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The structure of the univoque set in the small case

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Abstract. In this paper we generalize the results obtained in [1] and [2] for the determination of the Hausdorff dimension of the univoque set. We break down the general problem into two subcases, and accomplish the investigation completely in the simplest case. Finally we illustrate the theoretical results with an interesting example.

1. Introduction

The specification of the univoque numbers is one of the newest fields in the research of generalized number systems. In [1] and [2] Z. Daróczy and I. Kátai have specified the univoque sequences and have presented a method for the computation of the Hausdorff dimension of the univoque set in the cases $1 < \beta \leq 2$, where β is the base of the number system. In this paper we continue this investigation in the general case.

Let $\beta > 1$ be the base number of a number system, $\Theta = \frac{1}{\beta}$ and $\mathcal{A} = \{0, 1, \dots, [\beta]\}$ the set of the usable digits. The set of the fractions is

$$\mathcal{F} = \Big\{ x \mid x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n} = \sum_{n=1}^{\infty} a_n \Theta^n \Big\},$$

where $a = (a_1, a_2, \dots) \in \{0, 1, \dots, [\beta]\}^N$. In the sequel we work only on the set of the fractions. The largest number in $\mathcal{F}$ is

$$L = [\beta]\Theta + [\beta]\Theta^2 + \dots = \frac{[\beta]\Theta}{1 - \Theta}.$$

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For $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) \in \{0, 1, \dots, [\beta]\}^{\mathbb{N}}$ let

$$\langle \varepsilon, \Theta \rangle := \sum_{n=1}^{\infty} \frac{\varepsilon_n}{\beta^n} = \sum_{n=1}^{\infty} \varepsilon_n \Theta^n.$$

A sequence ε is said to be univoque (with respect to Θ), if $\langle \varepsilon, \Theta \rangle = \langle \delta, \Theta \rangle$ (for $\delta \in \{0, 1, \dots, [\beta]\}^{\mathbb{N}}$) is only true if $\varepsilon = \delta$, i.e. $\varepsilon_n = \delta_n$, for $n \in \mathbb{N}$. In this case the number $\langle \varepsilon, \Theta \rangle$ is said to be univoque, too.

We denote the set of the univoque sequences by

$$U(\Theta) := \{\varepsilon \mid \varepsilon \in \{0, 1, \dots, [\beta]\}^{\mathbb{N}},\ \varepsilon \text{ is univoque with respect to } \Theta\}.$$

The sequences

$$\underline{0} := (0, 0, \dots), \quad \underline{[\beta]} := ([\beta], [\beta], \dots)$$

are univoque, because every other expansion is clearly larger or smaller than these ones, respectively.

For $\varepsilon \in \{0, 1, \dots, [\beta]\}^{\mathbb{N}}$ let $\overline{\varepsilon} = \underline{[\beta]} - \varepsilon = ([\beta] - \varepsilon_1, [\beta] - \varepsilon_2, \dots)$, which we will call the complementary sequence. From this it follows that $\overline{\varepsilon} \in \{0, 1, \dots, [\beta]\}^{\mathbb{N}}$, and if $\langle \varepsilon, \Theta \rangle = x$, then $\langle \overline{\varepsilon}, \Theta \rangle = ([\beta] - \varepsilon_1)\Theta + ([\beta] - \varepsilon_2)\Theta^2 + \dots = L - x$.

Later we will make comparisons between two sequences using lexicographic ordering.

