{A})$, $\sup\{\|D^2\phi(\xi)\|_{L_1(H)}, \xi \in H\} < \infty$, we have $\psi_{1,n}, \psi_1 \in BC(H)$ and $\psi_{1,n} \longrightarrow \psi_1$ uniformly on H. Hence it follows from the weak* convergence of λ^n to λ^0 that for any $z \in L_2(\Omega, \mathcal{F}, P) = L_2(\Omega)$, and $t \in I$, we have

$$\int_{\Omega\times[0,t]} z\lambda_s^n(\psi_{1,n})ds\,dP \longrightarrow \int_{\Omega\times[0,t]} z\lambda_s^0(\psi_1)ds\,dP. \tag{3.6}$$

Define

$$\psi_{2,n}(\xi) \equiv (A_n^*(D\phi)(\xi), \xi) \quad \text{and} \quad \psi_2(\xi) \equiv (A^*(D\phi)(\xi), \xi).$$

Since $A_n \longrightarrow A$ on $D(A)$ in the strong operator topology and, for $\phi \in D(\mathcal{A})$, $D\phi(x) \in D(A^*)$, and further, by our choice of ϕ, $D\phi$ is continuous having compact support, we can deduce that $\psi_{2,n} \longrightarrow \psi_2$ uniformly on H. Hence, again we have

$$\int_{\Omega\times[0,t]} z\lambda_s^n(\psi_{2,n})ds\,dP \longrightarrow \int_{\Omega\times[0,t]} z\lambda^0(\psi_2)ds\,dP. \tag{3.7}$$

Similarly define

$$\psi_{3,n}(\xi) \equiv (F_n(\xi), D\phi(\xi)) \qquad \text{and} \quad \psi_3(\xi) \equiv (F(\xi), D\phi(\xi)).$$

Again since $\phi \in D(\mathcal{A})$ and $D\phi$ has compact support and by our assumption $F_n \longrightarrow F$ uniformly on compact subsets of H, it follows that $\psi_{3,n} \longrightarrow \psi_3$ in the topology of $BC(H)$. Thus we have

$$\int_{\Omega\times[0,t]} z\lambda_s^n(\psi_{3,n})ds\,dP \longrightarrow \int_{\Omega\times[0,t]} z\lambda_s^0(\psi_3)dsdP \tag{3.8}$$

for every $z \in L_2(\Omega)$. Combining (3.6)–(3.8) we conclude that for every $z \in L_2(\Omega)$ and $\phi \in D(\mathcal{A})$ with $D\phi, D^2\phi$ having compact supports,

$$\int_{\Omega\times[0,t]} z\lambda_s^n(\mathcal{A}_n\phi)ds\, dP \longrightarrow \int_{\Omega\times[0,t]} z\lambda_s^0(\mathcal{A}\phi)ds\, dP. \tag{3.9}$$

Since $x_{0,n} \xrightarrow{s} x_0$ a.s and $\phi \in BC(H^+)$, we have $\phi(x_{0,n}) \longrightarrow \phi(x_0)$ a.s. Then by Lebesgue dominated convergence theorem, for every $z \in L_2(\Omega)$, we have

$$\int_\Omega z\phi(x_{0,n})dP \longrightarrow \int_\Omega z\phi(x_0)dP \equiv \int_\Omega z\lambda_0(\phi)dP \tag{3.10}$$

where $\lambda_0(\phi) \equiv \int_H \phi(\xi)\delta_{x_0}(d\xi)$. For the stochastic integral in (3.5), note that since $D\phi$ is continuous having compact support, $\mathcal{B}_n\phi \in BC(H^+, E)$ and

$$E\int_I \|(\mathcal{B}_n\phi)(x_n(s))\|_E^2 ds < \infty.$$

Thus the stochastic integral in (3.5) is well defined and for any $z \in L_2(\Omega)$, it follows from the properties of conditional expectation and the martingale theory that

$$E\left(z\int_0^t \langle \lambda_s^n(\mathcal{B}_n\phi), dW(s)\rangle\right) = E\left(z_t\int_0^t \langle \lambda_s^n(\mathcal{B}_n\phi), dW(s)\rangle\right) \tag{3.11}$$

where $z_t \equiv E\{z|\mathcal{F}_t\}$ is a square integrable $\mathcal{F}_t$ martingale. Hence there exists an $\mathcal{F}_t$-adapted E-valued process $\eta(t)$, $t \in I$, and a square integrable random variable z_0 independent of the Brownian increments such that

$$E\int_I \|\eta(t)\|_E^2 dt < \infty,$$

and that $z_t = z_0 + \int_0^t \langle \eta(s), dW(s)\rangle$. Hence

$$E\left(z\int_0^t \langle \lambda_s^n(\mathcal{B}_n\phi), dW(s)\rangle\right) = E\left(\int_0^t \langle \eta(s), \lambda_s^n(\mathcal{B}_n\phi)\rangle ds\right). \tag{3.12}$$

Since $D\phi$ has compact support, $\mathcal{B}_n\phi \longrightarrow \mathcal{B}\phi$ in the topology of $BC(H^+, E)$ and hence

$$\langle \eta, \mathcal{B}_n\phi\rangle \xrightarrow{s} \langle \eta, \mathcal{B}\phi\rangle$$

in $M_1(I, L_2(\Omega, BC(H^+)))$. It follows from this and the fact that $\lambda^n \xrightarrow{w^*} \lambda^0$ that, for each $t \in I$,

$$E\bigg(\int_0^t \langle \eta(s), \lambda_s^n(\mathcal{B}_n\phi)\rangle ds\bigg) \longrightarrow E\bigg(\int_0^t \langle \eta(s), \lambda_s^0(\mathcal{B}\phi)\rangle ds\bigg) = E\bigg(z\int_0^t \langle \lambda_s^0(\mathcal{B}\phi), dW(s)\rangle\bigg). \tag{3.13}$$

