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Conformal flatness of complex Finsler structures

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Abstract. In the present paper, we shall be concerned with conformal flatness of convex Finsler structures. We introduce a complex Finsler connection and define its conformal curvature Θ. This curvature Θ is invariant by any conformal rescaling of the convex Finsler structure. Our main result is to show that this conformal curvature measures the conformal flatness.

1. Introduction

Let $\pi : E \to M$ be a holomorphic vector bundle of rank r over a complex manifold M of complex dimension n. The total space E is also a complex manifold of complex dimension $n+r$. The tangent vectors along the fibres define a holomorphic vector sub-bundle $\mathcal{V}$ of the holomorphic tangent bundle TE, that is, $\mathcal{V} = \ker d\pi$. Then we know that $\mathcal{V} \cong \pi^{-1}E$, and $\mathcal{V}$ is integrable. If a convex Finsler structure F is given on E, we can introduce a natural Hermitian structure h on $\mathcal{V}$. The Hermitian geometry of $(\mathcal{V}, h)$ has been investigated by Kobayashi [8], and a number of important results were obtained.

In this paper, however, we shall study the bundle $(\mathcal{V}, h)$ by using its Finsler connection, not the Hermitian connection. This connection is derived from the given Finsler structure F and a splitting of the following exact sequence of holomorphic vector bundles

$$0 \to \mathcal{V} \xrightarrow{i} TE \to \pi^{-1}TM \to 0, \tag{1.1}$$

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or equivalently its dual

$$0 \to \pi^{-1}TM^* \to TE^* \to \mathcal{V}^* \to 0. \tag{1.2}$$

Let $\sigma : \pi^{-1}TM \to TE$ be a splitting of (1.1). Putting $\mathcal{H} := \sigma(\pi^{-1}TM)$, it defines a transversal distribution of $\mathcal{V}$ which is C^∞ isomorphic to $\pi^{-1}TM$. Let $\{s_1, \dots, s_r\}$ be a local holomorphic frame field of E on an open set U. Then it induces a local complex coordinate system $(z^1, \dots, z^n, \xi^1, \dots, \xi^r)$ on the open set $\pi^{-1}(U)$ in E. Now, $\mathcal{H}$ has local frame field X_α on $\pi^{-1}(U)$ of the form

$$\sigma\left(\frac{\partial}{\partial z^\alpha}\right) = X_\alpha = \frac{\partial}{\partial z^\alpha} - \sum_{l=1}^{r} N^l_\alpha \frac{\partial}{\partial \xi^l}, \tag{1.3}$$

where $\{N^l_\alpha\}$ are local functions on $\pi^{-1}(U)$ satisfying some transformation law. Such a family $\{N^l_\alpha\}$ is called a *non-linear connection* on E.

If a splitting σ is given on the sequence (1.1), the co-tangent bundle TE^* has a C^∞-splitting $TE^* = \mathcal{H}^* \oplus \mathcal{V}^*$. Then, according to this splitting, the differential operator ∂ is also decomposed as $\partial = \partial_{\mathcal{H}} + \partial_{\mathcal{V}}$, where $\partial_{\mathcal{H}}$ is the natural projection to the transversal part $\mathcal{H}^*$, and $\partial_{\mathcal{V}} = \partial - \partial_{\mathcal{H}}$. If a convex Finsler structure F is given on E, we can take a non-linear connection satisfying $\partial^2_{\mathcal{H}} \equiv 0$, and by using this, we can introduce a canonical Finsler connection (cf. [1], [3], [4], [5]).

In a previous paper [3], we have discussed the conformal flatness of a Finsler structure in terms of Weyl connections. In this paper, we shall introduce a conformal invariant Θ which measures the conformal flatness of a complex Finsler structure, and we show that the vanishing of Θ is equivalent to the conformal flatness of F (Theorem 3.3). Our conformal invariant Θ is a natural generalization of the one in the Hermitian case (cf. [10]).

2. Finsler structures and Finsler connections

Let M be a connected complex manifold of dimension n, and E a holomorphic vector bundle of rank r over M. Each fibre E_z is a complex vector space of dimension r. In the case of $r = 1$, any complex Finsler metric is a Hermitian metric. Hence, in the sequel we assume that rank $E \geq 2$.

Definition 2.1 ([6]). A function $F(z,\xi)$ on E is said to be a *complex Finsler structure* if it satisfies the following conditions:

(2.1) $F(z,\xi)\geq 0$, and $F(z,\xi)=0$ iff $\xi=0$,

(2.2) $F(z,\xi)$ is C^∞ on the outside of the zero-section and continuous on E,

(2.3) $F(z,\lambda\xi)=|\lambda|^2 F(z,\xi)$ for an arbitrary $\lambda\in\mathbb{C}$.