The regular expansion. Let us define the following sequence $\varepsilon_n(x)$ for $x \in [0, L]$, by induction on n: $\varepsilon_n(x) = j$, if

$$\sum_{i=1}^{n-1} \varepsilon_i(x)\Theta^i + j\Theta^n \le x,$$

where $j \in \mathcal{A}$, but

$$\sum_{i=1}^{n-1} \varepsilon_i(x)\Theta^i + (j+1)\Theta^n > x$$

or $j + 1 > [\beta]$, i.e. we would use a non-usable digit for the expansion. The equality $x = \langle \varepsilon(x), \Theta \rangle$ is called the regular expansion of x, where $\varepsilon(x) = (\varepsilon_1(x), \varepsilon_2(x), \dots)$. We denote the sets of regular sequences by

$$R(\Theta) := \{\varepsilon(x) \mid x \in [0, L], \varepsilon(x) \text{ is regular}\}, \quad \text{and}$$

$$R_1(\Theta) := \{\varepsilon(x) \mid x \in [0, 1), \varepsilon(x) \text{ is regular}\}.$$

In the course of the regular expansion of an $x \in [0, L]$ we get the remainders x_i as follows:

$$x = \varepsilon_1\Theta + \Theta x_1, \text{ where } \varepsilon_1 \in \{0, 1, \dots, [\beta]\} \text{ and } \varepsilon_1 \text{ is maximum,}$$
$$\dots$$
$$x_{n-1} = \varepsilon_n\Theta + \Theta x_n, \text{ where } \varepsilon_n \in \{0, 1, \dots, [\beta]\} \text{ and } \varepsilon_n \text{ is maximum.}$$
$$\dots$$

Proposition 1. *If all of the remainders* $x_i \geq 1$ $(i = 1, \dots, n)$, *then* $\varepsilon_i = [\beta]$.

Proof. Let us assume that $\varepsilon_n \leq [\beta] - 1$. Then we would be able to write the remainder x_{n-1} in the form $x_{n-1} = (\varepsilon_n + 1)\Theta + \Theta(x_n - 1)$. But in this case in the expression $x_{n-1} = \varepsilon_n\Theta + \Theta x_n$ (which we get by the application of the definition) the coefficient ε_n would not be maximal.

Proposition 2. *For all* $x \in [0, 1]$ *the remainders resulting from the regular expansion fall into the interval* $[0, 1]$, *i.e. if – during the expansion of an arbitrary* x *– once the remainder is from the interval* $[0, 1]$, *then it remains there.*

Proof. The coefficient $\varepsilon_1(x)$ in the expansion would be maximal if we choose it as $[\beta x]$. Thus for $x \in [0, 1]$

$$x = [\beta x]\Theta + \Theta x_1, \tag{i}$$

and $0 \leq x_1$. Using that $[\beta x] \geq \beta x - 1$ we get from (i)

$$x \geq (\beta x - 1)\Theta + \Theta x_1 = \beta\Theta x - \Theta + \Theta x_1, \text{ i.e.}$$
$$0 \geq \Theta(x_1 - 1) \iff x_1 \leq 1.$$

By using the regular expansions we are able to decide whether a sequence is univoque or not. As a generalization of Theorem 2.1 in [1] we get

Lemma 1. $\varepsilon \in \{0, 1, \dots, [\beta]\}^{\mathbb{N}}$ *is univoque with respect to* $\Theta \iff \varepsilon$, $\underline{[\beta]} - \varepsilon \in R(\Theta)$.

Proof. $\Longrightarrow$ Like that in [1], but with $\underline{1}$ replaced by $\underline{[\beta]}$.

$\Longleftarrow$ The course of the proof is the same as in [1]. We use $\underline{[\beta]}$ instead of $\underline{1}$. In examining the Nth digits $\varepsilon_N > \delta_N$, we can thus choose $\delta_N = k$, $\varepsilon_N = k + j$, where k, $k + j \in \mathcal{A}$ and $j \geq 1$.

The quasiregular expansion. Let us define by induction on n the following sequence $\delta_n(x)$ for $x \in (0, L]$: $\delta_n(x) = j$, if

$$\sum_{i=1}^{n-1} \delta_i(x)\Theta^i + j\Theta^n < x$$

where $j \in \mathcal{A}$, but

$$\sum_{i=1}^{n-1} \delta_i(x)\Theta^i + (j+1)\Theta^n \geq x$$

or $j + 1 > [\beta]$, i.e. we would use a non-usable digit for the expansion. The equality $x = \langle \delta(x), \Theta \rangle$ is called the quasiregular expansion of x, where $\delta(x) = (\delta_1(x), \delta_2(x), \dots)$.