Thus multiplying both sides of equation (3.5) by arbitrary $z \in L_2(\Omega)$ and taking the limit of the expected values, it follows from (3.9), (3.10) and (3.13) that

$$E(z\lambda_t^0(\phi)) = E(z\lambda_0(\phi)) + E\bigg(z\int_0^t \lambda_s^0(\mathcal{A}\phi)ds\bigg) + E\bigg(z\int_0^t \langle \lambda_s^0(\mathcal{B}\phi), dW(s)\rangle\bigg).$$

Since this holds for arbitrary $z \in L_2(\Omega)$, for each $t \in I$ we have

$$\lambda_t^0(\phi) = \lambda_0(\phi) + \int_0^t \lambda_s^0(\mathcal{A}\phi)ds + \int_0^t \langle \lambda_s^0(\mathcal{B}\phi), dW(s)\rangle \ P\text{-a.s.} \tag{3.14}$$

By virtue of the fact that $\lambda^0 \in M_\infty^\omega(I, L_2(\Omega, \Sigma_{\mathrm{rba}}(H^+)))$, it is evident that for each $\phi \in D(\mathcal{A})$, $\lambda_t^0(\mathcal{A}\phi)$, $\lambda_t^0(\mathcal{B}\phi)$ are well defined $\mathcal{F}_t$ adapted processes and that $\lambda^0(\mathcal{A}\phi) \in L_2(I, L_2(\Omega))$, $\lambda^0(\mathcal{B}\phi) \in L_2(I, L_2(\Omega, E))$. Thus equation (3.14) holds for all $\phi \in D(\mathcal{A})$ and not just those having first and second Frechet differentials with compact supports. Hence λ^0 is a generalized solution of equation (3.2) in the sense of Definition 3.1. The proof of the last assertion of the theorem follows from the fact that the approximating sequence is a sequence of Dirac measures and clearly are all positive and that positivity is preserved under weak star limit. Thus $\lambda^0 \in M_\infty^w(I, L_2(\Omega, \Pi_{\mathrm{rba}}(H^+)))$. The a.s. weak star continuity $t \longrightarrow \lambda_t^0$ follows immediately from the expression (3.14). This completes the proof.

Remark 3.3. It is clear from the above result that for generalized (relaxed) solutions it suffices if the drift F and the dispersion σ are merely locally Lipschitz and bounded on bounded subsets of H. Thus these parameters may have polynomial growth. In contrast, for standard mild solutions, it is usually assumed that F is Lipschitz (or locally Lipschitz) and admits linear growth and σ is Hilbert–Schmidt, Lipschitz and has linear growth (see [5, Chapter 7]). This of course guarantees uniqueness and continuity of the path process. Our result sacrifices the path process and countable additivity and provides a stochastic finitely additive regular measure valued process as the solution. However it is countably additive on the compact Hausdorff space H^+ containing the original state space H as a dense subspace.

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.4. *Consider the forward Kolmogorov equation*

$$(d/dt)\nu(t) = \mathcal{A}^*\nu(t) \qquad \nu(0) = \mu_0, \tag{3.15}$$

where $\mathcal{A}^*$ is the dual of the operator $\mathcal{A}$ (see (3.1)) with F, σ satisfying the assumptions of Theorem 3.2. Then for every $\mu_0 \in \Pi_{\mathrm{rba}}(H)$ equation (3.15) has at least one weak solution $\nu \in L^w_\infty(I, \Pi_{\mathrm{rba}}(H^+)) \subset L^w_\infty(I, \Sigma_{\mathrm{rba}}(H^+))$ in the sense that for each $\phi \in D(\mathcal{A})$ the following equality holds

$$\nu(t)(\phi) = \mu_0(\phi) + \int_0^t \nu(s)(\mathcal{A}\phi)\, ds, \quad t \in I.$$

Proof. Since $\mu_0 \in \Pi_{\mathrm{rba}}(H)$ there exists a random variable x_0 taking values P-a.s in H (possibly on a Skorokhod extension) such that for each $\phi \in BC(H)$,

$$E\phi(x_0) = E\int_{H^+} \phi(\xi)\delta_{x_0}(d\xi) \equiv E\int_{H^+} \phi(\xi)\lambda_0(d\xi) = \int_{H^+} \phi(\xi)\mu_0(d\xi).$$

Here we have used ϕ itself to denote its extension from H to H^+. Using x_0 defined above as the initial state, it follows from Theorem 3.2 that equation (3.2) has at least one generalized solution $\lambda^0 \in M^w_\infty(I, L_2(\Omega, \Pi_{\mathrm{rba}}(H^+)))$ satisfying equation (3.14) for each $\phi \in D(\mathcal{A})$. Then the the map

$$\psi \longrightarrow E\biggl(\int_I \lambda^0_t(\psi)dt\biggr)$$

is a continuous linear functional on $L_1(I, BC(H^+))$. Hence there exists a unique $\nu \in L^w_\infty(I, \Pi_{\mathrm{rba}}(H^+))$ so that

$$E\biggl(\int_I \lambda^0_t(\psi)dt\biggr) = \langle \nu, \psi\rangle_{L^w_\infty(I,\Pi_{\mathrm{rba}}(H^+)), L_1(I,BC(H^+))} \equiv \int_I \nu_t(\psi)dt. \tag{3.16}$$