We shall fix an open covering $\{U\}$ with holomorphic frame field $\{s_U\}$ and the induced coordinate system $\{\pi^{-1}(U),(z,\xi)\}$ on E. A complex Finsler structure F is said to be *convex* if the Hermitian matrix $\left(F_{i\bar{j}}\right)$ defined by

$$F_{i\bar{j}}=\frac{\partial^2 F}{\partial\xi^i\partial\bar{\xi}^j}$$

is positive-definite. In this paper we always suppose the convexity of F. On $\pi^{-1}(U)$, the vertical bundle $\mathcal{V}$ is spanned by the *vertical* vector fields $Y_1:=\partial/\partial\xi^1,\ldots,Y_r:=\partial/\partial\xi^r$. We define a Hermitian structure h on $\mathcal{V}$ by

$$h\left(Y_i,Y_j\right)=F_{i\bar{j}}.$$

The transversal distribution $\mathcal{H}$ is locally spanned by the *transversal* vector fields $\{X_\alpha\}$ of the form (1.3) for a complex non-linear connection N^i_α. We shall determine a canonical non-linear connection N^i_α.

The connection form θ of the Hermitian connection ∇^h of $(\mathcal{V},h)$ is given by

$$\theta^i_j=\sum F^{i\bar{m}}\partial F_{j\bar{m}}=\sum F^{i\bar{m}}\left(\sum\frac{\partial F_{j\bar{m}}}{\partial z^\alpha}dz^\alpha+\sum\frac{\partial F_{j\bar{m}}}{\partial\xi^k}d\xi^k\right).$$

On the other hand, $\mathcal{V}$ has a canonical holomorphic section $\epsilon:(z,\xi)\to(z,\xi;\xi)$, or in local coordinates

$$\epsilon(z,\xi)=\sum_J\xi^jY_j.$$

Then we put

$$\nabla^h\epsilon=\sum_i\theta^i\otimes Y_i$$

where the $(1,0)$-form θ^j is defined by

$$\theta^i = d\xi^i + \sum_m \theta^i_m \xi^m = d\xi^i + \sum_{m,\alpha} F^{i\bar{m}} \frac{\partial F_{l\bar{m}}}{\partial z^\alpha} \xi^l dz^\alpha.$$

Here we used the identity $\sum_{k=1}^r \xi^k (\partial F_{i\bar{j}}/\partial \xi^k) \equiv 0$ which is derived from the homogeneity assumption (2.3). Now we shall define a morphism $\sigma^* : \mathcal{V}^* \to TM^*$ by

$$\sigma^*(d\xi^i) = \theta^i.$$

It is trivial that σ^* defines a splitting of the sequence (1.2). For this splitting σ^*, in local coordinates, the non-linear connection N^i_α is given by

$$N^i_\alpha = \sum F^{i\bar{m}} \frac{\partial F_{l\bar{m}}}{\partial z^\alpha} \xi^l. \tag{2.5}$$

The differential operators $\partial_{\mathcal{H}}$ and $\partial_{\mathcal{V}}$ are given by

$$\partial_{\mathcal{H}} f = \sum_\alpha X_\alpha f dz^\alpha = \sum_\alpha \left(\frac{\partial f}{\partial z^\alpha} - \sum_m N^m_\alpha \frac{\partial f}{\partial \xi^m} \right) dz^\alpha,$$

$$\partial_{\mathcal{V}} f = \sum_m Y_m f \theta^m = \sum_m \frac{\partial f}{\partial \xi^m} \theta^m$$

for an arbitrary function f on E. By using the facts that $\sum F_{i\bar{j}} \xi^i \bar{\xi}^j = F$ and $X_\alpha \xi^i = -N^i_\alpha$, we get the following identity

$$\partial_{\mathcal{H}} F = \sum_\alpha \left(\frac{\partial F}{\partial z^\alpha} - \sum_l N^l_\alpha \frac{\partial F}{\partial \xi^l} \right) dz^\alpha \equiv 0. \tag{2.6}$$

We denote by the index $\mathbb{C}$ the complexification of vector bundles, e.g., $T^{\mathbb{C}}M = TM \oplus \overline{TM}$, $\mathcal{V}^{\mathbb{C}} = \mathcal{V} \oplus \overline{\mathcal{V}}, \ldots$.

Definition 2.2 ([2]). A connection $\nabla : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V} \otimes T^{\mathbb{C}}E^*)$ defined by the following two properties is called the *Finsler connection* of (E, F) or $(\mathcal{V}, h)$.

(1) ∇ is a $(1,0)$-type connection,

(2) ∇ satisfies

$$d_{\mathcal{H}} h(Z, W) = h(\nabla Z, W) + h(Z, \nabla W)$$

for $\forall Z, W \in \Gamma(\mathcal{V})$, where we put $d_{\mathcal{H}} = \partial_{\mathcal{H}} + \bar{\partial}_{\mathcal{H}}$.

