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On the spectral radius of Coxeter transformations of trees

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Abstract. The spectral radius of a Coxeter transformation which plays an important role in the representation theory of hereditary algebras (see [DR]), is its important invariant. This paper provides both upper and lower bounds for the spectral radii of Coxeter transformations of the wild stars (i.e. the trees that have a single branching point and are neither of Dynkin nor of Euclidean type). In addition, the paper determines limit of the spectral radii of particular infinite sequences of wild stars.

1. Definitions and preliminary results

Let Δ be a tree, i.e. a finite non-oriented connected graph without cycles (multiple edges are allowed); let $\{1, 2, \dots, n\}$ be the set of its vertices. The *spectrum* $\mathrm{Spec}(\Delta)$ of Δ is the set of the eigenvalues of the adjancency matrix $A = A(\Delta) = (a_{ij})$ of Δ; here a_{ij} is the number of edges between the vertices i and j, and thus A is an integral symmetric matrix. Denote the *spectral radius* of Δ (i.e. the largest eigenvalue of A) by $\rho(\Delta)$.

Let Ω be an orientation of the tree Δ and $\mathcal{C} = \mathcal{C}_{\Omega(\Delta)} : \mathbb{C}^n \to \mathbb{C}^n$ the corresponding Coxeter transformation. Recall that the matrix $\Phi = \Phi_{\Omega(\Delta)}$ of $\mathcal{C}$ with respect to the standard basis can be written as $\Phi = -C^{-1}C^{tr}$, where $C = C_{\Omega(\Delta)} = (c_{ij})$ is an integral $n \times n$ matrix with c_{ji} equal to the number of paths from the vertex i to the vertex j in $\Omega(\Delta)$. The characteristic polynomial of Φ is called the *Coxeter polynomial* of $\mathcal{C}$. The *spectrum* $\mathrm{Spec}(\mathcal{C})$ is the set of all eigenvalues of Φ and the *spectral radius* of $\mathcal{C}$ is

$$\rho(\mathcal{C}_{\Omega(\Delta)}) = \max\{\|\lambda\| : \lambda \in \mathrm{Spec}(\mathcal{C}_{\Omega(\Delta)})\}.$$

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It is well-known (see [C]) that the characteristic polynomial of the Coxeter transformation is reciprocal and that the spectrum $\mathrm{Spec}(\mathcal{C}_{\Omega(\Delta)})$ of the Coxeter transformation $\mathcal{C}_{\Omega(\Delta)}$ does not depend on the orientation of Ω if Δ is a tree. Thus, we may write $\mathrm{Spec}(\mathcal{C}_\Delta) = \mathrm{Spec}(\mathcal{C}_{\Omega(\Delta)})$.

In the case when the graph is of Dynkin or Euclidean type then $\mathrm{Spec}(\mathcal{C})$ is well known. In general, A'Campo has proved the following relationship between the sets $\mathrm{Spec}(\Delta)$ and $\mathrm{Spec}(\mathcal{C}_\Delta)$.

Theorem 1.1 ([C]).
a) *Given* $0 \neq \lambda \in \mathbb{C}$ *then* $\lambda + \lambda^{-1} \in \mathrm{Spec}(\Delta)$ *if and only if* $\lambda^2 \in \mathrm{Spec}(\mathcal{C}_\Delta)$.
b) $\mathrm{Spec}(\mathcal{C}_\Delta) \subseteq S^1 \cup \mathbb{R}^+$, *where* $S^1 = \{\lambda \in \mathbb{C} : \|\lambda\| = 1\}$.
c) *If* Δ *is not Dynkin, then there exists a real number* $\lambda \geq 1$ *such that* $\rho(\Delta) = \lambda + \lambda^{-1}$ *and* $\rho(\mathcal{C}_\Delta) = \lambda^2$. *Moreover,* Δ *is Euclidean if and only if* $\lambda = 1$.

Since the Perron–Frobenius theorem for non-negative matrices yields that $\Delta' \subset \Delta$ implies $\rho(\Delta') < \rho(\Delta)$ (cf. [H]), we get immediately the following corollary.

Corollary 1.2. *If* Δ' *is a subtree of a tree* Δ, *neither of which is Dynkin, then* $\rho(\mathcal{C}_{\Delta'}) < \rho(\mathcal{C}_\Delta)$.