Clearly by equation (3.14), for $\phi \in D(\mathcal{A})$, the Ito differential of $\lambda_t^0(\phi)$ is given by

$$d\lambda_t^0(\phi) = \lambda_t^0(\mathcal{A}\phi)dt + \langle \lambda_t^0(\mathcal{B}\phi), dW(t)\rangle.$$

Evaluating the Ito differential of the scalar random process $\lambda_t^0(\xi(t)\phi)$ for any $\xi \in C_0^1(0,T)$, a C^1-function with compact support, and integrating, we have

$$-\int_I \lambda_t^0(\dot{\xi}(t)\phi)dt = \int_I \lambda_t^0(\mathcal{A}(\xi(t)\phi))dt + \int_I \langle \lambda_t^0(\mathcal{B}(\xi(t)\phi)), dW(t)\rangle. \tag{3.17}$$

Taking the expectation of either side of equation (3.17) and noting that, for $\phi \in D(\mathcal{A})$ and ξ as given,

$$(\dot{\xi}\phi), (\xi\phi), \mathcal{A}(\xi\phi), \quad \text{and} \quad \mathcal{B}(\xi\phi) \in L_1(I, BC(H^+)),$$

it follows from the representation (3.16) that

$$-\int_I \nu_t(\dot{\xi}(t)\phi)dt = \int_I \nu_t(\mathcal{A}(\xi(t)\phi))dt.$$

Since $\xi \in C_0^1$ is arbitrary it follows from this that

$$(d/dt)\nu_t(\phi) = \nu_t(\mathcal{A}\phi), \quad \text{for } t > 0, \tag{3.18}$$

in the sense of distribution. This holds for each $\phi \in D(\mathcal{A})$ and thus $\nu \in L_\infty^w(I, \Pi_{\text{rba}}(H^+))$ satisfies the differential equation of (3.15) in the sense of distribution and further $t \longrightarrow \nu_t$ is weak* continuous. For the initial condition, we use any $\xi \in C^1$ satisfying $\xi(T) = 0, \xi(0)(\neq 0)$ arbitrary, and compute the integrals

$$\int_I \xi(t)d\lambda_t^0(\phi), \quad \int_I \xi(t)d\nu_t(\phi).$$

This gives us

$$\begin{aligned}
-\lambda_0(\phi)\xi(0) &= \int_I \lambda_t^0(\dot{\xi}(t)\phi)dt + \int_I \lambda_t^0(\mathcal{A}(\xi(t)\phi)dt \\
&\quad + \int_I \langle \lambda_t^0(\mathcal{B}(\xi(t)\phi)), dW(t)\rangle \ P\text{-a.s.} \qquad \text{and} \\
-\nu_0(\phi)\xi(0) &= \int_I \nu_t(\dot{\xi}(t)\phi)dt + \int_I \nu_t(\mathcal{A}(\xi(t)\phi))dt,
\end{aligned}$$

respectively. Since $t \longrightarrow \lambda_t^0$ is weak* continuous P-a.s. and $t \longrightarrow \nu_t$ is weak* continuous, the leading terms in the above expressions are well defined. Taking expectation of either side of the first equation, exploiting the representation (3.16) again, and equating terms associated with $\xi(0)$, which is arbitrary, with those of the second equation we obtain

$$\nu_0(\phi) = E(\lambda_0(\phi)), \quad \phi \in D(\mathcal{A}).$$

Since by definition $E(\lambda_0(\phi)) = \mu_0(\phi)$, for all $\phi \in D(\mathcal{A})$ this completes the proof.

Martingale vs Generalized Solutions

As mentioned earlier, there is a similarity between the definition of martingale solution and our concept of generalized solution. In the formulation of martingale solution a probability space is constructed using the standard canonical sample space, for example, $\Omega \equiv C(I, H)$ equipped with the filtration $\mathcal{F}_t \equiv \sigma\{x(s),\ s \le t,\ x \in C(I, H)\}$ and then looking for the existence of a measure P satisfying certain initial or boundary conditions such that $(\Omega, \mathcal{F}_t \uparrow, P)$ is a filtered probability space and the functional $C_t(\phi)$, with values

$$\text{(3.19)} \quad C_t(\phi)(\omega) \equiv \phi(\omega(t)) - \phi(\omega(0)) - \int_0^t (\mathcal{A}\phi(\omega(s))ds, \quad t \in I,\ \omega \in \Omega$$

is a $P - \mathcal{F}_t$ martingale for each $\phi \in \Phi$ where Φ is a suitable class of test functions, such as cylinder functions on H. For more details including interesting application to control theory see Gatarek and Sobczyk [14, 15].

On the other hand, according to the notion of generalized solution, we are looking for solutions $\{\mu_t, t \ge 0\}$, which are $\mathcal{F}_t$ adapted w^*-continuous random processes with values in $\Pi_{\mathrm{rba}}(H) \subset \Pi_{\mathrm{rba}}(H^+) = \Pi_{\mathrm{rca}}(H^+)$ satisfying the identity in Definition 3.1.

In the martingale formulation represented by (3.19), the structure and regularity of the path space is imposed on the problem requiring that the process evolves continuously in H. If H is finite dimensional there is no problem, but if it is infinite dimensional the process may not continuously evolve in H. On the other hand, in the formulation of measure solution the process evolves in the space $\Sigma_{\mathrm{rba}}(H)$ in place of H and the temporal regularity such as w^* continuity is consequential not an imposition. This admits substantial relaxation in the whole notion of solutions. Further, in the martingale formulation the parameters F and σ are assumed to admit atmost linear growth where as in the formulation of generalized solutions polynomial growth (or even better) is allowed and further σ is not required to be Hilbert–Schmidt valued. Most important point however is not these generalities but the fact that even an innocent looking semilinear deterministic problem may have no H-valued solution but may very well possess a measure solution (see [6]).