Since ∇ is of $(1,0)$-type, we may put

$$\nabla Y_j = \sum_m \omega_j^m Y_m$$

for $(1,0)$-forms ω_j^m. Then we have $d_{\mathcal{H}} F_{i\bar{j}} = \sum F_{m\bar{j}} \omega_i^m + F_{i\bar{m}} \overline{\omega_j^m}$. Hence, since ω_j^i is of $(1,0)$-type, the connection form ω of ∇ is given by the following transversal form:

$$\omega_j^i = \sum F^{i\bar{m}} \partial_{\mathcal{H}} F_{j\bar{m}} = \sum_\alpha \Gamma^i_{j\alpha} dz^\alpha,$$

where the coefficients $\Gamma^i_{j\alpha}$ are given by

$$\Gamma^i_{j\alpha} = \sum F^{i\bar{m}} X_\alpha F_{j\bar{m}} = \sum F^{i\bar{m}} \left(\frac{\partial F_{j\bar{m}}}{\partial z^\alpha} - \sum N^l_\alpha \frac{\partial F_{j\bar{m}}}{\partial \xi^l} \right) \tag{2.7}$$

with the functions N^i_α defined by (2.5). Moreover, we get easily the following relation:

$$\Gamma^i_{j\alpha} = \frac{\partial N^i_\alpha}{\partial \xi^j}. \tag{2.8}$$

Remark 2.1. The following relation between the connection forms θ^i_j and ω^i_j is easily obtained:

$$\theta^i_j = \omega^i_j + \sum_{k,l} C^i_{jk} \theta^k,$$

where we put $C^i_{jk} = \sum F^{i\bar{m}} Y_k F_{j\bar{m}}$. Hence the Hermitian connection ∇^h corresponds to the so-called *Cartan connection* and our connection ∇ corresponds to the so-called *Rund connection* in real Finsler geometry (cf. [9]). We note that, from (2.8), our connection ∇ also corresponds to the *Berwald connection*.

We shall compute the curvature $\Omega = d\omega + \omega \wedge \omega$ and investigate its local expressions with respect to $\{dz^\alpha, \theta^i\}$. Since the non-linear connection N^i_α is given by (2.5), we have

Lemma 2.1. $\partial_{\mathcal{H}}\omega + \omega \wedge \omega \equiv 0$.

Proof. The proof is obtained by direct calculation. If we put

$$R^i_{j\,\alpha\beta} = X_\alpha \Gamma^i_{j\,\beta} - X_\beta \Gamma^i_{j\,\alpha} + \sum \Gamma^i_{m\alpha}\Gamma^m_{j\ \beta} - \sum \Gamma^i_{m\beta}\Gamma^m_{j\ \alpha},$$

the right hand side can be written as

$$\partial_{\mathcal{H}}\omega^i_j + \sum \omega^i_m \wedge \omega^m_j = -\frac{1}{2}\sum R^i_{j\,\alpha\beta} dz^\alpha \wedge dz^\beta.$$

Hence we must prove $R^i_{j\,\alpha\beta} \equiv 0$. Since $\Gamma^i_{j\,\alpha} = \sum F^{i\bar{m}} X_\alpha F_{j\bar{m}}$,

$$\begin{aligned} R^i_{j\,\alpha\beta} = \sum \Big\{ & X_\alpha F^{i\bar{m}} X_\beta F_{j\bar{m}} + F^{i\bar{m}} X_\alpha X_\beta F_{j\bar{m}} - X_\beta F^{i\bar{m}} X_\alpha F_{j\bar{m}} \\ & - F^{i\bar{m}} X_\alpha X_\alpha F_{j\bar{m}} + \Gamma^i_{m\alpha}\Gamma^m_{j\ \beta} - \Gamma^i_{m\beta}\Gamma^m_{j\ \alpha} \Big\} \\ = \sum \Big\{ & X_\alpha F^{i\bar{m}} X_\beta F_{j\bar{m}} - X_\beta F^{i\bar{m}} X_\beta F_{j\bar{m}} + F^{i\bar{m}} \sum R^l_{\alpha\beta} Y_i F_{j\bar{m}} \\ & + \Gamma^i_{m\alpha}\Gamma^m_{j\ \beta} - \Gamma^i_{m\beta}\Gamma^m_{j\ \alpha} \Big\} \\ = \sum & C^i_{jm} R^m_{\alpha\beta}, \end{aligned}$$

where we used $\Gamma^i_{j\,\alpha} = \sum F^{i\bar{m}} X_\alpha F_{j\bar{m}}$, and put $R^i_{\alpha\beta} = X_\alpha N^i_\beta - X_\beta N^i_\alpha$.

