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Pell numbers, squares and cubes

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Abstract. We consider the sequence of Pell numbers U_n ($n \geq 0$) and of associated Pell numbers V_n ($n \geq 0$) and we determine the finitely many indices n such that $U_{2n+1} = x^3 \pm 1$, $U_{2n} = x^3 \pm 2$, $V_{2n+1} = x^3 \pm 2$, or $V_{2n} = x^3 \pm 6$. We obtain results about the square classes in these sequences. We also show, among other facts, that for odd n, $U_n \neq \square \pm 1$ (except for $n = 3$), $U_n \neq \square \pm 5$, $V_n \neq \square \pm 2$ (except for $n = 3$), $V_n \neq \square \pm 14$. For even n, we show that $U_n \neq \square \pm 2$, $V_n \neq \square \pm 6$. Concerning cubes, we show for all n that $V_n \neq C \pm 2$ (except for $n = 2$), for odd n, $U_n \neq C \pm 1$ and for n even, $U_n \neq C \pm 2$, $V_n \neq C \pm 1$, $V_n \neq C \pm 6$.

1. Introduction

In this paper we consider the sequences of Pell numbers

$$U_n = 0, 1, 2, 5, 12, 29, 70, 169 \dots$$

and of "associated" Pell numbers

$$V_n = 2, 2, 6, 14, 34, 82 \dots .$$

These are second order linear recurrences with parameters 2, -1.

We are interested in products and certain sums and differences which are squares or cubes. The squares and cubes in these sequences were determined by Ljunggren, respectively Pethő (see [3], [6]).

After a review of the required facts we discuss when $U_m U_n$, $V_m V_n$ are squares. Even though our results are partial, it is possible to deduce many instances when necessarily $m = n$. For the applications, we deal

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also with products U_mV_n and $2U_mU_n$. By considering expressions like $U_{s+2k} \pm (-1)^k U_s$ we tackle the question of determining Pell numbers of the form $x^2 \pm 1$, $x^2 \pm 2$, etc. In the last two sections we consider cubes, investigate the analogous problems obtaining numerous results stated in detail in the text of the article.

It is very convenient to indicate any unspecified square by the symbol $\square$; thus $2\square$ stand for the double of a square. Similarly, an unspecified cube is denoted by C and $3C$ indicates the triple of a cube.

2. Preliminaries

Let P, Q be non-zero integers such that $D = P^2 - 4Q \neq 0$. Let $(U_n)_{n\geq 0}$, $(V_n)_{n\geq 0}$ be the Lucas sequences with parameters (P,Q), which are so defined:

$$\begin{cases} U_0 = 0,\ U_1 = 1, \\ U_n = PU_{n-1} - QU_{n-2} \text{ (for } n \geq 2) \end{cases} \tag{1}$$

$$\text{and} \quad \begin{cases} V_0 = 2,\ V_1 = P, \\ V_n = PV_{n-1} - QV_{n-2} \text{ (for } n \geq 2) \end{cases} \tag{2}$$

for $(P,Q) = (1,-1)$, U is the sequence of Fibonacci numbers and V is the sequence of Lucas.

For $(P,Q) = (2,-1)$, U and V are the sequences of Pell numbers of first, respectively second kind. In this paper we deal exclusively with these sequences:

$$\begin{aligned} &U : 0, 1, 2, 5, 12, 29, 70, 169, 408\ldots \\ &V : 2, 2, 6, 14, 34, 82, 198, 478\ldots \end{aligned}$$

We extend these sequences defining the terms with negative indices as follows:

$$\begin{cases} U_{-n} = -\dfrac{U_n}{(-1)^n} \\[2ex] V_{-n} = \dfrac{V_n}{(-1)^n} \end{cases} \tag{3}$$

With this definition, (1), (2) hold for all integers n. We note that $D = P^2 - 4Q = 8$. The following properties will be used:

$$V_n^2 - 8U_n^2 = 4(-1)^n \tag{4}$$

$$U_{m+n} = U_m V_n - (-1)^n U_{m-n} \tag{5}$$

$$V_{m+n} = V_m V_n - (-1)^n V_{m-n} = 8U_m U_n + (-1)^n V_{m-n} \tag{6}$$

In particular:

$$U_{2n} = U_n V_n \tag{7}$$

$$V_{2n} = V_n^2 - 2(-1)^n = 8U_n^2 + 2(-1)^n \tag{8}$$

and also

$$U_{3n} = U_n \left(V_n^2 - (-1)^n\right) = U_n \left(8U_n^2 + 3(-1)^n\right) \tag{9}$$

$$V_{3n} = V_n \left(V_n^2 - 3(-1)^n\right). \tag{10}$$

More generally:

2.1. *Let $k \geq 3$ be odd. Then there exist uniquely defined polynomials $f_k^+, f_k^- \in \mathbb{Z}[x]$ such that*

$$\deg(f_k^+) = \deg(f_k^-) = \frac{k-1}{2},$$
$$f_k^{\pm}(0) = (\pm 1)^{\frac{k-1}{2}} k$$

and

$$U_{kn} = \begin{cases} U_n f_k^+(U_n^2) & \textit{when } n \textit{ is even} \\ U_n f_k^-(U_n^2) & \textit{when } n \textit{ is odd.} \end{cases}$$

Proof. For $k = 3$ we have $f_3^+ = 8X + 3$ and $f_3^- = 8X - 3$, so the statement is true for $k = 3$ (by (9)). We proceed by induction on k:

$$\begin{aligned} U_{kn} &= U_{(k-2)n} V_{2n} - U_{(k-4)n} \\ &= U_{(k-2)n} \left[8U_n^2 + 2(-1)^n\right] - U_{(k-4)n} \\ &= U_n \left\{ f_{k-2}^{\pm}(U_n^2) \left[8U_n^2 + 2(-1)^n\right] - f_{k-4}^{\pm}(U_n^2) \right\}. \end{aligned}$$

Therefore we define $f_k^{\pm}(x) = (8x \pm 2) f_{k-2}^{\pm}(x) - f_{k-4}^{\pm}(x)$. We have $\deg f_k^{\pm} = \frac{k-1}{2}$, and

$$\begin{aligned} f_k^{\pm}(0) &= \pm 2 f_{k-2}^{\pm}(0) - f_{k-4}^{\pm}(0) \\ &= \pm 2(\pm 1)^{\frac{k-3}{2}}(k-2) - (\pm 1)^{\frac{k-5}{2}}(k-4) = (\pm 1)^{\frac{k-1}{2}} k. \end{aligned}$$

It is immediate to deduce that the above polynomials are unique satisfying the properties indicated. □

2.2. *Let $k \geq 3$ be odd. Then there exist uniquely defined polynomials g_k^+, $g_k^- \in \mathbb{Z}[x]$ such that*

$$\deg(g_k^{\pm}) = \frac{k-1}{2}, \qquad g_k^{\pm}(0) = \pm(-1)^{\frac{k-1}{2}} k$$

and

$$V_{kn} = \begin{cases} V_n g_k^+(V_n^2) & \text{when } n \text{ is even} \\ V_n g_k^-(V_n^2) & \text{when } n \text{ is odd.} \end{cases}$$