Remark 3.5. It may be tempting to construct a martingale solution using $C(I, \Sigma_{\mathrm{rba}}(H))$ as the canonical sample space for the evolution equation

$$\begin{aligned} d\mu_t &= \mathcal{A}^*\mu_t + \langle \mathcal{B}^*\mu_t, dW(t)\rangle, \quad t \in I \\ \mu_0 &= \nu_0. \end{aligned} \tag{3.20}$$

But again this imposes a strong temporal regularity. Further it requires characterization of the duals of the operators $\mathcal{A}$ and $\mathcal{B}$ and apparently a martingale formulation is not easy even if feasible. It is not clear how nonseparability of the Banach space $BC(H)$ would affect the choice of cylinder functions on $\Sigma_{\mathrm{rba}}(H)$. Any way if such a theory is feasible, it is not very clear how one can use the results in control applications.

4. Path process and equivalence of solutions

A generalized solution as considered in this paper is an $\mathcal{F}_t$-adapted stochastic process with values in the space of regular bounded finitely additive measures on H. In case the measure valued solution $\{\lambda_t^0, t \geq 0\}$ degenerates into a Dirac measure concentrated on a process $\{y^0(t), t \geq 0\}$, taking values possibly in H^+ then y^0 is the genuine path process. In general there is no path process. It is conceivable that a measure solution has a Dirac structure over certain periods of time while it is a genuine measure over others. At this time we do not know of any necessary or sufficient conditions required of the parameters F and σ so that such a phenomenon may be observed. However for generalized solutions of equation (3.2) we can still introduce (fabricate) a path process as follows. One could call this process as the stochastic mean flow. For each open ball $B_r \equiv \{x \in H : \|x\|_H < r\}$, of radius $r > 0$, define the function

$$\phi_r(\xi) \equiv \begin{cases} (\xi, h), & \text{for } \xi \in B_r; \\ (r/|\xi|_H)(\xi, h), & \text{for } \xi \in B_r' \end{cases} \tag{4.1}$$

where $h \in D(A) \subset H$ is arbitrary and $B'_r \equiv H^+ \setminus B_r$. Clearly this function is continuous and bounded on H and converges point wise to the linear functional $\phi(\xi) \equiv (\xi, h)$ as $r \to \infty$. For convenience of notation we use ϕ and ϕ_r also to denote their extensions to H^+. Introduce the process $x_r \equiv \{x_r(t),\, t \geq 0\}$ given by

$$(x_r(t), h) \equiv \int_{H^+} \phi_r(\xi) \lambda_t^0(d\xi). \tag{4.2}$$

Clearly, for each $r > 0$, x_r is H-valued and $x_r(t) \in B_r$ for all $t \in I$ with probability one. In other words for each $r > 0$, $\phi_r \in BC(H^+)$ and hence $\phi_r \in L_1(H^+, \lambda_t^0)$ for each $t \in I$; but its limit $\phi \notin BC(H^+)$ and hence may not be in $L_1(H^+, \lambda_t^0)$ unless it turns out that λ_t^0 has a bounded support. This can occur in the deterministic case ($\sigma \equiv 0$) with F being merely continuous and satisfying at most a linear growth. Note that even in this case the system does not have a pathwise solution (strong, mild or weak) (see [6]) but has a deterministic finitely additive measure valued solution. So again (4.2) can be used to fabricate a path process. Define

$$\Gamma^+ \equiv \{\xi \in H^+ : \phi(\xi) > 0\},$$
$$\Gamma^- \equiv \{\xi \in H^+ : \phi(\xi) \leq 0\}.$$

Clearly the limit of the path process as defined above may fail to exist if for any set of $t \in I$ having positive Lebesgue measure both of the following equalities hold

$$\text{(i): } \int_{\Gamma^+} \phi(\xi) \lambda_t^0(d\xi) = +\infty, \qquad \text{(ii): } \int_{\Gamma^-} \phi(\xi) \lambda_t^0(d\xi) = -\infty.$$

But if only one of them holds then the path process exists as an H^+-valued process. On the other hand, for each $h \in H$ and $s \in N$, $s \geq 1$, one can verify that

$$|\langle x_{r+s}(t) - x_r(t), h \rangle| \leq s|h|_H \lambda_t^0(B_r^{'}) \ P\text{-a.s.}$$

Clearly if $\lim_{r \to \infty} \lambda_t^0(B'_r) \longrightarrow 0$ in P-measure then for each $t \in I$, $\{x_r(t)\}$ is weakly sequentially Cauchy in P-measure. Hence there exists a subnet of the net $\{x_r(t),\, r > 0\}$ that converges weakly in P-measure to a limit $x^0(t)$ with values possibly in H^+.