Denote by $d(i)$ the degree of the vertex i; i.e. $d(i) = \sum_{j=1}^{n} a_{ij}$.

Theorem 1.3 (see [PT] and [X]). *Let* m *be the maximum of the degrees of all vertices of* Δ. *Then*
a) $\rho(\mathcal{C}_\Delta) \leq m^2 - 2$.
b) *If* Δ *is neither of Dynkin nor of Euclidean type, then* $\mu_0 \leq \rho(\mathcal{C}_\Delta)$; *where* μ_0 *is the largest (real) root of the polynomial*

$$f(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

2. Wild stars

Let $p = (p_1, p_2, \ldots, p_s)$, $s \geq 3$, be a sequence of positive integers p_i, $1 \leq i \leq s$ and let $n = \sum_{i=1}^{s} p_i + 1$. The wild star is a tree with simple edges which consists of paths with one common endpoint. Denote $\Delta_{[p_1,p_2,\ldots,p_s]}$ the wild star consisting of s paths of length $p_1, p_2, \ldots, p_s$, and denote by $\chi_{[p_1,p_2,\ldots,p_s]}(x)$ and $\rho(\mathcal{C}_{[p_1,p_2,\ldots,p_s]})$ the characteristic polynomial and the spectral radius of $\mathcal{C}_{\Delta_{[p_1,p_2,\ldots,p_s]}}$ respectively.

The following theorem is an answer to a problem concerning Coxeter polynomials posed by de la Pena, J.A. in his paper [P], for wild stars.

Theorem 2.1. *The Coxeter polynomial of a wild star has exactly two real roots and one irreducible non-cyclotomic factor.*

Proof. Let $f(x)$ be the Coxeter polynomial of a wild star. By [P] $f(x)$ has exactly two positive real roots δ and $1/\delta$. From Theorem 1.1/b it follows that these roots are its only real roots. Each non-real element of the spectrum of a wild star has absolute value equal to 1, thus the eigenvalues lie on the unit circle. Let $g(x)$ be the monic irreducible factor over $\mathbb{Q}$ having the root δ. If $1/\delta$ is not a root of $g(x)$ then the constant term of the polynomial $f(x)/g(x)$ has absolute value $1/\delta$ which is inpossible because $1/\delta$ is not an integer. Furthermore, the roots of $f(x)/g(x)$ lie on the unit cicle. A theorem of Kronecker states that if the roots of a monic polynomial with integer coefficients lie on the unit circle, then they are roots of unity. Thus, the only non-cyclotomic factor $g(x)$ of the Coxeter polynomial $f(x)$ is irreducible. □

Let us write

$$v_k = v_k(x) = (x^k - 1)/(x-1) \quad \text{for } k \in \mathbb{Z}_+.$$

It is known that among the trees with $s+1$ vertices the star of type $\Delta_{[1,1,\dots,1]}$ has the largest radius, viz. $\rho_{\Delta_{[1,1,\dots,1]}} = \sqrt{s}$ (see [CDS]); moreover by Theorem 1.1/a, we have $\rho(\mathcal{C}_{[1,1,\dots,1]}) < s-1$. The following theorem shows that upper bound of spectral radii depends on the degree of the branching point.

Theorem 2.2. *If $\Delta_{[p_1,p_2,\dots,p_s]} \neq \Delta_{[p_1,1,1,\dots,1]}$ is neither of Dynkin nor of Euclidean type, then*

$$s-2 < \rho(\mathcal{C}_{[p_1,p_2,\dots,p_s]}) < s-1 \quad \textit{if } 1 < p_i < \infty, \quad \textit{for all } 1 \le i \le s.$$

Proof. The bounds of spectral radii are determined by relation (see Corollary 1.2).

$$\rho(\mathcal{C}_{[p,p,\dots,p]} \le \rho(\mathcal{C}_{[p_1,p_2,\dots,p_s]}) \le \rho(\mathcal{C}_{[P,P,\dots,P]}),$$

where $p = \min\{p_1,p_2,\dots,p_s\}$ and $P = \max\{p_1,p_2,\dots,p_s\}$. For a wild star $\Delta_{[\underbrace{p,p,\dots,p}_{s\text{ times}}]}$, we get by Boldt's reduction formula [B]

$$\text{(1)} \qquad \begin{aligned} \chi_{[p,p,\dots,p]} &= v_{p+1}^{s-1}(x)(sv_{p+2}(x) - (s-1)(x+1)v_{p+1}(x)) \\ &= v_{p+1}^{s-1}(x)(x^{p+1} + 1 - (s-2)xv_p(x)). \end{aligned}$$