Proof. We take $g_3^+(x) = x - 3$ and $g_3^-(x) = x + 3$. We proceed by induction on k:

$$\begin{aligned} V_{kn} &= V_{(k-2)n} V_{2n} - V_{(k-4)n} \\ &= V_{(k-2)n}\left(V_n^2 - 2(-1)^n\right) - V_{(k-4)n} \\ &= V_n^3 g_{k-2}^{\pm}(V_n^2) \mp 2 V_n g_{g-2}^{\pm}(V^2) - V_n g_{k-4}^{\pm}(V_n^2) \\ &= V_n g_k^{\pm}(V_n^2) \end{aligned}$$

where $g_k^{\pm}(X) = X g_{k-2}^{\pm}(X) \mp 2 g_{k-2}^{\pm}(X) - g_{k-4}^{\pm}(X)$. So $\deg(g^{\pm}) = \frac{k-1}{2}$ and

$$\begin{aligned} g_k^{\pm}(0) &= \mp 2 g_{k-2}^{\pm}(0) - g_{k-4}^{\pm}(0) \\ &= \mp 2(-1)^{\frac{k-3}{2}}(k-2) \mp (-1)^{\frac{k-5}{2}}(k-4) \\ &= \pm(-1)^{\frac{k-1}{2}}\left[\pm 2(k-2) \mp (k-4)\right] = \pm(-1)^{\frac{k-1}{2}} k, \end{aligned}$$

as it was required. It is also immediate that the polynomials g_k^+, g_k^- are unique with the properties indicated. □

Concerning divisibility properties we shall require:

$$U_m \mid U_n \quad \text{if and only if } m \mid n$$
$$\text{while } V_m \mid V_n \quad \text{if and only if } m \mid n \text{ and } \frac{n}{m} \text{ is odd.}$$

Next we shall also need: Let $m \geq 1$, $n \geq 1$ and $d = \gcd(m,n)$. Then:

$$\gcd(U_m, U_n) = U_d \tag{11}$$

$$\gcd(V_m, V_n) = \begin{cases} V_d & \text{if } \frac{m}{d}, \frac{n}{d} \text{ are odd} \\ 2 & \text{otherwise} \end{cases} \tag{12}$$

$$\gcd(U_m, V_n) = \begin{cases} V_d & \text{if } \frac{m}{d} \text{ is even, } \frac{n}{d} \text{ is odd} \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases} \tag{13}$$

For each $n \neq 0$, $n = \pm 2^t g$ with g odd, we denote by $\mathrm{val}_2(n) = t$ the 2-adic value of n. We have $\mathrm{val}_2(U_{2^t g}) = \mathrm{val}_2(2^t g) = t$. In particular, U_n is even if and only if n is even. Similarly $\mathrm{val}_2(V_n) = 1$ for every n; in particular, $4 \nmid V_n$.

By considering the sequences U, V modulo 3, we observe that $3 \mid U_n$ if and only if $4 \mid n$ and $3 \mid V_n$ if and only if $n \equiv 2 \pmod 4$.

If p is any odd prime, there exists the smallest integer $\rho(p)$ such that $p \mid U_{\rho(p)}$; moreover, $p \mid U_n$ if and only if $\rho(p) \mid n$. For the sequence V there exist primes, like $p = 5$, such that $p \nmid V_n$ for all n.

We shall investigate powers in connection with Pell sequences. The following basic theorem was proved by Ljunggren [3]:

2.3. *The only solutions in positive integers of the equation* $X^2 - 8Y^4 = -4$ *are* $(2,1)$, $(478,13)$. *The equation* $X^2 - 8Y^4 = 4$ *has no solution in integers. Equivalently,* $U_n = \square$ *if and only if* $n = 1$, 7.

The difficult proof of this statement is omitted.

Concerning cubes, Pethő [6] showed:

2.4. U_n *is not a proper power with exponent, larger than* 2, *for all* n.

The proof of this theorem is also difficult.

The analogous results for the sequence V are:

2.5. V_n *is not a proper power, for all* n.

Proof. If $V_n = x^m$ (for some $m \geq 2$) and $U_n = y$ then $x^{2m} - 8y^2 = \pm 4$. Then x is even, say $x = 2z$ and $2^{2m}z^{2m} - 8y^2 = \pm 4$. Since $m \geq 2$ then 8 divides the left-hand side, which is impossible. $\square$

We shall also need the next two facts.

2.6. $U_n = 2\square$ *if and only if* $n = 2$.

Proof. Let $n > 2$ be the smallest integer such that $U_n = 2\square$ (if one such integer exists). Since $2 \mid U_n$ then $2 \mid n$. Let $n = 2m$, so $2\square = U_n V_m$ with $e = \gcd(U_m, V_m) = 1$ or 2. Thus we have:

$$\begin{cases} U_m = \square \\ V_m = \square \end{cases} \qquad \text{or} \qquad \begin{cases} = 2\square \\ = 2\square. \end{cases}$$

The first case is impossible by 2.5.

In the second case, by the minimality of n, we have $m = 2$, so $n = 4$. However, $U_4 = 12 \neq 2\square$. $\square$

The next result may be found in Sierpiński's book [10]:

2.7. *If* $V_n = 2\square$ *then* $n = 1$.

Proof. Let n be even, $V_n = 2v^2$, $U_n = u$, so $4v^4 - 8u^2 = 4$ hence $v^4 - 2u^2 = 1$. So v is odd, hence $v^2 = 8k + 1$ hence $8k(4k + 1) = y^2$. Since $\gcd(2k, 4k + 1) = 1$ then $8k = a^2$ and $v^2 = 9a^2 + 1$ which is impossible.

Let n be odd, let $V_n = 2v^2$, $U_n = u$, so $4v^2 - 8u^2 = -4$, hence $v^4 - 2u^2 = -1$ and therefore $u^4 - v^4 = (u^2 - 1)^2$. By the classical result of Fermat, $u = v = 1$, so $n = 1$.

3. Square classes

Let S be a set of positive integers. We say that $s_1, s_2 \in S$ are *square equivalent* if there exist non-zero integers x_1, x_2 such that $s_1 x_1^2 = s_2 x_2^2$. The equivalence classes are called the *square classes* of S. It is clear that s_1, s_2 are square equivalent if and only if $s_1 s_2 = \square$. When $1 \in S$ the square class of 1 consists of all the squares in S. A square class with only on element is said to be trivial.

In this section we give some results about the square classes of the sequences U, V of Pell numbers.

In [8] it was shown that if S is any Lucas sequence with positive discriminant, each square class of S is finite and its terms are effectively computable.

The determination of the square classes of the sequences U and V is difficult. We obtain here only very partial results and we illustrate with the determination of some special cases.

As already mentioned, $U_1 = 1$, $U_7 = 169$ are the only squares in the sequence U. Then $\{U_1, U_7\}$ is a square class. It is not known if every other square class of U is trivial.