On the other hand, by definition of $R^i_{j\,\alpha\beta}$ and $\sum \xi^j \Gamma^i_{j\alpha} = N^i_\alpha$, we get easily $R^i_{\alpha\beta} = \sum R^i_{j\alpha\beta}\xi^j$. The equation above and $\sum C^i_{jk}\xi^j \equiv 0$ imply $R^i_{\alpha\beta} = 0$, and so $R^i_{j\,\alpha\beta} \equiv 0$. □

By virtue of $\Omega = d\omega + \omega \wedge \omega = \bar{\partial}\omega + \partial_{\mathcal{V}}\omega + (\partial_{\mathcal{H}}\omega + \omega \wedge \omega)$ and Lemma 2.1, we have

Proposition 2.1. *The curvature form* Ω *of* ∇ *is given by* $\Omega = \bar{\partial}\omega + \partial_{\mathcal{V}}\omega = \bar{\partial}_{\mathcal{H}}\omega + \bar{\partial}_{\mathcal{V}}\omega + \partial_{\mathcal{V}}\omega$*:*

$$\Omega^i_j = \sum_{\alpha,\beta} R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \sum_{\alpha,k} R^i_{j\alpha\bar{k}} dz^\alpha \wedge \bar{\theta}^k + \sum_{\alpha,k} R^i_{j\alpha k} dz^\alpha \wedge \theta^k, \tag{2.9}$$

where we put $R^i_{j\alpha\bar{\beta}} = -X_{\bar{\beta}}\Gamma^i_{j\,a}$, $R^i_{j\alpha\bar{k}} = -Y_{\bar{k}}\Gamma^i_{j\,\alpha}$, $R^i_{j\alpha k} = -Y_k \Gamma^i_{j\,\alpha}$.

Remark 2.2. From this lemma, we can easily infer the identity $\partial_{\mathcal{H}}^2 \equiv 0$.

Here we describe a special class of Finsler structures. A complex Finsler bundle (E, F) is said to be *modeled on a complex Minkowski space* if its connection ∇ is projectable to a connection of E, that is, its connection coefficients $\Gamma^i_{j\alpha}$ are functions of position of $z \in M$ alone. Then we have

Theorem 2.1 ([2]). *Let (E, F) be modeled on a complex Minkowski space. Then there exists a Hermitian structure h_F on E, and the Finsler connection ∇ of (E, F) is given by the pull-back of the Hermitian connection of (E, h_F).*

It is trivial that (E, F) is modeled on a complex Minkowski space if and only if $\partial_{\mathcal{V}}\omega = \bar{\partial}_{\mathcal{V}}\omega = 0$, and, in this case, the first term of (2.9) is given by the curvature of its associated h_F.

3. Conformally flat Finsler structures

We begin with the following

Definition 3.1. A complex Finsler structure F is said to be *flat* if there exists an open covering $\{U\}$ and a suitably chosen holomorphic frame field $s_U = \{s_1, \ldots, s_r\}$ on each U such that with respect to $\{U, s_U\}$ the function F is independent at the base point $z \in M : F = F(\xi)$. Such a covering $\{U, s_U\}$ is said to be *adapted*.

This notion is a complex analogue of a *locally Minkowski space* in real Finsler geometry (cf. Definition 24.1 in [9]). We shall use, however, the term *flat* since the following theorem holds:

Theorem 3.1. *A complex Finsler structure F is flat if and only if its Finsler connection ∇ is flat, that is, the curvature Ω of ∇ vanishes identically.*

Proof. We suppose that F is independent at $z \in M$ with respect to an adapted $\{U, s_U\}$. Then, from (2.5), we have $N^i_\alpha = 0$, and so by virtue of (2.8) we get $\Gamma^i_{j\alpha} = 0$. Hence, with respect to an adapted $\{U, s_U\}$ the connection form ω vanishes on each U. This means the vanishing of its curvature.

Conversely we assume that the curvature Ω of ∇ vanishes identically. Then, (E, F) is modeled on a complex Minkowski space, and, by Theorem 2.1, ∇ is the Hermitian connection of the associated h_F. Since $\omega = 0$

is completely integrable, on a suitable open neighborhood U of each point $z \in M$, we can introduce a parallel frame field $s_U = \{s_1, \dots, s_r\}$. Since $ds_j = 0$, we have $\bar{\partial} s_j = 0$, and so s_U is holomorphic. Therefore the connection form ω with respect to $\{U, s_U\}$ vanishes on each U. Hence, by virtue of $\sum \xi^j \Gamma^i_{j\alpha} = N^i_\alpha$, we have $N^i_\alpha = 0$. Consequently, from (2.6) we have $F = F(\xi)$ with respect to $\{U, s_U\}$. □

By this theorem and Theorem 2.1, we have

Proposition 3.1. *A convex Finsler structure F on E is flat if and only if (E, F) is modeled on a complex Minkowski space and its associated Hermitian metric h_F is flat.*