The quasiregular expansion is always infinite, and if the regular expansion is infinite too, then the two expansions are the same.

Since $[\beta] \leq \beta < [\beta] + 1$, i.e. $[\beta]\Theta \leq 1 < ([\beta] + 1)\Theta$, both in the regular and the quasiregular expansion of 1 surely $\varepsilon_1 = [\beta]$ (and $\delta_1 = [\beta]$ if $[\beta] < \beta$).

If 1 has finite regular expansion of the form $\langle \varepsilon(1), \Theta \rangle = r_1\Theta + r_2\Theta^2 + \dots + r_k\Theta^k$, where $r_k \geq 1$, then the sequence $\delta(1)$ is periodic with a period of length k:

$$\delta(1) = (r_1, r_2, \dots, r_{k-1}, r_k - 1, r_1, r_2, \dots, r_{k-1}, r_k - 1, \dots).$$

We shall use the notation $\ell = \ell(\Theta) = (\ell_1, \ell_2, \dots) \in \{0, 1, \dots, [\beta]\}^{\mathbb{N}}$ for the quasiregular expansion of 1.

By Lemma 1, in order to decide whether a sequence is univoque or not we need the regular expansion of the sequence and that of the complementerary sequence. To establish that an expansion producing a number less than 1 is regular, we use the result of W. Parry [3]. Reformulating his Theorem 3 according to our notations we get the following

Parry condition. A finite or infinite sequence $b = (b_1, b_2, \dots)$, which contains non-negative integers and produces a number x less than 1 (where $x = \langle b, \Theta \rangle$) is regular if and only if for all $n \geq 1$ the subsequence $(b_n, b_{n+1}, \dots)$ is less than the quasiregular expansion of 1, i.e.

$$b = (b_1, b_2, \dots) \in R_1(\Theta) \iff (b_n, b_{n+1}, \dots) < (\ell_1, \ell_2, \dots) \quad \forall\, n \geq 1,$$

where $0 \leq x = b_1\Theta + b_2\Theta^2 + \dots < 1$.

Remark. Using only the Parry condition we are not able to decide the univoque or regular property of a sequence representing a number in the interval $[1, L]$. To accomplish this we shall use Propositions 1 and 2.

2. The set of the univoque numbers

Let

$$H = \{\langle\varepsilon, \Theta\rangle \mid \langle\varepsilon, \Theta\rangle \in [0, L] \text{ and } \varepsilon \text{ univoque with respect to } \Theta\}$$

be the set of the univoque numbers of the interval $[0, L]$, and similarly H^* and H_1 the set of the univoque numbers of the intervals $[\Theta, 1)$, $[0, 1)$ respectively.

Proposition 3. *We have*

$$H_1 = \{0\} \cup \bigcup_{n=0}^{\infty} \Theta^n H^*.$$

Proof. We prove this equality in three steps.

a) First we prove that multiplying the elements of H^* with Θ^n ($n = 0, 1, 2, \dots$) we again get univoque numbers.

Let $\langle\varepsilon, \Theta\rangle = \varepsilon_1\Theta + \varepsilon_2\Theta^2 + \dots$ be a univoque number in H^*. If the number $\langle\varepsilon, \Theta\rangle \cdot \Theta = 0 + \varepsilon_1\Theta^2 + \varepsilon_2\Theta^3 + \dots$ is not univoque, then we can choose a sequence δ, which produces the same number: $\langle\delta, \Theta\rangle = \langle\varepsilon, \Theta\rangle \cdot \Theta$.

If $\delta_1 = 0$, then $\langle\delta, \Theta\rangle/\Theta = \langle\varepsilon, \Theta\rangle$, but this is a contradiction, since ε is univoque.