We may prove:

3.1. *Let $m = 2^e g$, $n = 2^f h$ be distinct non-zeo integers, with $e, f \geq 0$, g, h odd. Let $d = \gcd(g,h)$ and $g = dr$, $h = ds$. Let $U_m U_n = \square$. Then*

a) $e = f$.

b) $U_g U_h = \square$, $V_g V_h = \square, \ldots, V_{2^{e-1}g} V_{2^{e-1}h} = \square$.

c) If p is a prime dividing $\gcd(U_d, r)$ (respectively $\gcd(U_d, s)$) then $\mathrm{val}_p(r) \neq 1$ (respectively $\mathrm{val}_p(s) \neq 1$).

Proof. a) We assume for example that $0 \leq f < e$, then $\square = U_m U_n = U_g V_g V_{2g} \cdots V_{2^{e-1}g} U_{2^f h}$. We have $\gcd(V_{2^{e-1}g},\ U_g V_g \cdots V_{2^{e-2}g} U_{2^f h}) = 1$ or a power of 2. Then $V_{2^{e-1}g} = \square$ or $2\square$. The first case is impossible by (2.5), while the second case, $2^{e-1}g = 1$ by (2.7). Then $e = 1$, $g = 1$, $f = 0$, so $\square = U_m U_n = 2U_h$ with h odd, U_h is odd. This is impossible. Therefore $e = f$.

b) From $\square = U_m U_n = U_g V_{2g} \cdots V_{2^{e-1}g} \cdot U_h V_h \cdots V_{2^{e-1}h}$ and $\gcd(U_g U_h,$ $V_g \cdots V_{2^{e-1}g} V_h \cdots V_{2^{e-1}h}) = 1$ then $U_g U_h = \square$. Also $V_g \cdots V_{2^{e-1}g} V_h \cdots V_{2^{e-1}h} = \square$. But $\gcd(V_{2^{e-1}g} V_{2^{e-1}h}, V_g \cdots V_{2^{e-2}g} V_h \cdots V_{2^{e-1}h})$ is a power of 2, so

$$\begin{cases} V_{2^{e-1}g} V_{2^{e-1}h} = \square \\ V_g \cdots V_{2^{e-2}g} V_h \cdots V_{2^{e-2}h} = \square \end{cases} \quad \text{or} \quad \begin{cases} = 2\square \\ = 2\square. \end{cases}$$

The second case is impossible, because $4 \nmid V_g$, $4 \nmid V_h$. So $V_{2^{e-1}g} V_{2^{f-1}h} = \square$ and the argument may be repeated leading to the stated conclusion.

c) We have $U_d = \gcd(U_g, U_h)$, so

$$\frac{U_g}{U_d} \cdot \frac{U_h}{U_d} = \square,$$

hence $U_d U_g = U_d U_{dr} = \square$ and also $U_d U_h = U_d U_{ds} = \square$. $\square$

By 2.1, $U_d^2 f_r^-(U_d^2) = \Box$, so $f_r^-\left(U_d^2\right) = \Box$. The constant term of $f_r^- \in \mathbb{Z}[X]$ is $\pm r$. If $p \mid \gcd(U_d, r)$ and $v_p(r) = 1$ since $p^2 \mid U_d^2$ then $v_p(f_r^-(U_d^2)) = 1$, which is impossible because $f_q^-(U_d^2) = \Box$.

We also prove:

3.2. *If g, h are odd and $U_g U_h = \Box$ then $g \equiv \pm h \mod 8$ and $\mathrm{val}_2(g-h) \neq 2$.*

Proof. We consider the sequence U modulo 8,

$$U/8 : 1\ 2\ 5\ 4\ 5\ 6\ 1\ 0 \cdots$$

So $U_g U_h = \Box$ implies that $g \equiv \pm h \mod 8$. It follows that $\mathrm{val}_2(g-h) \neq 2$. $\Box$

We use particular arguments to determine certain square classes. For the next two results, see also Robbins [9]:

3.3. *Let*

$$S = \{3\Box, 5\Box, 6\Box, 10\Box, 15\Box, 30\Box\}.$$

Then $U_n \in S$ if and only if $n = 3, 4$. In particular, the square class of $U_3 = 5$ is trivial.

Proof. $U_1, U_2 \notin S$, $U_3 = 5$. Assume that there exists the smallest $n > 3$ such that $U_n \in S$.

First case. $5 \mid U_n$ then $3 \mid n$. Let $n{=}3m$, so $U_n{=}U_m\left(8U_m^2 + 3(-1)^m\right)$. Let

$$d = \gcd\left(U_m, 8U_m^2 + 3(-1)^m\right),$$

hence $d = 1$ or 3. If $d = 1$ then $U_m \in \{\Box, 2\Box\} \cup S$. If $U_m = \Box, 2\Box$ then $m = 1, 7, 2$ so $n = 3, 21, 6$. But $U_{21}, U_6 \notin S$ as verified by calculation (note that $4 \nmid 21$ so $3 \nmid U_{21}$). If $U_m \in S$ by minimality of n then $m = 3$, so $n = 9$, however $U_9 \notin S$ as seen by calculation. If $d = 3$ then

$$\frac{U_m}{3} \cdot \frac{8U_m^2 + 3(-1)^m}{3} \in S,$$

hence $U_m \in \{\Box, 2\Box\} \cup S$. We note that $3 \mid U_m$ implies that $4 \mid m$. If $U_m \in S$ by the minimality, $m \leq 3$, which is impossible; similarly $U_m \neq \Box, 2\Box$.

Second case. $5 \nmid U_m$. Then $3 \mid U_n$ so $4 \mid n$. Let $n = 2k$ with k even. So $U_k V_k \in S$, with $e = \gcd(U_k, V_k) = 2$, since k is even. Thus $\frac{U_k}{2}\frac{V_k}{2} \in S$

and therefore $U_k \in \{\Box, 2\Box\} \cup S$. Since k is even and by the minimality of n, $k = 2$ and $n = 4$. $\Box$

3.4. *The square class of U_5 is trivial, or equivalently if $U_n = 29\Box$ then $n = 5$.*

Proof. Let n be minimal such that $U_n = 29\Box$. By 3.1, n is odd. Since $29 = U_5$ divides U_n then $5 \mid n$. Let $n = 5m$, so by 2.1 $U_n = U_m f_5(U_m^2)$ where $f_5 \in \mathbb{Z}[x]$ with constant term ± 5. Then $d = \gcd\left(U_m, f_5(U_m^2)\right) = 1$ or 5. If $d = 1$ then $U_m = \Box$ or $29\Box$. The second case is not possible. By the minimality of n. In the first case by Ljunggren's result, $m = 1$ or 7, so n =5 or 35. But, we see by direct calculation that $U_{35}/29 \neq \Box$.