Furthermore, the non-cyclotomic factor $f(x) = x^{p+1} + 1 - (s-2)xv_p(x)$ for $p \geq 1$ satisfies $f(s-1) = s > 0$ and, for $p > 1$ in the case that $\Delta_{[p_1,p_2,\dots,p_s]}$ is neither of Dynkin nor of Euclidean type, we have $f(s-2) < 0$. Theorem 2.1 implies that the Coxeter polynomial has only one real zero greater than 1, thus $\rho(\mathcal{C}_{[p_1,p_2,\dots,p_s]}) < s - 1$.

Consider the graph $\Gamma = \Delta_{[2,2,\underbrace{1,\dots,1}_{s-2\text{ times}}]}$, $(s > 4)$ which is neither Dynkin nor Euclidean. One can calculate that

$$\chi_\Gamma = v_3(x)v_2(x)^{s-3}(x^4 - (s-3)x^3 - (s-2)x^2 - (s-3)x + 1),$$

and $s - 2 < \rho(\Gamma) < s - 1$.

If $p = 1$ then for $s = 3$ the wild star $\Delta_{[1,2,6]}$, for $s = 4$ the wild star $\Delta_{[3,2,1,1]}$ and for $s > 4$ the wild star $\Delta_{[2,2,1,\dots,1]}$ is a subgraph of $\Delta_{[p_1,p_2,\dots,p_s]} \neq \Delta_{[p_1,1,1,\dots,1]}$, consequently, its spectral radius is greater than $s - 2$. □

Remark. Consider the graph $\Gamma = \Delta_{[2,\underbrace{1,\dots,1}_{s-1\text{ times}}]}$, $s \geq 4$, which is no Dynkin. Easy calculation shows that the non-cyclotomic irreducible factor of χ_Γ is $x^4 - (s-3)x^3 - (s-3)x^2 - (s-3)x + 1$, and $s - 3 < \rho(\Gamma) < s - 2$.

Write

$$\boldsymbol{p}(t) = (p_1(t), p_2(t), p_3(t)) \quad \text{and} \quad p(t) = \min\{p_1(t), p_2(t), p_3(t)\}.$$

Theorem 2.3. *If* $\{\Delta_{[\boldsymbol{p}(t)]} \mid t \geq 1\}$ *is a sequence of wild stars and* $\lim_{t\to\infty} p(t) = \infty$ *then*

$$\lim_{t\to\infty} \rho(\mathcal{C}_{[\boldsymbol{p}(t)]}) = 2.$$

Proof. In order to prove the theorem, we apply Corollary 1.2. Indeed, consider $\Delta_{[p_1(t),p_2(t),p_3(t)]}$ as a substar of $\Delta_{[\boldsymbol{p}(t)]}$ and show that

$$\lim_{t\to\infty} \rho(\mathcal{C}_{[p(t),p(t),p(t)]}) = 2.$$

Write $p = p(t)$ and apply (1):

$$\chi_{[p,p,p]} = v_{p+1}^2(x)(x^{p+1} + 1 - xv_p(x)).$$

For $f(x) = x^{p+1} + 1 - xv_p(x)$ and $p > 2$ we have $f(1) = 2 - p < 0$ and $f(2) = 3 > 0$, i.e. its (only) real root which is greater than 1 lies in the interval $(1, 2)$. If $x_0 = \frac{2p-1}{p+1} = 2 - \frac{3}{p+1}$ and $p \geq 4$ then

$$f(x_0) = \frac{(2p-1)^{p+1}}{(p+1)^{p+1}} + 1 - \frac{2p-1}{p+1} \frac{\left(\frac{2p-1}{p+1}\right)^p - 1}{\frac{2p-1}{p+1} - 1}$$

$$= \frac{\left(\frac{p-2}{p+1} - 1\right)\left(\frac{2p-1}{p+1}\right)^{p+1} + \frac{p-2}{p+1} + \frac{2p-1}{p+1}}{\frac{p-2}{p+1}} = \frac{\frac{-3}{p+1}\left(\frac{2p-1}{p+1}\right)^{p+1} + 3}{\frac{p-1}{p+2}} < 0,$$

$$\text{since} \qquad \frac{(2p-1)^{p+1}}{(p+1)^{p+2}} > 1.$$

Consequently, the real root α of the reciprocal polynomial $f(x) = x^{p+1} + 1 - xv_p(x)$ which is greater than 1, satisfies $\alpha \in (2 - \frac{3}{p+1}, 2)$; this completes the proof. □

