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Diameter preserving linear bijections of $C_0(X)$

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Abstract. The purpose of this paper is to solve a linear preserver problem on the function algebra $C_0(X)$. In [GyM], we determined all diameter preserving linear bijections of $C(X)$ in the case when X is a first countable compact Hausdorff space. In this paper we generalize this result to the case of first countable locally compact Hausdorff spaces.

Linear bijections of $C(X)$ preserving some given norm have been studied in several papers; for references see e.g. [GyM]. Recently, we determined with L. Molnár [GyM] all linear bijections of $C(X)$ on a first countable compact Hausdorff space X which preserve the seminorm $f \mapsto \operatorname{diam}(f(X)) = \sup\{|f(x) - f(y)| \mid x, y \in X\}$. These linear maps are called diameter preserving. The aim of the present article is to generalize this theorem of [GyM] to $C_0(X)$ where X is a first countable locally compact Hausdorff space and $C_0(X)$ denotes the algebra of all continuous complex valued functions on X which vanish at ∞.

In our Theorem below we distinguish three cases, according as X is compact, σ-compact but not compact and not σ-compact, respectively. If in particular X is compact, our Theorem gives the above mentioned result of [GyM].

For a first countable locally compact Hausdorff space X with topology Λ, let X_0 denote $X \cup \{\infty\}$ if X is not compact, and X if X is compact. Then X_0 endowed with the topology

$$\Lambda_0 = \Lambda \cup \{X_0 \setminus K \mid K \subseteq X \text{compact}\}$$

is a compact Hausdorff space, and X is a subspace of X_0.

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Theorem. *Let X be a first countable locally compact Hausdorff space.*

1) *If X is compact, then a bijective linear map $\phi : C_0(X) \to C_0(X)$ is diameter preserving if and only if there exists a complex number τ of modulus 1, a homeomorphism $\varphi : X \to X$ and a linear functional $t : C_0(X) \to \mathbb{C}$ with $t(1) \neq -\tau$ such that ϕ is of the form*

$$\phi(f) = \tau \cdot f \circ \varphi + t(f)1 \qquad (f \in C_0(X)). \tag{1}$$

2) *If X is not σ-compact then a bijective linear map $\phi : C_0(X) \to C_0(X)$ is diameter preserving if and only if there exists a complex number τ of modulus 1 and a homeomorphism $\varphi : X \to X$ such that ϕ is of the form*

$$\phi(f) = \tau \cdot f \circ \varphi \qquad (f \in C_0(X)). \tag{2}$$

3) *If the space X is σ-compact but not compact, then a bijective linear map $\phi : C_0(X) \to C_0(X)$ is diameter preserving if and only if there exists a complex number τ of modulus 1 and a homeomorphism $\varphi : X_0 \to X_0$ such that ϕ is of the form*

$$\phi(f) = \tau \cdot f \circ \varphi - \tau f(\varphi(\infty)) \qquad (f \in C_0(X)), \tag{3}$$

where $f(\infty) = 0$ for every $f \in C_0(X)$.

Remark1. If in (3) $\varphi(\infty) = \infty$, then ϕ is of the same form as in (2).

Remark 2. If ϕ is of the form (2), then it is obviously a surjective isometry.

Remark 3. Our Theorem also holds for the algebra of all continuous real valued functions on X. In this situation we have $\tau = \pm 1$ and in (1) $t : C_0(X) \to \mathbb{R}$. In this case the proof is more simple.

Our proof consists of several steps. Some of them are similar to those of [GyM]. We shall detail only those steps which differ essentially from the corresponding arguments of [GyM].

Proof of Theorem. It is easy to verify that under the assumptions of the Theorem, the linear map ϕ of the form (1), (2) or (3), respectively, is a diameter preserving linear bijection of $C_0(X)$.

Now suppose that $\phi : C_0(X) \to C_0(X)$ is a linear bijection which preserves the diameter of the ranges of functions in $C_0(X)$.

Because of the natural isomorphism, we shall not make difference between $C(X_0)$ and $C_0(X)$, defining every function $f \in C_0(X)$ at the point ∞ as $f(\infty) = 0$. We note that $\operatorname{diam}(f(X)) = \operatorname{diam}(f(X_0))$ for any $f \in C_0(X)$.