If $d = 5$ then $3 \mid m$ so $3 \mid n$; let $n = 3k$ hence $U_n = U_k\left(8U_k^2 - 3\right)$. Since k is odd then $3 \nmid U_k$ so the above factors are coprime. Thus $U_k = \Box$ or $29\Box$. The second case is excluded by the minimality of n; by Ljunggren's result, $n = 3$ or 21. However, $5 \nmid 3$, $5 \nmid 21$, so $29 \nmid U_3, U_{21}$. $\Box$

3.5. *Let*

$$S = \{3\Box, 6\Box, 197\Box, 2 \times 197\Box, 5 \times 197\Box, 10 \times 197\Box, 3 \times 197\Box,$$
$$6 \times 197\Box, 15 \times 197\Box, 30 \times 197\Box\}.$$

Then $U_n \in S$ if and only if $n = 4$ or 9. In particular, the square class of $U_9 = 5 \times 197$ is trivial.

Proof. If $n \leq 9$ and $U_n \in S$ then $n = 4, 9$. Let $n > 9$ be the smallest index such that $U_n \in S$.

First case. $197 \mid U_n$. Then $9 \mid n$. Let $n = 3m$, so $U_n = U_m\left(8U_n^2 + 3(-1)^m\right)$. Let

$$d = \gcd\left(U_m, 8U_m^2 + 3(-1)^m\right)$$

so $d = 1$ or 3. If $d = 1$ then $U_m = S \cup \{\Box, 2\Box, 5\Box, 10\Box\}$. By minimality of n, if $U_m \in S$ then $m = 9$ so $n = 27$. However, $U_{27} \notin S$, which may be verified by direct calculation. If $U_m \in \{\Box, 2\Box, 5\Box, 10\Box\}$ then by the previous results, $m = 1, 7, 2, 3$ hence $n = 3, 21, 6, 9$; only 21 is not excluded, but $U_{21} \notin S$, as seen by direct calculation. If $d = 3$ then $3 \mid U_m$ so $4 \mid m$ and

$$\frac{U_m}{3} \cdot \frac{8U_m^2 + 3(-1)^m}{3} \in S$$

so $U_m \in S \cup \{\Box, 2\Box, 5\Box, 10\Box\}$. If $U_m \in S$ then by minimality of n, $m = 9$, so $n = 27$, however $U_{27} \notin S$, as seen by calculation. If $U_m \notin S$ then by the previous results $m = 4$, so $n = 12$, however $U_{12} \notin S$ (since $9 \nmid 12$ then $197 \nmid U_{12}$).

Second case. $197 \nmid U_n$. Then $U_n \in \{3\Box, 6\Box\}$ so $4 \mid n$. Let $n = 2k$ with k even. Then $U_k V_k \in \{3\Box, 6\Box\}$ with $\gcd(U_k, V_k) = 2$. So $U_k \in \{\Box, 2\Box, 3\Box, 6\Box\}$. By the minimality of n and k even then $k = 2$, $n = 4$, which was found already as a possibility. $\Box$

Concerning the sequence V, me may prove:

3.6. *Let m, n be distinct integers such that $V_m V_n = \Box$, let $d = \gcd(m, n) = 1$. Then:*

a) $\dfrac{m}{d}, \dfrac{n}{d}$ *are odd,*

b) $m \equiv n \pmod 8$,

c) *for any primep* $\mid V_d$, *both* $\operatorname{val}_p\left(\dfrac{m}{d}\right) \neq 1$ *and* $\operatorname{val}_p\left(\dfrac{n}{d}\right) \neq 1$

Proof. a) Let $V_m V_n = \Box$ and $e = \gcd(V_m, V_n)$. If $e = 2$ then $V_m = 2\Box$, $V_n = 2\Box$, so $m = n = 1$, which is contrary to the hypothesis. Thus $e = V_d$, where $d = \gcd(m, n)$ and $\frac{m}{d}$, $\frac{n}{d}$ are odd.

b) If d is even then m, n are even, while if d is odd, then m, n are odd. Considering the sequence $\frac{1}{2}V$ modulo 4, namely

$$1,\ 3,\ 3,\ 1,\ 1,\ 3,\ 3,\ 1\ldots$$

if $V_m V_n = \Box$ and if m, n are odd then $m \equiv n \pmod 4$; similarly if m, n are even, then $m \equiv n \pmod 4$. Let $n = m + 4t$. If t is odd, from $V_n = V_{m+2t} V_{2t} - V_m$ then

$$V_n V_m = -V_m^2 \left(\bmod \frac{1}{2} V_2 = 3\right)$$

so $\left(\frac{-1}{3}\right) = +1$, which is absurd. This shows that $m \equiv n \pmod 8$.

c) We have also $V_m V_d = \Box$ and by 2.2

$$V_m = V_d g^{\pm}_{m/d}\left(V_d^2\right) \quad \text{where } g^{\pm}_{m/d} \in \mathbb{Z}[x],$$

$g^{\pm}_{m/d}$ have degree $(\frac{m}{d}-1)/2$ and constant term $\pm m/d$. So $g^{\pm}_{m/d}(V_d^2) = \square$. If p is any prime dividing V_d, if $\mathrm{val}_p(\frac{m}{d}) = 1$ then $g^{\pm}_{m/2} \neq \square$; thus $\mathrm{val}_p(\frac{m}{p}) \neq 1$ and $\mathrm{val}_p(\frac{n}{p}) \neq 1$. We note in passing that either $\mathrm{val}_p(\frac{m}{d})$ or $\mathrm{val}_p(\frac{n}{d})$ is 0. $\square$

3.7. *The square class of $V_3 = 14$ is trivial.*

Proof. Let $n > 3$ be the smallest index such that $V_n = 14\square$. By considering the sequence V modulo 14, we deduce that $n \equiv 3 \pmod 6$ so $n = 6m-3$ and $V_n = V_{2m-1}(V_{2m-1}^2 + 3)$. Let $d = \gcd(V_{2m-1}, V_{2m-1}^2 + 3)$ so $d = 1$ or 3. But $3 \mid V_k$ if and only if $k \equiv 2 \pmod 4$. Thus $d = 1$. From $V_n = 14\square$ it follows that $V_{2m-1} = 14\square$ (impossible by the minimal choice of n) or $V_{2m-1} = 7\square$ (impossible since $4 \nmid V_{2m-1}$), or $V_{2m-1} = 2\square$ (this implies that $2m-1 = 1$, so $m = 1$, $n = 3$, contrary to the assumption) or $V_{2m-1} = \square$ (impossible). $\square$

3.8. *The square class of $V_5 = 82$ is trivial.*

Proof. By considering the sequence $\frac{1}{2}V$ modulo 41, we observe that $41 \mid V_k$ if and only if $k \equiv 5 \pmod{10}$. Thus if n is the smallest integer $n > 5$, such that $V_n = 82\square$ we have $n = 5m$. Thus $V_n = V_m g_5^{\pm}(V_m^2)$, where $g_5^{\pm} \in \mathbb{Z}[x]$ with constant term ± 5. Thus $d = \gcd\left(V_m, g_5^{\pm}(V_m^2)\right)$ is 1 or 5. However, by considering the sequence $\frac{1}{2}V$ modulo 5, we note that $5 \nmid V_k$ for all k. So $d = 1$, hence $V_m = 82\square$, which implies that $m = 5$ so $n = 25$; or $V_m = 41\square$ (impossible since $4 \nmid V_m$), $V_m = 2\square$ (so $m = 1$ and $n = 5$, which is contrary to the assumption), or $V_m = \square$ (impossible). Finally, by direct numerical computation, we verify that $V_{25} \neq 82\square$, concluding the proof that the square class of V_5 is trivial. $\square$

We shall require the explicit determination of the square class of V_7.