Theorem 2.4. *Let $p_1, p_2, \dots, p_{s-1}$ be a fixed sequence of positive integers. Then*

$$\lim_{k \to \infty} \rho(\mathcal{C}_{[p_1 p_2,, \dots, p_{s-1}, k]}) = x_o,$$

where x_o is the only positive real root of the polynomial $\chi_2(x) - \chi_1(x)$. Here $\chi_k(x) = \chi_{[p_1, p_2, \dots, p_{s-1}, k]}(x)$.

Proof. Write $\rho_k = \rho(\mathcal{C}_{[p_1, p_2, \dots, p_{s-1}, k]})$. By Theorem 1.2/a we have $\rho_k \leq s^2 - 2$ and the sequence of $\{\rho_k \mid k = 1, 2, \dots\}$ is monotone. Hence, there exists a limit point $\lim_{k \to \infty} \rho_k = x_0$. By Corollary 1.2,

$$(2) \qquad 1 < \rho_1 < \rho_2 < \dots \rho_{k+1} < \cdots < s^2.$$

Using the reduction formula of [B])

$$\chi_k(x) = (x+1)\chi_{k-1}(x) - x\chi_{k-2}(x),$$

we get

$$v_{k-1}(x)(\chi_2(x) - \chi_1(x)) + \chi_1(x) = \chi_k(x).$$

Taking $x = \rho_k$, we have $v_{k-1}(\rho_k)(\chi_2(\rho_k) - \chi_1(\rho_k)) = -\chi_1(\rho_k)$. Since the polynomial $\chi_1(x)$ is bounded on the interval $[\rho_1, s^2 - 1]$, we get

$$|\chi_2(\rho_k) - \chi_1(\rho_k)| = \frac{|\chi_1(\rho_k)|}{v_{k-1}(\rho_k)} \leq \frac{c}{v_{k-1}(1)} \leq \frac{c}{k},$$

where $c > 0$ is the maximal value of $|\chi_1(x)|$ for $x \in [\rho_1, s^2 - 2]$. It follows that for arbitrary $\epsilon > 0$, there exists $k \in \mathbb{N}$, such that $|\chi_2(\rho_k) - \chi_1(\rho_k)| < \epsilon$. Consequently $\lim_{k\to\infty} \chi_2(\rho_k) - \chi_1(\rho_k) = \chi_2(\rho) - \chi_1(\rho) = 0$. Thus, $x_0 = \lim_{k\to\infty} \rho_k$ is a positive root of $\chi_2(x) - \chi_1(x)$.

Using (2) (see Lemma 2.8 in [P]), we can show by induction on the number of edges that

$$\chi_2(x) = 1 + x - a_2x^2 - \cdots - a_{n-2}x^{n-2} + x^{n-1} + x^n,$$

$$\chi_1(x) = 1 + x - b_2x^2 - \cdots - b_{n-3}x^{n-3} + x^{n-2} + x^{n-1},$$

and $a_i \geq b_i > 0$ for $2 \leq i \leq n-3$. Therefore the coefficients of the polynomial $\chi_2(x) - \chi_1(x)$ have one change of sign, and it has only one positive root. □