In case the system is linear, the solution (strong, weak or mild) is always a genuine path process or equivalently a Dirac measure concentrated on this path process. Thus in this case there is no need to construct an artificial path process. But one may be interested to see how the definition of the fabricated path process as introduced here may lead to the true solution process at least for linear systems. For example let us consider the simple linear system

$$dx = -Axdt + dW$$

in the Hilbert space H and suppose W is a cylindrical Wiener process in this space and A is a positive selfadjoint (unbounded) operator with domain and range in H generating a C_0-semigroup $T(t)$, $t \geq 0$, in H. The mild solution for this system with initial state x_0 is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)dW(s), \quad t \geq 0. \tag{4.3}$$

Since W is cylindrical and the semigroup is not necessarily Hilbert–Schmidt, the process x also is cylindrical in H irrespective of whether or not the initial state has nuclear covariance operator. Assuming that x_0 has a covariance operator $P_0 \in \mathcal{L}(H)$, not necessarily nuclear, one can verify that the covariance operator $P(t)$ for $x(t)$ satisfies the following estimate

$$0 \leq (P(t)h, h) \leq M_t^2 \|P_0\|_H |h|_H^2 + tM_t^2 |h|_H^2 < \infty$$

for every $h \in H$ where $M_t \equiv \sup\{\|T(s)\|_H,\ 0 \leq s \leq t\}$ for $0 \leq t < \infty$. This ofcourse also means that for any $h \in H$ (and not just for $h \in D(A)$), $E(x(t), h)^2 < \infty$. Thus $P(t) \in \mathcal{L}(H)$ but not nuclear and hence the corresponding measure μ_t which is deterministic is only finitely additive on H, more precisely on the algebra $\Phi_c(H)$ of closed subsets of H. But for this simple example one can construct a suitably larger Hilbert space containing H as a dense subspace on which $P(t)$ is nuclear and the corresponding μ_t is countably additive. This is based on the property that A is positive selfadjoint. In general for an arbitrary semigroup generator A and nonlinear operators F and σ this is not always possible. Hence the introduction of the Stone–Cech compactification H^+ of H, which is very large, is useful. Further it also offers a compact Hausdorff space on which our measure solutions are supported and are actually countably additive. For the linear example, using the positive selfadjoint operator A we can easily construct a larger space (containing H) on which the measures are countably additive. Thus given that A is a positive selfadjoint operator we introduce the hierarchie of Hilbert spaces using the graph norms given by

$$\begin{aligned} H_\beta &\equiv [D(A^\beta)], && \beta \geq 0 \\ \|\xi\|_\beta &\equiv \|A^\beta \xi\|_H, && \xi \in D(A^\beta) \\ (\xi, \eta)_\beta &\equiv (A^\beta \xi, A^\beta \eta)_H, && \xi, \eta \in H_\beta. \end{aligned}$$

Clearly $H_0 = H$. Corresponding to these family there exists a family of dual spaces denoted by

$$H_{-\beta} \equiv H_\beta^*$$

which are in fact completion of H with respect to the norm topology

$$\|\zeta\|_{-\beta} \equiv \|A^{-\beta}\zeta\|_H.$$

Identifying H with its own dual we have the following inclusions

$$H_\gamma \hookrightarrow H_\beta \hookrightarrow H \hookrightarrow H_{-\beta} \hookrightarrow H{-\gamma}$$

for $\gamma \geq \beta \geq 0$ with the injections being continuous and dense. Now returning to the process x one can easily show that it is a well defined (P-a.s) continuous process with values in $H_{-\beta}$ for $\beta > (1/2)$, and the corresponding covariance operator $P(t) \in \mathcal{L}_n^+(H_\beta, H_{-\beta})$. Further, since every mild solution is a weak solution, x is also a weak solution and for each $h \in D(A^*) = D(A)$

$$(x(t), h) = (x_0, h) - \int_0^t (Ah, x(s))ds + (W(t), h), \quad t \geq 0. \tag{4.4}$$

Define the stochastic measure process as the Dirac measure concentrated along the solution process, that is, $\lambda_t(d\xi) \equiv \delta_{x(t)}$ This measure valued process off course satisfies our general expression (3.14). Using the function ϕ_r as defined by equation (4.1) for any $h \in D(A^*) = D(A)$, we fabricate the path process according to the formula given by equation (4.2). This yields, for each $t \geq 0$,

$$(x_r(t), h) = \phi_r(x(t)), \tag{4.5}$$

and that $x_r(t) \in B_r \subset H$ with probability one. Thus by virtue of point wise convergence of ϕ_r to ϕ it follows from (4.5) that for any $t \geq 0$, and $h \in D(A^*) = D(A)$

$$\lim_{r\to\infty} (x_r(t), h) = \lim_{r\to\infty} \phi_r(x(t)) = \phi(x(t)) \equiv (x(t), h) \; P\text{-a.s.} \tag{4.6}$$

In this sense the fabricated path process converges to the true solution process. No such implication has any meaning in case of genuine measure solutions. Only if the nonlinear system (3.2) has a strong, or mild or a weak solution (a genuine path process) one can consider the Dirac measure concentrated along the solution process x as the generalized solution and the expression (3.14) holds. The standard weak form given by

$$\begin{aligned}(x(t), \eta) = (x_0, \eta) &+ \int_0^t \{(x(s), A^*\eta) + (F(x(s)), \eta)\} ds \\ &+ \int_0^t \langle \sigma^*(x(s))\eta, dW(s)\rangle, \; P\text{-a.s,}\end{aligned} \tag{4.7}$$

which holds for each $\eta \in D(A^*)$ follows from (3.14) as a special case. This is easily proved by using ϕ_r and limiting arguments noting that the Frechet derivatives $D\phi_r$ has B_r as its support and equals h on B_r and the second Frechet derivative vanishes every where. However in general for nonlinear systems the limit of the fabricated path process, if it exists, has no other implication. But it may have interesting applications. For example, in control problems if one wishes to control the support of the measure process $\{\lambda_t^0, \, t \geq 0\}$, so as to capture a target set, or equivalently have its support concentrated on the target set with high probability, one may instead use and control the mean flow process to hit the target.