We introduce the following notation. Let $\tilde{X}$ stand for the collection of all subsets of X having exactly two elements, and $\tilde{X_0}$ stand for the collection of all subsets of X_0 having exactly two elements. Let $\mathfrak{X}$ denote X_0 if X is σ-compact and X if X is not σ-compact. Similarly, let $\tilde{\mathfrak{X}}$ denote $\tilde{X_0}$ if X is σ-compact and $\tilde{X}$ if X is not σ-compact. For convenience of the reader, we follow the notation of [GyM] where it is possible. For any $f \in C_0(X)$ let

$$\begin{aligned} S(f) &= \{\{x,y\} \in \tilde{X_0} : |f(x) - f(y)| = \operatorname{diam}(f(X))\}, \\ P(f) &= \{(x,y) \in X_0 \times X_0 : |f(x) - f(y)| = \operatorname{diam}(f(X))\}, \\ T(f) &= \{(x,y,u) \in X_0 \times X_0 \times \mathbb{C} : |f(x) - f(y)| = \operatorname{diam}(f(X)), \\ &\qquad u = f(x) - f(y)\}. \end{aligned}$$

Further, for every $\{x,y\} \in \tilde{X_0}$ and $u \in \mathbb{C}$ let

$$\begin{aligned} \mathcal{S}(\{x,y\}) &= \{f \in C(X_0) : \{x,y\} \in S(f)\}, \\ \mathcal{S}_s(\{x,y\}) &= \{f \in C(X_0) : \{\{x,y\}\} = S(f)\}, \\ \mathcal{T}(x,y,u) &= \{f \in C(X_0) : (x,y,u) \in T(f)\}, \\ \mathcal{T}_s(x,y,u) &= \{f \in C(X_0) : \{(x,y,u),(y,x,-u)\} = T(f)\}. \end{aligned}$$

Finally, we define

$$\begin{aligned} G(\{x,y\}) &= \cap\{S(\phi(f)) : f \in C_0(X),\ \{x,y\} \in S(f)\}, \\ H(x,y,u) &= \cap\{T(\phi(f)) : f \in C_0(X),\ (x,y,u) \in T(f)\}. \end{aligned}$$

Let

$$\mathcal{D} = \{f \in C(X_0) : \exists \{x,y\} \in \tilde{\mathfrak{X}} : \{\{x,y\}\} = S(f)\}.$$

It is clear that for every nonconstant function $f \in C(X_0)$, the sets $S(f)$, $P(f)$ and $T(f)$ are nonempty. Since X is first countable, using Uryson's lemma it is easy to see that for every distinct $x, y \in X$ there exists a continuous real valued function $f \in C(X_0)$ from X_0 into $[-1,1]$ such that

$f(x) = 1, f(y) = -1$ and $-1 < f(z) < 1$ $(z \in X,\ z \neq x,\ z \neq y)$. If X is σ-compact, then X_0 is first countable and similarly, for every distinct $x, y \in X_0$ there exists a real valued function $f \in C(X_0)$ from X_0 into $[0,1]$ such that $f(x) = 1,\ f(y) = 0$ (we may assume that $x \neq \infty$) and $0 < f(z) < 1$ $(z \in X,\ z \neq x,\ z \neq y)$. This shows that for any element $\{x,y\} \in \tilde{\mathcal{X}}$ and any non-zero $u \in \mathbb{C}$, the sets $\mathcal{S}_s(\{x,y\})$, $\mathcal{T}_s(x,y,u)$ are also nonempty. It is obvious that the sets $\mathcal{S}(\{x,y\})$, $\mathcal{T}(x,y,u)$ are nonempty for any $\{x,y\} \in \tilde{X}_0$ and any non-zero $u \in \mathbb{C}$.

We begin now the proof of the necessity of our statements which will be carried out through a series of steps. The following lemma will be used repeatedly in our proof. Its proof as well as the proofs of Steps 1 and 2 below are the same as the proofs of the Lemma and Steps 1 and 2 in [GyM], it suffices only to replace X and $\tilde{X}$ by X_0 and $\tilde{X}_0$, respectively.