By this proposition, we know that if (E, F) is a flat Finsler vector bundle, then it admits a flat Hermitian structure h_F. Conversely, the norm function derived from a flat Hermitian structure is also a flat Finsler structure. Hence we have (cf. Proposition 4.21 on p. 14 of [7])

Theorem 3.2. *The following conditions are equivalent:*

(1) *E admits a flat Finsler structure.*

(2) *E admits a flat unitary structure.*

(3) *E is defined by a representation $\rho : \pi_1(M) \to U(r)$: $E \cong \tilde{M} \times_\rho \mathbb{C}^r$,*

where $\pi_1(M)$ is the fundamental group of M and $\tilde{M}$ is the universal covering of M.

We shall consider a conformal rescaling $F \to \tilde{F} = e^{\sigma(z)} F$ of the Finsler metric F for a differentiable function $\sigma(z)$ on M. We shall calculate the connection form $\tilde{\omega}$ of $(E, \tilde{F})$. Because of $\tilde{F}_{i\bar{j}} = e^{\sigma(z)} F_{i\bar{j}}$ and (2.5), the non-linear connection is changed as $\tilde{N}^i_\alpha = N^i_\alpha + (\partial\sigma/\partial z^\alpha)\, \xi^i$. Hence, by (2.8) we get

$$\tilde{\omega} = \omega + \partial\sigma \otimes I_{\mathcal{V}} \tag{3.1}$$

for the identity endomorphism $I_{\mathcal{V}}$ of $\mathcal{V}$. Then we have

Lemma 3.1. *Let (E, F) be modeled on a complex Minkowski space with an associated Hermitian structure h_F. Then, for any confomal rescaling $F \to \tilde{F} = e^{\sigma(z)} F$, $(E, \tilde{F})$ is also modeled on a complex Minkowski space, and the conformal rescaling $e^{\sigma(z)} h_F$ of h_F associates with $(E, \tilde{F})$.*

Proof. The Finsler connection $\tilde{\nabla}$ of $(E, \tilde{F})$ is given by (3.1). Hence the first assertion is trivial. Moreover, we have

$$\tilde{\omega} = \omega + \partial\sigma \otimes I_{\mathcal{V}} = (e^{\sigma(z)} h_F)^{-1} \partial (e^{\sigma(z)} h_F) = {h_{\tilde{F}}}^{-1} \partial h_{\tilde{F}}.$$

Hence $e^{\sigma(z)}h_F$ associates with $(E,\tilde{F})$. □

From (3.1) and Proposition 2.1, the curvature form is transformed as follows:

$$\tilde{\Omega} = \Omega + \bar{\partial}\partial\sigma \otimes I_{\mathcal{V}}. \tag{3.2}$$

From (3.1), the $(1,0)$-form θ^j is transformed as $\tilde{\theta}^j = \theta^j + \xi^j \otimes \partial\sigma$. Then we have

Lemma 3.2. *By any conformal rescaling $F \to \tilde{F} = e^{\sigma(z)}F$, the forms $\partial_{\mathcal{V}}\omega$ and $\bar{\partial}_{\mathcal{V}}\omega$ are invariant.*

Proof. By definition,

$$\partial_{\mathcal{V}}\omega^i_j = \sum_{\alpha,k} \frac{\partial \Gamma^i_{j\alpha}}{\partial \xi^k}\theta^k \wedge dz^\alpha.$$

With respect to the new function $\tilde{F}$, we shall compute the right hand side:

$$\begin{aligned}\sum_{\alpha,k} \frac{\partial \tilde{\Gamma}^i_{j\alpha}}{\partial \xi^k}\tilde{\theta}^k \wedge dz^\alpha &= \sum_{\alpha,k} \frac{\partial}{\partial \xi^k}\left(\Gamma^i_{j\alpha} + \frac{\partial\sigma}{\partial z^\alpha}\right)(\theta^k + \xi^k\partial\sigma) \wedge dz^\alpha \\ &= \sum_{\alpha,k} \frac{\partial \Gamma^i_{j\alpha}}{\partial \xi^k}\theta^k \wedge dz^\alpha,\end{aligned}$$

since, by the homogeneity of F, we have $\sum_k \xi^k Y_k \Gamma^i_{j\alpha} \equiv 0$. This means that the form $\partial_{\mathcal{V}}\omega$ is invariant under any conformal rescaling.

The proof for the invariance of $\bar{\partial}_{\mathcal{V}}\omega$ is similar. □

By this lemma, we have

Proposition 3.2. *The $End(\mathcal{V})$-valued $(1,1)$-form*

$$\Theta = \Omega - \frac{1}{r}\rho \otimes I_{\mathcal{V}} \tag{3.3}$$

is invariant by any conformal rescaling, where ρ is defined by $\rho = Tr.\bar{\partial}_{\mathcal{H}}\omega$.