3.9. *The square class of $V_7 = 2 \times 239$ is trivial.*

Proof. We assume that there exists the smallest $n > 7$ such that

$$V_n \in \{2 \times 239\square, 2 \times 239 \times 7\square\}.$$

If $V_n = 478\square$ then by 3.6 $n \equiv 7 \pmod 8$, so n is odd. Similarly, if $V_n = 7 \times 478\square$ since $7 \mid V_n$ then $n \equiv 3$ or $9 \pmod{12}$, so n is odd also in this case.

Since $239 \mid V_n$ then $7 \mid n$. Let $n = 7m$ hence m is odd and by 2.2 $V_n = V_m g_7^{\pm}(V_m^2)$ where $g_7^{\pm} \in \mathbb{Z}[x]$ with constant term ± 7. Let $d = \gcd\left(V_m, g_7^{\pm}(V_n^2)\right)$, so $d = 1$ or 7.

If $d = 1$ then $V_m \in \{\square,\ 2\square,\ 7\square,\ 239\square,\ 2\times 7\square,\ 2\times 239\square,\ 7\times 239\square,\ 2\times 7\times 239\square\}$. First we note that $V_m \notin \{\square, 7\square, 239\square, 7\times 239\square\}$ because $4 \nmid V_m$. Also by minimality of n, $V_m \notin \{2\times 239\square, 2\times 7\times 239\square\}$. If $V_m = 14\square$ by 3.7 $m = 3$ hence $n = 21$, however $V_{21} \notin \{2\times 239\square, 2\times 7\times 239\square\}$, as seen by direct calculation. Finally, if $V_m = 2\square$ then $m = 1$ so $n = 7$.

Now let $d = 7$ so $7 \mid V_m$ and $V_m/7$ is a factor of $2\times 239\square$ or $2\times 7\times 239\square$, that is $V_m \in \{\square,\ 2\square,\ 7\square,\ 239\square,\ 14\square,\ 2\times 239\square,\ 7\times 239\square,\ 2\times 7\times 239\square\}$, hence we are in the preceding situation, leading to $n = 7$. $\square$

We shall also require the following result:

3.10. *Let $1 \leq m < n$ and assume that $U_m U_n = 2\square$. Then $(m,n) = (1,2)$ or $(2,7)$.*

Proof. It is clear that $U_1U_2 = 2\square$, $U_2U_7 = 2\square$. Now let $0 \leq e \leq f, g, h$ odd and $U_{2^e g} U_{2^f h} = 2\square$. We have $\mathrm{val}_2(U_{2^e g}) = e$, $\mathrm{val}_2(U_{2^f h}) = f$ so $e + f$ is odd. So $0 \leq e < f$. If $e = 0$ then

$$U_g U_{2^f h} = U_g U_h V_h \cdots V_{2^{f-1}h} = 2\square.$$

We have $\gcd\left(U_g U_h, V_h \cdots V_{2^{f-1}h}\right) = 1$. Since f is odd then $V_h \cdots V_{2^{f-1}h} = 2\square$. But $\gcd\left(V_h \cdots\ V_{2^{f-2}h}, V_{2^{f-1}h}\right) = 2$ so $V_{2^{f-1}h} = 2\square$, thus $2^{f-1}h = 1$ and $U_{2^f h} = U_2 = 2$. Thus $2U_g = 2\square$ so $g = 1$ or 7.

We show that if $1 \leq e$ then $U_{2^e g} U_{2^f h} \neq 2\square$. Assuming the contrary, let $1 < e$ be smallest such that $U_{2^e g} U_{2^f h} = 2\square$ (for g, h odd and $e < f$). Then

$$U_{2^{e-1}g} V_{2^{e-1}g} U_{2^{f-1}h} V_{2^{f-1}h} = 2\square$$

with

$$\gcd(U_{2^{e-1}g} U_{2^{f-1}h}, V_{2^{e-1}g} V_{2^{f-1}h}) = 1, 2, \text{ or } 4.$$

Then

$$\begin{cases} U_{2^{e-1}g} U_{2^{f-1}h} = \square \\ V_{2^{e-1}g} V_{2^{f-1}h} = 2\square, \end{cases} \quad \text{or} \quad \begin{cases} = 2\square \\ = \square. \end{cases}$$

Since $e + f$ is odd, the first case is impossible. In the second case, by the minimality of e, we have $e = 1$ so $U_g U_{2^{f-1}h} = 2\square$. By the preceding proof

$2^{f-1}h = 2$ so $f = e = 1$, and this is a contradiction, proving the statement. $\square$

Now we shall determine when a product $U_m\ V_n$ is a square.

3.11. *If* m, n *are non-zero integers and* $U_m V_n = \square$ *then* $(m,n) = (2,1)$ *or* $(14,7)$.

PROOF. Let $U_m V_n = \square$ and $e = \gcd(U_m, V_n)$ so $e = 1, 2$ or V_d where $d = \gcd(m,n)$ and m/d is even.

If $e = 1$ then $U_m = \square$ and $V_n = \square$, which is impossible. If $e = 2$ then $U_m = 2\square$ and $V_n = 2\square$ so $m = 2$, $n = 1$.

Let $e = V_d$ with m/d even, hence n/d is odd. Then m is even and we write $m = 2^f g$ with $f \geq 1$, g odd, hence $d = 2^l h$ with $0 \leq l < f$, h odd and h divides g. Now $U_m V_d = \square$ and we have

$$U_g\ V_g\ V_{2g}\ \cdots\ V_{2^{f-1}g}\ V_{2^l h} = \square.$$

Since $\gcd(U_g, V_{2^i g}) = 1$, $\gcd(U_g, V_{2^l h}) = 1$ then $U_g = \square$, hence $g = 1$ or 7, and $V_g \cdots V_{2^{f-1}g}\ V_{2^l h} = \square$. But $\gcd(V_{2^i g}, V_{2^j g}) = 2$ (for $i < j$), $\gcd(V_{2^i g}, V_{2^l h}) = 2$ (for $i \neq l$) and $\gcd(V_{2^l g}, V_{2^l h}) = V_{2^l h}$, then $V_{2^l g}, V_{2^l h} = \square$ or $2\square$ and

$$\frac{V_g \cdots V_{2^{f-1}g}}{V_{2^l g}} = \square \text{ or } 2\square.$$

The second case cannot happen.