Equivalence of weakened, weak and mild solutions for stochastic differential equations of the form

$$dx = Axdt + dM_t(x)$$

where M_t is a state dependent martingale process was thoroughly studied in an important paper [13] by Michalik. The same conclusion remains valid for the system (3.2) provided the coefficients F, σ satisfy the standard (slightly weakened) assumptions. In other words, under the standard assumptions guaranteeing existence of weakened or mild or weak solutions, all the notions consisting of the generalized, weak, mild and weakened solutions coincide and in this case the classical concept of solutions as path processes holds. However existence of generalized or measure solutions does not imply existence of solution in any one of the other senses as demonstrated by the deterministic example given above.

Remark. Since A is a positive selfadjoint operator, the map A^β is a homeomorphism of H onto $H_{-\beta}$ and hence these spaces are topologically equivalent. Therefore the Stone-Cech compactification H^+ of H is equivalent to the Stone–Cech compactification $H^+_{-\beta}$ of $H_{-\beta}$. In fact they are identical. In other words $H_{-\beta} \subset H^+$ also.

Explosion Time

Let $\mathcal{S}(\lambda)$ denote the support of any measure λ. It is well known that a cylindrical H-valued Gaussian random variable has nonnuclear covariance operator and hence the corresponding $\mathcal{S}(\lambda) \not\subseteq H$. Suppose $\mathcal{S}(\lambda_0) \subset H$ where λ_0 is the law of the initial state x_0. This, off course, holds if x_0 is either a fixed element of H or an H-valued random variable having finite second moment. Consider the system (3.2) with the initial state as specified above. In this case

$$P\{\{t \geq 0, \mathcal{S}(\lambda_t^0) \subset H\} \neq \emptyset\} > 0.$$

We define the explosion time as

$$\begin{aligned} \tau_\triangle &\equiv \mathrm{Inf}\{t \geq 0 : \mathcal{S}(\lambda_t^0) \not\subseteq H\} \\ &= \mathrm{Inf}\{t \geq 0 : x^0(t) \in \triangle\}, \end{aligned}$$

where x^0 denotes the path process corresponding to λ^0. Note that the event $\{\tau_\triangle \rangle t\}$ is $\mathcal{F}_t$ measurable. The mean explosion time is given by the solution of an elliptic problem in Hilbert space with Dirichlet boundary condition as stated below.

Proposition 4.1. *Suppose there exists a $\tilde{\phi} \in D(\mathcal{A})$ such that*

$$\begin{aligned} (\mathcal{A}\tilde{\phi})(x) &= -1, \quad \textit{for } x \in H \\ \tilde{\phi}|_\triangle &= 0. \end{aligned}$$

Then

$$E\tau_\triangle = \int_{H^+} \tilde{\phi}(x)\mu_0(dx).$$

Proof. This is easily verified by the result of Theorem 3.2. Replacing ϕ by $\tilde{\phi}$ and t by the explosion time $\tau_\triangle$ in (3.14) and taking expectation on either side we obtain the result as stated. Here μ_0 is the initial data for the measure equation (3.15).

The explosion time problem can of course be approximated by a sequence of hitting time problems as follows. For $r > 0$, consider the Dirichlet problem:

$$(\mathcal{A}\phi)(x) = -1, \quad x \in B_r$$
$$\phi|_{\partial B_r} = 0,$$

and let $\phi_r \in BC(B_r)$ denote the corresponding solution. Let τ_r denote the first exit time of the path process x^0 from the ball B_r. Then

$$E(\tau_r) \equiv \int_H \phi_r(x)\mu_0(dx),$$

and $E(\tau_\triangle) = \lim_{r\to\infty} E(\tau_r)$.

Remark 4.2. The operator $\mathcal{A}$ has unbounded coefficients. Thus the question of existence of solution of the Dirichlet problem as stated above is very delicate and remains open. For results in this direction on finite dimensional problems with unbounded coefficients see [9, 10].

5. Three illustrative examples

For illustration of Theorem 3.2, we present the following three examples.

Example 1. First we provide a general characterization of drift and dispersion parameters which satisfy our basic assumptions and for which our results obviously hold. Let $H^k \equiv H \times H \times H \times \cdots \times H$ denote the k-fold cartesian product of H and let $\mathcal{L}(H^k, H)$ denote the class of bounded linear operators from H^k to H completed with respect to the norm topology induced by

$$\|L_k\|_{\mathcal{L}(H^k,H)} \equiv \sup\{\|L_k(h_1, h_2, \ldots, h_k)\|_H,\ \|h_i\|_H = 1,\ i = 1, 2, \ldots, k\},$$

where $L_k \in \mathcal{L}(H^k, H)$. For $k = 0$, set $\mathcal{L}(H^0, H) \equiv H$. Define

$$\mathcal{P}_k(x) \equiv L_k(x, x, x, \ldots, x) \quad \text{and}$$
$$\mathcal{P}^m(x) \equiv \sum_{0 \le k \le m} \mathcal{P}_k(x), \quad x \in H.$$

Then we introduce the class

$$\mathcal{F} \equiv \{\mathcal{P}^m,\ m \in \mathbb{N},\ m < \infty\},$$

as the class of admissible drifts. We show that this class satisfies our basic assumptions. Let Q_r denote the retraction of the ball $B_r \subset H$ of radius r centered at the origin. That is

$$Q_r(x) \equiv \begin{cases} x, & \text{for } x \in B_r; \\ (r/\|x\|)x, & \text{otherwise.} \end{cases}$$