Proof. By Lemma 3.1, the forms $\partial_{\mathcal{V}}\omega$ and $\bar{\partial}_{\mathcal{V}}\omega$ are invariant by any conformal rescaling. Hence the $(1,1)$-form ρ is transformed as

$$\begin{aligned}\tilde{\rho} &= Tr.(\tilde{\Omega} - \partial_{\mathcal{V}}\omega - \bar{\partial}_{\mathcal{V}}\omega) = Tr.(\Omega + \bar{\partial}\partial\sigma \otimes I_{\mathcal{V}} - \partial_{\mathcal{V}}\omega - \bar{\partial}_{\mathcal{V}}\omega) \\ &= Tr.\partial_{\mathcal{H}}\omega + r\bar{\partial}\partial\sigma = \rho + r\bar{\partial}\partial\sigma.\end{aligned}$$

Thus we have $\bar{\partial}\partial\sigma = (\tilde{\rho} - \rho)/r$. Substituting this into (3.2), we see that Θ is invariant by any conformal rescaling. □

We call the Θ given by (3.2) the *conformal curvature* of (E, F). It is trivial that if Ω vanishes, then Θ also vanishes.

Definition 3.2. A Finsler structure F is (locally) *conformally flat* if every $z \in M$ has an open neighborhood U and a differentiable function $\sigma_U : U \to \mathbb{R}$ such that $\tilde{F}_U = e^{\sigma_U} F$ is a flat Finsler structure on U.

Now we shall prove our main theorem:

Theorem 3.3. *Let F be a convex Finsler structure on a holomorphic vector bundle E. Then F is conformally flat if and only if the conformal curvature Θ vanishes identically.*

Proof. We shall fix a frame field $\{U, s_U\}$ for E, and use the local expressions with respect to $\{U, s_U\}$.

We suppose that Θ vanishes identically. Then, since $\partial_{\mathcal{V}}\omega = \bar{\partial}_{\mathcal{V}}\omega = 0$, (E, F) is modeled on a complex Minkowski space. By Theorem 2.1 there exists an associated Hermitian structure h_F, and Ω is given by the pull-back of the one Ω_F of h_F. Hence ρ is the Ricci curvature of h_F:

$$\rho = \bar{\partial}\partial \log \det(h_{i\bar{j}}),$$

where we put $h_{i\bar{j}} = h_F(s_i, s_j)$. On each U, we put $\sigma_U(z) = \frac{1}{r} \log \det(h_{i\bar{j}})$, and we consider the conformal rescaling $F \to \tilde{F}_U = e^{\sigma_U(z)} F|_U$. Then, $\tilde{F}_U$ is also modeled on a complex Minkowski space, and its curvature $\tilde{\Omega}$ is given by

$$\begin{aligned} \tilde{\Omega} &= \frac{1}{r}\tilde{\rho} \otimes I_{\mathcal{V}} = \frac{1}{r}\left(Tr.\Omega_F + r\bar{\partial}\partial\sigma_U\right) \otimes I_{\mathcal{V}} \\ &= \frac{1}{r}\left(-\bar{\partial}\partial\sigma_U + \bar{\partial}\partial\sigma_U\right) \otimes I_{\mathcal{V}} = 0, \end{aligned}$$

which shows that $\tilde{F}_U$ is flat. Hence F is conformally flat.

The converse is trivial. □

The conformal flatness of a Hermitian structure has been studied in [10], where the conformal flatness of a Hermitian structure has been characterized by the vanishing of a conformally invariant curvature tensor. Our conformal curvature Θ coincides with that of Matsuo if the given Finsler structure F is the norm function associated to a Hermitian structure h, that is, $F(z, \xi) = \sum h_{i\bar{j}}(z)\xi^i\bar{\xi}^j$. Then, from Theorem 3.3, we have

Proposition 3.3. *A convex Finsler structure on E is conformally flat if and only if (E, F) is modeled on a complex Minkowski space, and its associated Hermitian structure is conformally flat.*

Let P be the $GL(r,\mathbb{C})$-principal bundle associated to E. We denote by $PGL(r,\mathbb{C})$ the projective linear group $GL(r,\mathbb{C})/\mathbb{C}^* I_r$, where $\mathbb{C}^* I_r$ is the center of $GL(r,\mathbb{C})$. A vector bundle E is said to be *projectively flat* if the $PGL(r,\mathbb{C})$-principal bundle $\hat{P} = P/\mathbb{C}^* I_r$ is provided with a flat structure (cf. [7]).