If $\delta_1 \neq 0$, then $\sum_{i=1}^{\infty} \delta_i\Theta^i \geq \delta_1\Theta^1 \geq \Theta$. Since $1 > \sum_{i=1}^{\infty} \varepsilon_i\Theta^i \geq \Theta$, multiplying by Θ we get $\Theta > \sum_{i=1}^{\infty} \varepsilon_i\Theta^{i+1} \geq \Theta^2$. Using the first and the last inequality $\Theta \leq \langle\delta, \Theta\rangle = \langle\varepsilon, \Theta\rangle \cdot \Theta < \Theta$, but this means that such a δ does not exist.

Thus we have proved that multiplying a univoque sequence in H^* by Θ we again get a univoque sequence. Similarly, repeating the multiplication the univoque property remains true.

b) In this part we prove that thus we get all univoque numbers in $(0, 1)$.

Let us assume that the number $\langle\varepsilon, \Theta\rangle \cdot \Theta$ is univoque. In this case the number $\langle\varepsilon, \Theta\rangle$ is univoque too, since if there is a sequence δ for which $\langle\delta, \Theta\rangle = \langle\varepsilon, \Theta\rangle$, then multiplying by Θ we would get $\langle\delta, \Theta\rangle \cdot \Theta = \langle\varepsilon, \Theta\rangle \cdot \Theta$, which is a contradiction. This – together with the result of part a) – means that

$$\langle \varepsilon, \Theta \rangle \text{ is univoque} \iff \langle \varepsilon, \Theta \rangle \cdot \Theta \text{ is univoque.}$$

Thus, multiplying the univoque numbers of the interval $[\Theta, 1)$ by Θ, we get exactly the univoque numbers of the interval $[\Theta^2, \Theta)$, and similarly – by repeated multiplication – the univoque numbers of the interval $[\Theta^{j+1}, \Theta^j)$ using multiplication by Θ^j. Since the interval $(0,1)$ can be represented in the form $(0,1) = [\Theta, 1) \cup [\Theta^2, \Theta) \cup [\Theta^3, \Theta^2) \cup \dots$, by this method we get all the univoque numbers in $(0,1)$.

c) 0 is a univoque number, since the sequence $\underline{0}$ is univoque. The assertion of the proposition follows now from a) and b). □

By Lemma 1, the location of the univoque numbers in the interval $[0, L]$ is symmetrical, since a sequence $\varepsilon \in \{0, 1, \dots, [\beta]\}^N$ is univoque with respect to Θ if and only if the sequences ε and $\underline{[\beta]} - \varepsilon$ are regular. This is equivalent however to the fact, that the sequence $\underline{[\beta]} - \varepsilon$ is univoque. Thus if $L < 2$, then from the univoque numbers of the interval $[0, 1)$ by reflection we can get the univoque numbers in $[1, L]$, and so eventually the univoque numbers of the whole interval $[0, L]$.

$L = \frac{[\beta]\Theta}{1-\Theta} < \frac{1}{1-\Theta}$, and the fraction on the right side is less than 2 if $1 - \Theta \geq \frac{1}{2}$, i.e. if $\beta \geq 2$. This will be assumed in the sequel, since the properties of the univoque set in the cases $1 < \beta \leq 2$ are already well-known ([1], [2]).

Thus we can specify all univoque numbers (the set H) if we know the univoque numbers in the interval $[\Theta, 1)$, i.e. the set H^*.