1) Let $f > 1$. If $i \neq l$, $0 \leq i \leq f-1$ then $V_{2^i g} = \square$ or $2\square$, so $V_{2^i g} = 2\square$ hence $2^i g = 1$, so $i = 0, g = 1, f = 2, l = 1, h = 1$. Thus $m = 4, d = 2$ but $U_4 V_2 = 12 \times 6 \neq \square$.

2) Let $f = 1$ then $l = 0$. If $g = h = 1$ then $m = 2$. If $U_m V_n = \square$ then $V_n = 2\square$ so $n = 1$. If $g = 7$ and $h = 1$ then $V_7 V_1 = \square$ so $V_7 = 2\square$, which is absurd. If $g = h = 7$ then $m = 14$, $d = 7$ so n is odd. From $U_{14} V_n = U_7 V_7 V_n = \square$, $U_7 = \square$ then $V_7 V_n = \square$. By 3.9 $n = 7$, $m = 14$. $\square$

Using the fundamental relation $V_n^2 - 8U_n^2 = \pm 4$ or equivalently $v^2 - 2U_n^2 = \pm 1$ (where $v = \frac{1}{2}V_n$), we may apply the above result to elliptic curves. Thus we obtain: The only solutions in positive integers of the following equations are the ones indicated:

$X^2 - 2Y^4 = 1$ No solution

$X^2 - 2Y^4 = -1$	$(x,y) = (1,1), (239,7)$
$X^2 - 4Y^4 = 1$	$(x,y) = (3,2)$
$X^2 - 4Y^4 = -1$	No solution
$X^2 - 50Y^4 = 1$	No solution
$X^2 - 50Y^4 = -1$	$(x,y) = (7,1)$
$X^2 - 288Y^4 = 1$	$(x,y) = (17,1)$
$X^2 - 288Y^4 = -1$	No solution
$X^2 - 1682Y^4 = 1$	No solution
$X^2 - 1682Y^4 = -1$	$(x,y) = (41,1)$
$X^2 - 9800Y^4 = 1$	$(x,y) = (99,1)$
$X^2 - 9800Y^4 = -1$	No solution

and also

$X^4 - 2Y^2 = 1$	No solution
$X^4 - 2Y^2 = -1$	$(x,y) = (1,1)$
$9X^4 - 2Y^2 = 1$	$(x,y) = (1,2)$ is the only known solution
$9X^4 - 2Y^2 = -1$	No solution
$49X^4 - 2Y^2 = 1$	No solution
$49X^4 - 2Y^2 = -1$	$(x,y) = (1,5)$
$289X^4 - 2Y^2 = 1$	$(x,y) = (1,12)$ is the only known solution
$289X^4 - 2Y^2 = -1$	No solution
$1681X^4 - 2Y^2 = 1$	No solution
$1681X^4 - 2Y^2 = -1$	$(x,y) = (1,29)$
$9801X^4 - 2Y^2 = 1$	$(x,y) = (1,70)$ is the only known solution
$9801X^4 - 2Y^2 = -1$	No solution

$57121X^4 - 2Y^2 = 1$ No solution

$57121X^4 - 2Y^2 = -1$ $(x, y) = (1, 169)$

4. Sums or differences which are squares

We shall determine, wherever possible, the indices s, k such that the expressions below are squares:

$$U_{s+2k} \pm (-1)^k U_s = \square$$
$$V_{s+2k} \pm (-1)^k V_s = \square$$

4.1. *Let* $s \geq 1$, $k \geq 1$. *Then*

$$U_{s+2k} - (-1)^k U_s \neq \square. \tag{14}$$

Proof. Let $s \geq 1$, $k \geq 1$ be such that (14) does not hold. Then

$$\square = U_{s+2k} - (-1)^k U_s = U_k V_{s+k}$$

By 3.11 $(k,\, s+k) = $ (2,1) or (14,7) which is impossible. $\square$

4.2. *Let* $s \geq 1$, $k \geq 1$. *Then*

$$U_{s+2k} + (-1)^k U_s = \square \tag{15}$$

if and only if $(s, k) = (1, 1)$, *or* $(7, 7)$.

Proof. Let $s \geq 1$, $k \geq 1$ be such that (15) holds. Then

$$\square = U_{s+2k} + (-1)^k U_s = U_{s+k} V_k$$

It follows from 3.11 that $(s + k, k) = (2, 1)$ or $(14, 7)$ hence $(s, k) = (1, 1)$ or $(7, 7)$. Both solutions satisfy (15). $\square$

As particular cases of 4.1, 4.2, with $s = 1, 2, 3$, we deduce: If m is odd then $U_m \neq \square \pm 1$ (except $U_3 = \square + 1$) and $U_m \neq \square \pm 5$. If m is even, then $U_m \neq \square \pm 2$.

4.3. *Let $s \geq 1$, $k \geq 1$. Then*

$$V_{s+2k} - (-1)^k V_s = \square \tag{16}$$

if and only if $(s,k) = (1,1), (5,2)$.

Proof. It is clear that if $(s,k) = (1,1)$ or $(5,2)$ then (16) holds. Conversely if $\square = V_{s+2k} - (-1)^k V_s = 8U_{s+k}U_k$, hence $U_{s+k}U_k = 2\square$. By 3.10, $(k, s+k) = (1,2)$ or $(2,7)$, hence $(s,k) = (1,1)$ or $(5,2)$. $\square$

4.4. *Let $s \geq 1$, $k \geq 1$ be integers such that*

$$V_{s+2k} + (-1)^k V_s = \square \tag{17}$$

Then the square classes of V_k and V_{s+k} are not trivial and $8 \mid s$. Moreover, if $d = \gcd(s,k)$ then k/d is odd, s/d is even. Also, if p is any prime dividing V_d then $\mathrm{val}_p(\frac{k}{d}) \neq 1$ and $\mathrm{val}_p(\frac{s+k}{d}) \neq 1$.

Proof. If (17) holds then $\square = V_{s+2k} + (-1)^k V_s = V_k V_{s+k}$ so the square classes of V_k, V_{s+k} are not trivial and by 3.6 we deduce that $8 \mid s$ and that if $d = \gcd(s,k) = \gcd(s+k,k)$ then k/d, $(s+d)/d$ are odd. So s/d is even. Moreover if $p \mid V_d$ then $\mathrm{val}_p(\frac{k}{d}) \neq 1$, $\mathrm{val}_p(\frac{s+k}{d}) \neq 1$. $\square$

Combining 4.3 and 4.4 we deduce as particular cases ($s = 1, 2, 3$): If m is odd then $V_m \neq \square \pm 2$, (except $V_3 = \square - 2$) and also $V_m \neq \square \pm 14$. If m is even then $V_m \neq \square \pm 6$.

5. Pell sequences and cubes

In this section we treat problems similar to the ones of the preceding sections, but concerned with cubes. Like $\square$ designated an arbitrary square, we shall denote an arbitrary cube by the letter C. In this connection we quote the following fundamental result of Pethő [6]:

5.1. *If U_n is a cube, then $n = 1$.*

The proof involves Baker's bounds for linear forms in logarithms.

As it was indicated in 2.5, V_m is not a cube, for all n.