For $F \in \mathcal{F}$, by definition there exists an integer $m \in N$ such that $F = \mathcal{P}^m$. Let $\rho(A)$ denote the resolvent set of A and $R(\lambda, A)$ the resolvent corresponding to $\lambda \in \rho(A)$. Define

$$F_n(x) \equiv nR(n, A)F(Q_n(x)) = nR(n, A)\mathcal{P}^m(Q_n(x)), \quad n \in \rho(A).$$

Clearly $\{F_n\}$ is a sequence of continuous and bounded maps in H and for each $x \in H$, $F_n(x) \in D(A)$ and $F_n(x) \longrightarrow F(x)$ point wise in H and hence uniformly on compact subsets of H. It is straight forward to verify that for any fixed $m \in N$, there exist constants $\{\alpha_n = \alpha_n(m) > 0\}$, dependent on m, such that $\lim_n \alpha_n = \infty$ and for each n

$$\|F_n(x)\| \le \alpha_n(1 + \|x\|_H), \qquad \text{for all } x \in H$$
$$\|F_n(x) - F_n(y)\| \le \alpha_n(\|x - y\|_H) \quad \text{for all } x, y \in H.$$

For the diffusion parameters we introduce the set $\mathcal{K}$ a subset of $C(H, \mathcal{L}(E, H))$ to denote the class of locally Lipschitz maps. Let P_n be any increasing sequence of finite dimensional (possibly orthogonal) projections in the Hilbert space E converging strongly to the identity. For each $\sigma \in \mathcal{K}$ define

$$\sigma_n(x) \equiv nR(n, A)\sigma(Q_n x)P_n.$$

It is easy to verify that the sequence $\{\sigma_n\}$ satisfies our basic hypotheses and hence the class $\mathcal{K}$ is covered by our result. In view of this characterization, for each $F \in \mathcal{F}$ and $\sigma \in \mathcal{K}$, the system (3.2)

$$\begin{aligned} &dx = Axdt + F(x)dt + \sigma(x)dW, \\ &x(0) = x_0, \end{aligned}$$

has generalized (measure) solutions but not classical, weak or mild solutions. Note $\sigma\sigma^*$ is not required to be Hilbert–Schmidt.

Example 2. For a more specific example, consider the system (3.2) with F given by $F(x) \equiv \|x\|^{p-1}x$, for any $p > 1$. Clearly F is locally Lipschitz but not dissipative. For $\gamma > 0$, define

$$G_\gamma(x) = \big(\|x\|^{p-1}/(1 + \gamma\|x\|^{p-1})\big)x, \quad \gamma > 0.$$

It is easy to verify that

$$\begin{aligned} \|G_\gamma(x)\| &\le (1/\gamma)\|x\|, \\ \|G_\gamma(x) - G_\gamma(y)\| &\le (p/\gamma)\|x - y\|, \quad \gamma > 0. \end{aligned}$$

Then define

$$F_n(x) = nR(n, A)G_{1/n}(x), \quad n \in N \cap \rho(A).$$

One can easily check that, for $\alpha_n \equiv 2np\|nR(n, A)\|$,

$$\begin{aligned} &\|F_n(x)\| \le \alpha_n(1 + \|x\|) && \text{for all } x \in H \\ &\|F_n(x) - F_n(y)\| \le \alpha_n\|x - y\|, && \text{for all } x, y \in H. \end{aligned}$$

Clearly $\alpha_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Note that $F_n(x) \in D(A)$ for each $n \in \mathbb{N}$ and $F_n \longrightarrow F$ uniformly on compact sets of H. Let σ, given by

$$\sigma(x) = \beta(x)\Gamma,$$

denote the diffusion operator where $\beta \in BC(H)$ and Lipschitz and $\Gamma \in \mathcal{L}(E, H)$ (not necessarily Hilbert–Schmidt). Then the sequence of operators $\{\sigma_n\}$ given by

$$\sigma_n(x) \equiv \beta(x)nR(n, A)\Gamma P_n$$

satisfy all the hypothesis of Theorem 3.2. Indeed one can easily verify the existence of a constant K such that

$$\|\sigma_n(x)\|_{\mathcal{L}_2(E,H)} \leq K\sqrt{n}\|nR(n,A)\|\|\Gamma\|_{\mathcal{L}(E,H)}(1+\|x\|) \leq \beta_n(1+\|x\|)$$
$$\text{and} \quad \|\sigma_n(x)-\sigma_n(y)\|_{\mathcal{L}_2(E,H)} \leq \beta_n(\|x-y\|),$$

and that $\beta_n \longrightarrow \infty$. Further $\sigma_n : H \mapsto \mathcal{L}(E, D(A))$ and $\sigma_n(x) \longrightarrow \sigma(x)$ strongly in $\mathcal{L}(E,H)$ uniformly on compact subsets of H. In other words the sequence $\{\sigma_n\}$ is an Hilbert–Schmidt approximation of σ in $\mathcal{L}(E,H)$.

Thus both F and σ satisfy all the hypothesis of Theorem 3.2, and hence equation (3.2), with these F and σ, has a generalized solution.