Breaking down the problem into two cases. Let us assume now that

$$\frac{1}{K+1} < \Theta \leq \frac{1}{K}, \quad \text{i.e. } K = [\beta].$$

Clearly, in this interval there exists a Θ_K for which $K\Theta_K + {\Theta_K}^2 = 1$, since for all Θ in this interval $K\Theta \leq 1$ but $(K+1)\Theta > 1$. The value of this number is

$$\Theta_K = \frac{-K + \sqrt{K^2 + 4}}{2},$$

and if we use the notation $\beta_K = \frac{1}{\Theta_K}$, then $\beta_K = K + \Theta_K$. The Θ_K and β_K values for $K = 1, \dots, 5$ are the following:

Form of 1	$\Theta + \Theta^2$	$2\Theta + \Theta^2$	$3\Theta + \Theta^2$	$4\Theta + \Theta^2$	$5\Theta + \Theta^2$
β_K	$\dfrac{\sqrt{5}+1}{2}$	$\sqrt{2}+1$	$\dfrac{\sqrt{13}+3}{2}$	$\sqrt{5}+2$	$\dfrac{\sqrt{29}+5}{2}$
Θ_K	$\dfrac{\sqrt{5}-1}{2}$	$\sqrt{2}-1$	$\dfrac{\sqrt{13}-3}{2}$	$\sqrt{5}-2$	$\dfrac{\sqrt{29}-5}{2}$

The case $K+1 > \beta > K+\Theta_K$ will be called from now on the "big case" (when the fraction part is larger than Θ_K), and the case $K \le \beta \le K+\Theta_K$ the "small case" (when the fraction part is smaller than Θ_K). Below we examine the small case (this is the simpler one), to the big case we will revert in a subsequent paper.

3. Specification of the univoque sequences

In this case

$$\frac{1}{1+K} < \Theta_K \le \Theta \le \frac{1}{K}, \quad \text{i.e. } K \le \beta \le \beta_K < K+1.$$

Let $1 = t_1\Theta + t_2\Theta^2 + \dots$ be the quasiregular expansion for all Θ-s in this interval. Then $t_1 = K$ because $K\Theta \le 1$, and $t_2 = 0$ since $K\Theta + \Theta^2 \ge 1$, and with the quasiregular expansion we cannot reach equality. Thus $\underline{t} = t_1t_2t_3\dots \le K0^\infty$. Let

$$Z = \{z = \varepsilon_1\Theta + \varepsilon_2\Theta^2 + \varepsilon_3\Theta^3 + \cdots \mid 1 \le \varepsilon_i \le K-1\}.$$

Theorem 1. *All elements of Z are univoque numbers.*

Proof. We prove that for a sequence ε corresponding to an arbitrary $z \in Z$ the sequence itself and the complementary sequence are regular, and represent a number less than 1, i.e. ε and $[\underline{\beta}] - \varepsilon \in R_1(\Theta)$.

a) Choosing an arbitrary $z \in Z$ the inequality $\varepsilon_i\varepsilon_{i+1}\dots < t_1t_2\dots$ is always true, since $t_1 = K$ and $\varepsilon_i \le K-1$. Thus $\varepsilon \in R_1(\Theta)$.

b) By the definition of the set Z clearly $L - Z = Z$, therefore $(K - \varepsilon_i)(K-\varepsilon_{i+1})\dots < t_1t_2\dots$ is satisfied too. Thus $\overline{\varepsilon} \in R_1(\Theta)$.

Briefly, using the shift operator $\sigma^i(\varepsilon) = \varepsilon_{1+i}\varepsilon_{2+i}\dots$ for a sequence ε corresponding to an arbitrary $z \in Z$, the inequalities

$$\bar{t}_1\bar{t}_2 \dots < \sigma^i(\varepsilon) < t_1t_2\dots$$

are true for all $i = 0, 1, \dots$.

However there are also other univoque sequences. By Lemma 1, a sequence $a = a_1a_2\dots$ is univoque $\iff$ the sequences a and $\underline{[\beta]} - a$ are regular. By Propositions 1, 2 and the Parry condition a sequence $a = a_1a_2\dots$ is regular in the following two cases:

a) $a_i = [\beta]$ for all $i \in \mathbb{N}$,
b) there is a smallest index $i \geq 1$, such that $a_i \neq [\beta]$ (i.e. the remainder falls into the interval $[0, 1]$ after the ith step), and then for the shifted sequence $a_ia_{i+1}\dots$ the Parry condition is satisfied.