We shall need:

5.2. *If $U_n = 2C$ or $4C$ then $n = 2$.*

Proof. Assume that there exists the smallest $n > 2$ such that $U_n \in \{2C, 4C\}$. Then n is even, $n = 2m$, and $U_m V_m \in \{2C, 4C\}$. Let $\gcd(U_m, V_m) = d$, so $d = 1$ or 2. If $d = 1$ then $U_m \in \{2C, 4C\}$ with $m < n$, so $m = 2$ and $n = 4$, but $U_4 = 12 \notin \{2C, 4C\}$. If $d = 2$ then $\frac{U_m}{2}\frac{V_m}{2} \in \{2C, 4C\}$, so $U_m \in \{C, 2C, 4C\}$; since $U_m \neq C$, $m < n$. Then $m = 2$, but $U_4 \neq 2C, 4C$. □

5.3. *If* $U_n \in \{3C, 6C, 12C\}$ *then* $n = 4$.

Proof. Assume that there exists the smallest $n > 4$ such that

$$U_n \in \{3C, 6C, 12C\}.$$

Since $3 \mid U_n$ then $4 \mid n$ so $n = 4m$ and

$$U_{2m}V_{2m} \in \{3C, 6C, 12C\}.$$

We have $\gcd(U_{2m}, V_{2m}) = 2$, so

$$\frac{U_{2m}}{2}\frac{V_{2m}}{2} \in \{3C, 6C, 12C\}.$$

Therefore, $U_{\frac{2m}{2}} \in \{C, 2C, 3C, 4C, 6C, 12C\}$ and $U_{2m} \in \{C, 2C, 3C, 4C, 6C, 12C\}$. By the minimality of n and 5.2 this implies that $2m = 2$ so $n = 4$. □

5.4. $U_n \notin \{9C, 18C, 36C\}$ *for all* n.

Proof. Let n be the smallest integer such that $U_n \in \{9C, 18C, 36C\}$. Since $3 \mid U_n$ then $4 \mid n$. Let $n = 4m$ so $U_{2m}V_{2m} \in \{9C, 18C, 36C\}$. Since $\gcd(U_{2m}, V_{2m}) = 2$ then $\frac{U_{2m}}{2}\frac{V_{2m}}{2} \in \{9C, 18C, 36C\}$. So $U_{2m} \in \{C, 2C, 4C, 9C, 18C, 36C\}$. By the minimality of n and (5.2) $U_{2m} = 2C$ so $2m = 2$, $n = 4$. However, $U_4 = 12 \neq 9C, 18C, 36C$. □

As a further example of the method, we show:

5.5. *If* $U_n \in \{5C, 10C, 20C, 15C, 30C, 60C, 45C, 90C, 180C\}$ *then* $n = 3$.

Proof. Let $n > 3$ be the smallest index such that

$$U_n \in \{5C, 10C, 20C, 15C, 30C, 60C, 45C, 90C, 180C\}.$$

Since $5 \mid U_n$ then $3 \mid n$. Let $n = 3m$, so $U_n = U_m\left(8U_m^2 + 3(-1)^m\right)$ with $d = \gcd\left(U_m, 8U_m + 3(-1)^m\right) = 1$ or 3.

If $d = 1$ then $U_m \in \{C, 5C, 2C, 10C, 4C, 20C, 3C, 15C, 6C, 30C, 12C, 60C, 9C, 45C, 18C, 90C, 36C, 180C\}$. By the previous results and the minimality of n, we have $m = 2$, 3 or 4. Hence $n = 6, 9$ or 12; however, U_6, U_9, U_{12} are not of the form under consideration.

If $d = 3$ then

$$\frac{U_m}{3} \cdot \frac{8U_m^2 + 3(-1)^m}{3} \in \{15C, 30C, 60C, 45C, 90C, 180C, 5C, 10C, 20C\},$$

so $\frac{U_m}{3} \in \{C, 3C, 5C, 15C, 2C, 6C, 10C, 30C, 4C, 12C, 20C, 60C, 9C, 45C, 18C, 90C, 36C, 180C\}$ and $U_n \in \{3C, 9C, 15C, 45C, 6C, 18C, 30C, 90C, 12C, 36C, 60C, 180C, C, 5C, 2C, 10C, 4C, 20C\}$. By the preceding results and the minimality of n, this is only possible when $m = 2$, so $n = 6$. However, $U_6 = 70$ is not of the required form. $\square$

The above results may be translated in terms of elliptic curves, by using the fundamental relation

$$V_n^2 - 8U_n^2 = \pm 4$$

hence

$$v_n^2 - 2U_n^2 = \pm 1$$

where $v_n = \frac{1}{2}V_n$.

Thus for example 5.1 and 5.2:

5.1'. The equation $x^2 - 2y^6 = \pm 1$ has no solutions in integers.

5.2'. The only solution in positive integers of $x^2 - 8y^6 = 1$ is $(x, y) = (3, 1)$. The equations $x^2 - 8y^6 = -1$, and $x^2 - 32y^6 = \pm 1$ have no solution in integers.

We shall require the following fact:

5.6. *Let $m \geq 1$, $n \geq 1$ with* $\gcd(m, n) = 1$ *or* 2. *Then* $U_m V_n \neq C$.

Proof. Assume that $U_m V_n = C$ and let $e = \gcd(U_m, V_n)$. If $e = 1$ then $U_m = C$, $V_n = C$ which is impossible. If $e = 2$ then $\frac{U_m}{2}\frac{V_n}{2} = 2C$ so

$$\begin{cases} U_m/2 = 2C \\ V_n/2 = C \end{cases} \quad \text{or} \quad \begin{cases} = C \\ = 2C \end{cases}$$

hence

$$\begin{cases} U_m = 4C \\ V_n = 2C \end{cases} \quad \text{or} \quad \begin{cases} = 2C \\ = 4C. \end{cases}$$

By 5.2 the first case is not possible, while the second case is impossible since $4 \nmid V_n$. Let $d = \gcd(m,n)$ and assume that m/d is even, so n/d is odd. Then $V_d = \gcd(U_m, V_n)$. If $d = 1$ then $V_d = 2$ and this case was already considered. If $d = 2$ then $V_2 = 6$ hence $\frac{U_m}{6}\frac{V_n}{6} = 6C$; so

$$\begin{cases} U_m/6 = 6C \\ V_n/6 = C \end{cases} \quad \text{or} \begin{cases} = 3C \\ = 2C \end{cases} \quad \text{or} \begin{cases} = 2C \\ = 3C \end{cases} \quad \text{or} \begin{cases} = C \\ = 6C \end{cases}$$

hence

$$\begin{cases} U_m = 36C \\ V_n = 6C \end{cases} \quad \text{or} \begin{cases} = 18C \\ = 12C \end{cases} \quad \text{or} \begin{cases} = 12C \\ = 18C \end{cases} \quad \text{or} \begin{cases} = 6C \\ = 36C \end{cases}.$$

Since $4 \nmid V_n$ then cases 2, 4 are impossible. By 5.4 the first case is impossible. In case 3, by 5.3 $m = 4$ so $C = 4V_n = 4 \times 18C = 9C$ which is absurd. □

6. Sums or differences which are cubes

As in §4, we shall consider expressions

$$U_{s+2k} \pm (-1)^k U_s \quad \text{and} \quad V_{s+2k} \pm (-1)^k V_s$$

and determine indices s, k for which the above expressions are cubes.