Example 3 (Nonlinear Beams). Here we give a more practical example arising from random vibration of structures. To admit moderately large vibrations, the following model has been considered to be more appropriate [12] for beam dynamics:

$$\begin{gathered}\rho\left(\frac{\partial^2 y}{\partial t^2}\right)+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)-N(y)\left(\frac{\partial^2 y}{\partial x^2}\right)+Ky_t=q(t,x),\\ t>0,\ x\in(0,\ell)\\ y(t,0)=0, \quad Dy(t,0)\equiv y_x(t,0)=0\\ EID^2y(t,\ell)=u_1, \qquad D\big(EI(D^2y(t,\ell))\big)-Ny_x(t,\ell)=u_2\end{gathered} \tag{5.1}$$

where, in general, the nonlinear operator N is given by

$$N \equiv a + b\int_0^\ell \left(\frac{\partial y}{\partial x}\right)^2 dx.$$

The nonlinear term represents membrane force. Here a and b are constants. If $a>0$ and $b=0$, it represents a linear extensible beam. The parameters ρ, EI and K denote the mass density (per unit length), flexural rigidity and aero-dynamic damping coefficient respectively. Normally feedback controls of the form

$$\begin{aligned} u_1 &\equiv -\delta D y_t(t,\ell),\\ u_2 &\equiv \gamma y_t(t,\ell) - N(y)Dy(t,\ell)\end{aligned} \tag{5.2}$$

are used to stabilize the system [11, 12]. The term q represents random load. We write the system (5.1)–(5.2) as an abstract (ordinary) differential equation on a Hilbert space. The most suitable space is the energy space given by

$$H \equiv H_0^2 \times L_2(0,\ell),$$

where

$$H_0^2 \equiv \left\{\varphi \in L_2 : \frac{\partial\varphi}{\partial x} \equiv D\varphi \in L_2,\ D^2\varphi \in L_2 \text{ and } \varphi(0)=0,\ D\varphi(0)=0\right\}.$$

Let B denote the formal beam operator given by

$$B\psi \equiv (1/\rho)\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2}{\partial x^2}\psi\right) \equiv (1/\rho)D^2\left(EID^2\psi\right).$$

Define the state as

$$z \equiv \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \equiv \begin{pmatrix} y \\ y_t \end{pmatrix}.$$

Then the system (5.1) can be written as

$$(d/dt)z = Az + \mathcal{F}(z) + \tilde{q} \tag{5.3}$$

where the operator A is given by the restriction of the formal differential operator

$$\mathcal{L} \equiv \begin{pmatrix} 0 & 1 \\ -B & 0 \end{pmatrix},$$

to the domain $D(A)$ given by

$$\begin{aligned} D(A) \equiv \{z \in H : \mathcal{L}z \in H \text{ and } & EID^2 z_1(\ell) + \delta D z_2(\ell) = 0, \\ & D\left(EID^2 z_1(\ell)\right) - \gamma z_2(\ell) = 0\}; \end{aligned}$$

and

$$\tilde{q} \equiv \begin{pmatrix} 0 \\ q \end{pmatrix}.$$

The operator A as defined above is dissipative and generates a contraction semigroup in H.

The nonlinear operator $\mathcal{F}$ is given by

$$\mathcal{F}(z) \equiv -(1/\rho)\begin{pmatrix} 0 \\ N(z_1)D^2 z_1 + K z_2 \end{pmatrix}.$$

Since the modulus of rigidity EI and the mass density ρ are strictly positive, in view of the given boundary conditions the space H is a Hilbert space with respect to the scalar product,

$$\langle \varphi, \psi \rangle \equiv (EID^2\varphi_1, D^2\psi_1) + (\rho\varphi_2, \psi_2),$$

where the first term corresponds to elastic potential energy and the last term to a measure of kinetic energy. For the random load, we assume that $\{e_i,\ i = 1, 2, 3, \dots\}$ is a complete orthonormal basis of $E \equiv L_2(0, \ell)$ and $\{\beta_i(t),\ t \geq 0,\ i = 1, 2, 3, \dots\}$ is a sequence of one dimensional standard Brownian motions. Define

$$W(t) \equiv \sum_{i \geq 1} \sqrt{\lambda_i}\, \beta_i(t) e_i, \quad t \geq 0, \tag{5.4}$$

where the sequence of numbers $\{\lambda_i\}$ are nonnegative. This is an E-valued Wiener process. It is easy to verify that

$$E\{(W(t), f)(W(s), g)\} = (t \wedge s) \sum_{i \geq 1} \lambda_i\, (f, e_i)(g, e_i).$$

For the random load q we choose the model

$$q(t, \xi) \equiv \sum_{i \geq 1} \sqrt{\lambda_i} \dot{\beta}_i(t) e_i(\xi), \quad t \geq 0,\ \xi \in (0, \ell), \tag{5.5}$$

where $\dot{\beta}_i$ is the distributional derivative of the Brownian motion β_i and is called the white noise. If $\{\lambda_i = 1,\ i = 1, 2, 3, \dots\}$ then W is called a cylindrical Brownian motion in E. Using this model for the random load, equation (5.3) can be rigorously interpreted as the Stochastic differential equation given by

$$dz = (Az + F(z))dt + \sigma dW, \tag{5.5}$$

where $\sigma \equiv \binom{0}{I} \in \mathcal{L}(E, H)$ with I denoting the identity operator in E. It is easy to verify that F is Locally Lipschitz and maps bounded sets of H into bounded sets of H. For the approximating sequence we choose $\sigma_n \equiv nR(n, A)\sigma P_n$ and $F_n \equiv nR(n, A)F(Q_n(.))$ which satisfy the basic hypotheses of our Theorem 3.2. Hence for any given initial measure $\mu_0 \in \Pi_{\mathrm{rba}}(H)$, the system (5.5) has generalized solutions in $M^w_\infty(I, L_2(\Omega, \Pi_{\mathrm{rba}}(H^+)))$.

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