Thus the univoque sequences are the following:

a) The sequences $\underline{[\beta]} = \underline{K}$ and $\underline{0}$,
b) the sequences of type $[\beta]\dots[\beta]b_ib_{i+1}\dots$ and $0\dots0b_ib_{i+1}\dots$, where for the tail $b = b_ib_{i+1}\dots$

$$\bar{t}_1\bar{t}_2 \dots < \sigma^j(b) < t_1t_2\dots$$

is true for all $j = 0, 1, \dots$.

So we can represent the whole univoque set from Z as follows:

$$H = \{0\} \cup \{L\} \cup Z \cup \bigcup_{j=1}^{\infty} \Theta^j Z \cup \bigcup_{j=1}^{\infty} (K\Theta + K\Theta^2 + \dots + K\Theta^j + \Theta^j Z).$$

4. The Hausdorff dimension of the set H

The Hausdorff dimensions of the sets $\Theta^j Z$ and $K\Theta + \dots + K\Theta^j + \Theta^j Z$ ($j = 1, 2, \dots$) are clearly the same as the dimension of the set Z. Thus the Hausdorff dimension of the whole set H equals the dimension of the base set Z.

To compute the dimension of the set Z, we first specify the self-similarity dimension, and after this we check the fulfilment of the open set condition, which guarantees that the Hausdorff dimension equals the self-similarity dimension, according to the method presented by G. A. Edgar [4].

We can construct the set Z from itself with $K-1$ pieces of projections

$$Z = \bigcup_{a=1}^{K-1} (a\Theta + \Theta Z),$$

thus its self-similarity dimension s is computable from the equation $1 = (K-1)\Theta^s$, and so

$$s = \frac{-\log(K-1)}{\log\Theta} = \frac{\log(K-1)}{\log\beta}.$$

As to the open set condition, if we prove that the sets $a\Theta + \Theta Z$ $(a = 1, \ldots, K-1)$ with different a values are disjoint, then open sets "little bit" larger than Z are a good choice for each projection.

Let us consider for example the sets belonging to the values $a = 0$ and $a = 1$. If these are disjoint, then the other couples of sets are disjoint too, because from the set ΘZ we get all of the sets $a\Theta + \Theta Z$ $(a = 1, \ldots, K-1)$ with shifts by $\Theta, \ldots, (K-1)\Theta$. The smallest and largest elements of Z are

$$\Theta + \Theta^2 + \cdots = \frac{\Theta}{1-\Theta}, \quad \text{and } (K-1)\Theta + (K-1)\Theta^2 + \cdots = \frac{(K-1)\Theta}{1-\Theta}$$

respectively, thus

$$\Theta Z \subseteq \left[\frac{\Theta^2}{1-\Theta}, \frac{(K-1)\Theta^2}{1-\Theta}\right].$$

The sets ΘZ and $\Theta + \Theta Z$ have a non-empty intersection if

$$\frac{(K-1)\Theta^2}{1-\Theta} \geq \Theta + \frac{\Theta^2}{1-\Theta}, \quad \text{i.e. if } (K-1)\Theta^2 \geq \Theta \iff \Theta \geq \frac{1}{K-1}.$$

But, since $\frac{1}{K+1} < \Theta < \frac{1}{K}$, this does not hold. So finally we have proved the following

Theorem 2. *The Hausdorff dimension of the univoque set is*

$$\dim H = \frac{\log(K-1)}{\log\beta}.$$

Remark. To represent the univoque sequences we can use a graphic model. We build a directed graph, the nodes of which are the usable digits in the number system, and draw an edge from the node a to b if the digit b is allowable (in a univoque sequence) after the digit a. We label the edges by Θ. Thus we get a directed graph called Mauldin–Williams graph [4]. Wandering over all the digits of the graph, we can construct all univoque sequences.