Remark. Using Boldt's formulas, the notation $v_k = v_k(x)$ and the relations

$$(x+1)v_{p_i} - xv_{p_i-1} = v_{p_i+1}$$

and

$$v_{p_2}v_{p_1+1} - xv_{p_1}v_{p_2-1} = v_{p_1+p_2},$$

we obtain the following formulae for $\chi_{[p_1,p_2,p_3]}$

$$\begin{aligned}
&(x+1)v_{p_1+1}v_{p_2+1}v_{p_3+1} - x(v_{p_1}v_{p_2+1}v_{p_3+1} + v_{p_1+1}v_{p_2}v_{p_3+1}) - xv_{p_1+1}v_{p_2+1}v_{p_3}\\
&\quad = ((x+1)v_{p_1+1}v_{p_2+1} - xv_{p_1}v_{p_2+1} - v_{p_1+1}v_{p_2})v_{p_3+1} - xv_{p_1+1}v_{p_2+1}v_{p_3}\\
&\quad = (v_{p_1+2}v_{p_2+1} - xv_{p_1+1}v_{p_2})v_{p_3+1} - xv_{p_1+1}v_{p_2+1}v_{p_3}\\
&\quad = v_{p_1+p_2+2}v_{p_3+1} - xv_{p_1+1}v_{p_2+1}v_{p_3}\\
&\quad = v_{p_1+p_2+2}v_{p_3+1} - xv_{p_1+p_2+1}v_{p_3} + xv_{p_1+p_2+1}v_{p_3} - xv_{p_1+1}v_{p_2+1}v_{p_3}\\
&\quad = v_{p_1+p_2+p_3+2} + xv_{p_3}(v_{p_1+p_2+1} - v_{p_2+1}v_{p_3}) = v_{p_1+p_2+p_3+2} - x^2v_{p_1}v_{p_2}v_{p_3}.
\end{aligned}$$

Thus, $\chi_{[p_1,p_2,2]}(x) - \chi_{[p_1,p_2,1]}(x) = v_{p_1+p_2+4} - v_{p_1+p_2+3} - x^2v_{p_1}v_{p_2}v_2 + x^2v_{p_1}v_{p_2} = x^{p_1+p_2+3} - x^3v_{p_1}v_{p_2}$.

For example, we have

$$\lim_{m\to\infty} \rho(\mathcal{C}_{[1,3,m]}) = \text{the real root of the polynomial} \quad x^3 - x^2 - 1 \ (\sim 1.465).$$

By the above statement a particular case of Theorem 2.4 yield the following results ([CDS]):

$$\lim_{m\to\infty} \rho(\mathcal{C}_{[1,2,m]}) = \text{the real root of the polynomial } x^3 - x - 1 \ (\sim 1.3241).$$

and

$$\lim_{m\to\infty} \rho(\mathcal{C}_{[2,2,m]}) = \text{the positive real root of the polynomial } x^2 - x - 1.$$

Using Theorem 1.1/a and Proposition 3.6 in [H] we have

$$\lim_{m\to\infty} \rho(\mathcal{C}_{[2,2,m]}) = \lim_{m\to\infty} \rho(\mathcal{C}_{[1,m,m]}) = (1+\sqrt{5})/2 \ (\sim 1.61803).$$

References

[B] A. Boldt, Methods to determine Coxeter polynomials, *Linear Algebra Appl.* **230** (1995), 151–164.

[C] N. A'Campo, Sur les valeurs propres de la transformation de Coxeter, *Invent. Math.* **33** (1972), 61–67.

[CDS] D. M. Cvetkovič, M. Doob and H. Sachs, Spectra of graphs, *Johann Ambrosius Borth Verlag*, 1995.

[DR] V. Dlab and C. M. Ringel, Eigenvalues of Coxeter transformations and the Gelfand–Kirillov dimension of the preprojective algebras, *Proceedings AMS* **83** (1990), 228–232.

[H] A. J. Hoffman, On limit points of spectral radii of non-negative symmetric integral matrices, Graph theory and applications (Kalamazoo, MI, 1972), Lecture Notes in Math., vol. 303, *Springer*, *Berlin*, 1972, 165–172.

[P] J. A. de la Pena, Coxeter transformations and the representation theory of algebras, Finite dimensional algebras and related topics (Dlab, V. et al., ed.), *Ottawa*, *Canada*, August 10–18, 1992, Proceedings of the NATO Advanced Research Workshop on Representations of algebras and related topics; *Kluwer Academic Publishers, NATO ASI Ser., Ser. C, Math. Phys. Sci.* **424** (1994), 223–253.

[PT] J. A. de la Pena and M. Takane, Some bounds for the spectral radius of a Coxeter transformation, *Tsukuba J. Math.* **17** no. 1 (1993), 193–200.

[X] Ch. Xi, On wild algebras with the small growth number, *Comm. in Algebra* **18** (1987), 3413–3422.

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