If the conformal curvature satisfies $\Theta \equiv 0$, then the curvature Ω of the associated h_F satisfies

$$\Omega = \frac{1}{r}\rho \otimes I_{\mathcal{V}}.$$

Hence, according to Proposition 2.8 in [7], the bundle E is projectively flat, and $\hat{P}$ is defined by a representation $\rho : \pi_1(M) \to PU(r)$, where $PU(r) = U(r)/U(1)I_r$ is the projective unitary group. This means that, if we consider the universal covering space $\tilde{M}$ as a $\pi_1(M)$-principal bundle $\tilde{M} \to M$, the bundle $\hat{P}$ is defined by the representation $\rho : \pi_1(M) \to PU(r)$. The flat structure of $\hat{P}$ is induced by the natural flat structure of $\tilde{M} \to M$.

By Proposition 3.3, we know that, if (E, F) is a conformally flat Finsler vector bundle, it admits a conformally flat Hermitian structure h_F. Conversely, the norm function defined by a conformally flat Hermitian structure is also a conformally flat Finsler structure. Hence we have (cf. Proposition 4.22 in p. 14 of [7])

Theorem 3.4. *The following conditions are equivalent:*

(1) *E admits a conformally flat Finsler structure.*

(2) *E admits a conformally flat Hermitian structure.*

(3) *The bundle $\hat{P} = P/\mathbb{C}^* I_r$ is defined by a representation $\rho : \pi_1(M) \to PU(r) : \hat{P} \cong \tilde{M} \times_\rho PU(r)$.*

Example 3.1. Let M be a so-called *Hopf manifold* $\{\mathbb{C}^n - 0\}/\Delta_\lambda$, where Δ_λ is the group generated by the holomorphic transformations $(z^1, \ldots, z^n) \to (\lambda z^1, \ldots, \lambda z^n)$ on $\mathbb{C}^n - \{0\}$ for $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$. Then there exists a standard Hermitian structure on TM:

$$ds^2 = \frac{1}{\|z\|^2} \sum_\alpha dz^\alpha \otimes d\bar{z}^\alpha = e^{-\log \|z\|^2} \sum_\alpha dz^\alpha \otimes d\bar{z}^\alpha, \tag{3.3}$$

where $\|z\|^2 = \sum_\alpha z^\alpha \bar{z}^\alpha$. This metric is locally conformal Kähler-flat (l.c.K_0 in short, cf. [12]). Its Hermitian connection is given by

$$\omega = -\partial(\log \|z\|^2) \otimes I_{TM}. \tag{3.4}$$

The norm function defined by the metric above is $F_0(z,\xi) = e^{-\log \|z\|^2}\|\xi\|^2$.

To obtain a conformally flat Finsler structure F, we shall modify F_0 into the form

$$F(z,\xi) = e^{-\log \|z\|^2} f(\xi) \tag{3.5}$$

for a positive function $f(\xi)$ on $\mathbb{C}^n$ satisfying $f(\lambda\xi) = |\lambda|^2 f(\xi)$ and the Hermitian matrix $\left(\partial^2 f/\partial\xi^i\partial\bar{\xi}^j\right)$ is positive definite. Since F is also invariant by the action of Δ_λ, it defines a convex Finsler structure on TM. It is trivial that this Finsler structure F is conformally flat. We shall check this by computing its conformal curvature Θ.

This complex Finsler manifold (M,F) is modeled on a complex Minkowski space, and its associated Hermitian metric is given by (3.1). We shall show this. If we put $f_{i\bar{j}}(\xi) = \partial^2 f/\partial\xi^i\partial\bar{\xi}^j$, we have $F_{i\bar{j}} = e^{-\log \|z\|^2} \times f_{i\bar{j}}(\xi)$. Hence, by (2.5), the non-linear connection N^i_α of (M,F) is given by

$$N^i_\alpha = -\frac{\bar{z}^\alpha}{\|z\|^2}\xi^i. \tag{3.6}$$

Moreover, from (2.8), the connection coefficients of ∇ are given by

$$\Gamma^i_{j\alpha} = -\frac{\partial \log \|z\|^2}{\partial z^\alpha}\delta^i_j.$$

Hence the Finsler connection ∇ is given by (3.4).

Since the curvature form Ω of ∇ is given by $\Omega = -\bar{\partial}\partial(\log \|z\|^2)\otimes I_{TM}$, we get $\rho = -n\bar{\partial}\partial(\log \|z\|^2)$. From these equations and the definition of Θ, we get $\Theta \equiv 0$.