In the present case the self-similarity dimension of the graph is the same as the self-similarity dimension of the set H (because the part of the graph representing the elements of the set Z is strongly connected and strongly contracting), and the open set criterion guarantees that this is the Hausdorff dimension of the set H.

On the basis of our investigations, the graph of the univoque sequences in the "small case" is the following:

a) The graph part containing the nodes $1, 2, \dots, K-1$ is totally connected (there is an edge from an arbitrary node to an arbitrary node).

b) From the nodes 0 and K there are edges to the nodes $0, 1, \dots, K-1$ and $1, 2, \dots, K$ respectively. This allows the sequence-parts $00\dots0$ and $KK\dots K$ at the beginning of the sequences. If we come out of these loops, then we (finally) enter the graph part a).

The graph model is a useful means to demonstrate the univoque sequences, but it is not absolutely necessary.

Finally, we illustrate the theoretical results by an interesting example. In this part we compute the self-similarity dimension by a slightly different method [4], using the Mauldin–Williams graph.

The number system with base number $\frac{3+\sqrt{13}}{2}$. In this number system $[\beta] = 3$, $\mathcal{A} = \{0, 1, 2, 3\}$,

$$\Theta = \frac{-3+\sqrt{13}}{2}, \quad L = \frac{3\Theta}{1-\Theta} = \frac{3}{\beta - 1} = \frac{6}{1+\sqrt{13}}.$$

Since $3\Theta + \Theta^2 = 1$, the sequences belonging to the regular and the quasiregular expansion of 1 are 31 and $(30)^\infty$, respectively.

As we know, the univoque sequences are the following:

a) The sequences $\underline{3}$ and $\underline{0}$,

b) the sequences of type $3 \dots 3 b_i b_{i+1} \dots$ and $0 \dots 0 b_i b_{i+1} \dots$, where $b_j = 1$ or $b_j = 2$ for all $j = i, i+1, \dots$.

Let us denote the set of the univoque numbers beginning with i with H_i, where $i = 0, 1, 2, 3$. The set of all univoque numbers is $H = H_0 \cup H_1 \cup H_2 \cup H_3$. Thus

$$H_0 = (0 + \Theta H_0) \cup (0 + \Theta H_1) \cup (0 + \Theta H_2),$$

$$H_1 = (\Theta + \Theta H_1) \cup (\Theta + \Theta H_2),$$

$$H_2 = (2\Theta + \Theta H_1) \cup (2\Theta + \Theta H_2),$$

$$H_3 = (3\Theta + \Theta H_1) \cup (3\Theta + \Theta H_2) \cup (3\Theta + \Theta H_3).$$

The structure of the univoque set is representable by the Mauldin–Williams graph shown in Figure 1.

Figure 1: The graph representing the structure of the set H

This graph is not strongly connected, but its strongly connected part (which represents the set Z) has self-similarity dimension equal to the Hausdorff dimension of the set H, if the open set condition is satisfied. This part specifies the following equation system:

$$q_1^s = \Theta^s \cdot q_1^s + \Theta^s \cdot q_2^s$$

$$q_2^s = \Theta^s \cdot q_1^s + \Theta^s \cdot q_2^s,$$

where q_1 and q_2 are the Perron numbers belonging to the (reduced) nodes 1 and 2 respectively, and s is the self-similarity dimension of the (reduced) graph. Now the two Perron numbers are the same, so we can choose them to be 1. From this we get

$$1 = 2 \cdot \left(\frac{-3+\sqrt{13}}{2} \right)^s, \quad \text{thus } s = \frac{\log \frac{1}{2}}{\log \frac{-3+\sqrt{13}}{2}} = \frac{\log(K-1)}{\log \beta}.$$

Choosing open sets $U_1 = U_2$ a "little bit" larger than Z, the required condition is satisfied, so s is the Hausdorff dimension of the univoque set. The set H approximately has the form shown on Figure 2.

Figure 2: The approximate form of the set H

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