6.1. *Let $k \geq 1$, $s \geq 1$ be integers with $d = \gcd(s,k) = 1$ or 2. Then $U_{s+2k} - (-1)^k U_s \neq C$.*

Proof. If $C = U_{s+2k} + (-1)^k U_s = U_k V_{s+k}$ since $\gcd(s+k,k) = 1$ or 2, it follows from 5.6 that this is impossible. □

6.2. *Let $s \geq 1$, $k \geq 1$ be integers such that $d = \gcd(s,k) = 1$ or 2. Then $U_{s+2k} + (-1)^k U_s \neq C$*

Proof. If $C = U_{s+2k} + (-1)^k U_s = U_{s+k} V_k$ since $d = \gcd(s+k,k) = 1$ or 2, thus impossible by 5.6. □

Now we prove similar results for the sequence V.

6.3. *Let $s \geq 1$, $k \geq 1$ be integers such that $d = \gcd(s,k) = 1$ or 2. Then $V_{s+2k} - (-1)^k V_s \neq C$.*

Proof. Let $C = V_{s+2k} - (-1)^k V_s = 8U_{s+k}U_k$ and let $d = \gcd(s+k,k)$ so $U_d = \gcd(U_{s+k}, U_k)$. If $d = 1$ then $U_{s+k} = C$, $U_k = C$, which is impossible. If $d = 2$, so $U_2 = 2$, hence

$$\begin{cases} U_{s+k}/2 = 2C \\ U_k/2 = C. \end{cases} \quad \text{or} \quad \begin{cases} = C \\ = 2C \end{cases}.$$

Both cases are impossible by 5.2. $\square$

6.4. *If $s \geq 1$, $k \geq 1$ then $V_{s+2k} + (-1)^k V_s \neq C$.*

Proof. Assume that $C = V_{s+2k} + (-1)^k V_s = V_{s+k}V_k$. Let $e = \gcd(V_{s+k}, V_k)$. If $e = 2$ then

$$\begin{cases} V_{s+k}/2 = 2C \\ V_k/2 = C \end{cases} \quad \text{or} \quad \begin{cases} = C \\ = 2C \end{cases}.$$

Both cases are impossible, since $4 \nmid V_n$ for every $n \geq 1$. Let $d = \gcd(s+k,k)$ and $\frac{s+k}{d}$, $\frac{k}{d}$ odd. So $V_d = \gcd(V_{s+k}, V_k)$. Then

$$\begin{cases} V_{s+k}/V_d = 2aC \\ V_k/V_d = bC \end{cases} \quad \text{or} \quad \begin{cases} = aC \\ = 2bC \end{cases}$$

where $ab = \frac{1}{2}V_d$. Then again both cases are impossible, because $4 \nmid V_n$ for all $n \geq 1$. $\square$

As special cases of the above results, we note:

$$\begin{aligned} U_n &\neq C \pm 1 \quad \text{for all odd } n, \\ U_n &\neq C \pm 2 \quad \text{for all even } n, \\ V_n &\neq C \pm 2 \quad \text{for all odd } n, \\ V_n &\neq C \pm 6 \quad \text{for all even } n \geq 1. \end{aligned}$$

For the next result we require:

6.5. Let $x \neq 0$, $y > 0$ be integers.

a) If $x^3 = y^2 - 1$ *then* $(x, y) = (2, 3)$.

b) $x^3 \neq y^2 + 1$.

c) If $x^3 = y^2 - 3$ *then* $(x, y) = (1, 2)$.

d) $x^3 \neq y^2 + 3$.

e) $x^3 \neq y^2 - 4$.

f) If $x^3 = y^2 + 4$ *then* $(x, y) = (2, 2), (5, 11)$.

See [1], [4], and [6].

6.6. *If* n *is even then* $V_n \neq C \pm 1$, *except* $n = 0$, $C = 1$.

Proof. Let $V_n = C + 1$ with $n = 2m$. Then $V_m^2 - 2(-1)^m = C + 1$. If m is even $V_m^2 = C + 3$. So $V_m = 2$, which is absurd. If m is odd then $V_m^2 = C - 1$ which is impossible.

If $V_n = C - 1$ then $V_m^2 - 2(-1)^m = C - 1$. If m is even then $V_m^2 = C + 1$ so $V_m = 3$, which is absurd. Finally, if m is odd, then $V_m^2 = C - 3$ which is impossible. □

6.7. *If* $V_{2n} = C \pm 2$ *then* $n = 1$.

Proof. If $V_n^2 - 2(-1)^m = V_{2n} = C + 2$ and n is even then $V_n^2 = C + 4$ which is impossible. If n is odd, $V_n^2 = C$ so $V_n = C$ which is again impossible.

If $V_n^2 - 2(-1)^n = V_{2n} = C - 2$ and n is even, then $V_n^2 = C$ so $V_n = C$, which is not true.

If n is odd, then $V_n^2 = C - 4$ so $V_n = 2$, hence $n = 1$. □

References

[1] M. Lal, M. F. Jones and W. J. Blundon, Numerical solutions of the diophantine equation $y^2 - x^3 = k$, *Math. Comp.* **20** (1966), 322–325.

[2] W. Ljunggren, Zur Theorie der Gleichung $x^2 + 1 = Dy^4$, *Avh. Norsk. Vid. Akad. Oslo* **1** no. 5 (1942), 1–27.

[3] W. Ljunggren, Über die Gleichung $x^4 - Dy^2 = 1$, *Arch. Mat. Naturvid.* **45** no. 5 (1942), 61–70.

[4] W. L. McDaniel and P. Ribenboim, The square terms in Lucas sequences, *J. Number Theory* **58** (1996), 104–123.

[5] L. J. Mordell, Diophantine Equations, *Academic Press*, *London*, 1969.

[6] A. Pethő, The Pell sequence contains only trivial powers, *Coll. Mat. Soc. János Bolyai 60 (Sets, Graphs, and Numbers)*, *Budapest*, 1991, 561–568.

[7] P. Ribenboim, The Fibonacci Numbers and the Arctic Ocean, *Proceedings of the 2nd Gauss Symposium, Conference A: Mathematics and Theoretical Physics* (Munich, 1993), 41–83.

[8] P. Ribenboim and W.L. McDaniel, Square classes of Lucas sequences, *Port. Math.* **48** (1991), 469–473.

[9] N. Robbins, On Pell numbers of the form px^2, where p is prime, *Fibonacci Q.* **22** no. 4 (1984), 340–348.

[10] W. Sierpiński, Elementary Theory of Numbers (A. Schinzel, ed.), *North-Holland, Amsterdam*, 1987.

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