Let (E,F) be a complex Finsler bundle over a compact Kähler manifold (M,g). Assume that (E,F) is conformally flat. Then, since $\Theta = 0$, (E,F) is modeled on a complex Minkowski space, and its curvature Ω is given by $\Omega = \frac{1}{r}\rho\otimes I_{\mathcal{V}}$. Now, it is easily proved that the associated Hermitian vector bundle (E,h_F) satisfies the *weak Einstein condition*. Moreover, if (M,g) is compact Kähler, by suitable conformal rescaling $h_F \to ah_F$, we can obtain that ϕ is constant (cf. Proposition 2.4 in Chapter IV of [7]). Hence the associated Hermitian bundle (E,h_F) is Einstein–Hermitian over (M,g). Consequently we have

Proposition 3.4. *Let (E, F) be a convex Finsler vector bundle over a compact Kähler manifold (M, g). If (E, F) is conformally flat, then the associated Hermitian vector bundle (E, h_F) satisfies the Einstein condition.*

4. Some remarks from Hermitian geometry

By Proposition 3.3, some geometric properties of a conformally flat (E, F) are obtained from those of (E, h_F). We shall show some results directly obtained from Hermitian or Kählerian geometry.

Let M be a complex manifold of $\dim_{\mathbb{C}} M = n$, and F a convex Finsler structure on TM. The pair (M, F) is called a *complex Finsler manifold*. Suppose that (M, F) is conformally flat. Then (TM, F) is modeled on a complex Minkowski space, and its associated Hermitian metric h_F is conformally flat. Hence, there exists an open covering $\{U\}$ and a family of local functions $\{\sigma_U\}$ such that $h_U = e^{\sigma_U} h_F$ is a flat metric on U. Moreover, if each h_U is a flat Kähler metric on U, (M, h_F) is l.c.K_0. (Example 1 is just of this type). Then, applying Theorem 2.2 in [12] (see also Theorem 6.8 in [11]), we see that the universal covering $\tilde{M}$ of M is $\mathbb{C}^n - \{0\}$, and h_F is globally conformal to the metric induced by (3.3). Applying this fact, we have

Theorem 4.1. *Let (M, F) be a compact complex Finsler manifold of $\dim_{\mathbb{C}} M = n$ which is conformally flat. Suppose that its associated Hermitian manifold (M, h_F) is (not globally) l.c.K_0. Then the universal covering $\tilde{M}$ of M is given by $\mathbb{C}^n - \{0\}$, and F is globally conformal to the Finsler structure induced by the one of the form* (3.5).

Proof. The fact that $\tilde{M} = \mathbb{C}^n - \{0\}$ is trivial from Vaisman's theorem. We shall prove the second part of the theorem. Since the Finsler connection ∇ of (M, F) is given by the form ω in (3.2), its non-linear connection N^i_j is given by (3.6). Now, from (2.6), we have $X_\alpha F = 0$. Hence, in this case, we have

$$\frac{\partial F}{\partial z^\alpha} + \frac{\bar{z}^\alpha}{\|z\|^2} F = 0.$$

This equation implies

$$\frac{\partial}{\partial z^\alpha}\left(e^{\log \|z\|^2} F\right) = \frac{\partial}{\partial z^\alpha}\left(\|z\|^2 F\right) = \bar{z}^\alpha F + \|z\|^2 \frac{\partial F}{\partial z^\alpha} = 0.$$

In the same way as above, we have $\partial\left(e^{\log\|z\|^2}F\right)/\partial\bar{z}^\alpha = 0$. Hence we get

$$e^{\log\|z\|^2}F = f(\xi)$$

for a function f which depends only on ξ. It is trivial that f satisfies the homogeneity and the convexity conditions. Consequently F must be of the type (3.5). □

In a previous paper [2], we have introduced the notion of *Finsler–Kähler manifold*. We shall recall its definition. We use the Greek letters $j, k, \dots$ for the indices of the local coordinates of M. If a complex Finsler manifold (M, F) is given, its non-linear connection N^i_j is given by (2.5):

$$N^i_j = \sum_{l,m} F^{i\bar{m}} \frac{\partial F_{l\bar{m}}}{\partial z^j} \xi^l,$$

and the connection coefficients of its Finsler connection ∇ are given by $\Gamma^i_{jk} = \partial N^i_k/\partial\xi^j$. Then (M, F) is said to be Finsler–Kähler if the condition

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

is satisfied. In [1], such a manifold is called *strongly Finsler–Kähler*. By Theorem 2.1 it is trivial that, if a Finsler–Kähler manifold (M, F) is modeled on a complex Minkowski space, then its associated h_F is Kähler.

Any conformally flat Kähler manifold is flat (cf. Theorem 4.1 in [13], see also Corollary 4.3 in [10]). In our case, we have

Theorem 4.2. *Let (M, F) be a Finsler–Kähler manifold. If (M, F) is conformally flat, then (M, F) is flat.*

Proof. By Proposition 3.3, (M, F) is conformally flat if and only if it is modeled on a complex Minkowski space, and moreover its associated h_F is conformally flat. By the assumption of Finsler–Kähler, the associated (M, h_F) is a conformally flat Kähler manifold. Hence (M, h_F) is flat. Consequently, by Proposition 3.1, (M, F) is flat. □

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