Lemma. *Let $f_1, \dots, f_n \in C_0(X)$ be arbitrary functions. Then*

$$\operatorname{diam}((f_1 + \dots + f_n)(X)) = \operatorname{diam}(f_1(X)) + \dots + \operatorname{diam}(f_n(X))$$

holds if and only if there exists an $\{x,y\} \in \tilde{X}_0$ and a complex number v of modulus 1 such that $f_i \in \mathcal{T}(x,y,\lambda_i v)$ holds for every $i = 1, \dots, n$, where $\lambda_i = \operatorname{diam}(f_i(X))$ $(i = 1, \dots, n)$.

Step 1. *For arbitrary $\{x,y\} \in \tilde{X}_0$ and $0 \neq u \in \mathbb{C}$, we have $G(\{x,y\}) \neq \emptyset$ and $H(x,y,u) \neq \emptyset$.*

Step 2. *If $\{x_1,y_1\}, \{x_2,y_2\} \in \tilde{X}_0$ and $\{x_1,y_1\} \neq \{x_2,y_2\}$, then we have $G(\{x_1,y_1\}) \cap G(\{x_2,y_2\}) = \emptyset$.*

Step 3. *We have $f \in \mathcal{D}$ if and only if $\phi(f) \in \mathcal{D}$.*

Let $f \in \mathcal{D}$. Then there exists $\{x,y\} \in \tilde{\mathcal{X}}$ such that $f \in \mathcal{S}_s(\{x,y\})$. Let $f_0 = \phi^{-1}(f)$ and let $\{x_0,y_0\} \in S(f_0)$ be arbitrary. Then

$$\emptyset \neq G(\{x_0,y_0\}) \subseteq S(f) = \{\{x,y\}\},$$

and so

$$G(\{x_0,y_0\}) = \{\{x,y\}\}.$$

Since $\{x_0,y_0\} \in S(f_0)$ is arbitrary, $S(f_0)$ has exactly one element by Step 2, thus $\phi^{-1}(f) \in \mathcal{D}$. Applying this result to the diameter preserving bijection ϕ^{-1} instead of ϕ, the proof of Step 3 is complete.

Step 4. *For every $\{x,y\} \in \tilde{\mathfrak{X}}$, the set $G(\{x,y\})$ has exactly one element which is contained in $\tilde{\mathfrak{X}}$. The function $G' : \tilde{\mathfrak{X}} \to \tilde{\mathfrak{X}}$ defined by $\{G'(\{x,y\})\} = G(\{x,y\})$ is a bijection.*

Let $\{x,y\} \in \tilde{\mathfrak{X}}$ and $f \in \mathcal{S}_s(\{x,y\})$. Since $f \in \mathcal{D}$, by Step 3 we have $\phi(f) \in \mathcal{D}$, thus $S(\phi(f))$ has exactly one element which is in $\tilde{\mathfrak{X}}$. Hence from

$$\emptyset \neq G(\{x,y\}) \subseteq S(\phi(f))$$

we deduce that $G(\{x,y\})$ has also exactly one element which is contained in $\tilde{\mathfrak{X}}$.

We now prove that the function G' is bijective. In view of Step 2 the injectivity is obvious. To prove the surjectivity, let $\{x,y\} \in \tilde{\mathfrak{X}}$ and pick $f \in C_0(X)$ for which $\phi(f) \in \mathcal{S}_s(\{x,y\})$. Then $\phi(f) \in \mathcal{D}$, so by Step 3 we infer that $f \in \mathcal{D}$. Thus there exists $\{x_0,y_0\} \in \tilde{\mathfrak{X}}$ for which $S(f) = \{\{x_0,y_0\}\}$. Hence we have $G'(\{x_0,y_0\}) \in S(\phi(f)) = \{\{x,y\}\}$, thus $G'(\{x_0,y_0\}) = \{x,y\}$ verifying our claim.

Step 5. *Let $\{x,y\} \in \tilde{\mathfrak{X}}$ and $f \in C(X_0)$ be arbitrary. If $\phi(f) \in \mathcal{S}_s(G'(\{x,y\}))$, then $f \in \mathcal{S}_s(\{x,y\})$.*

If $\{x_0,y_0\} \in S(f)$ is arbitrary, then $G'(\{x_0,y_0\}) \in S(\phi(f)) = \{G'(\{x,y\})\}$. Thus by Step 4 $\{x_0,y_0\} = \{x,y\}$, hence $S(f) = \{\{x,y\}\}$.

Step 6. *Defining the function G'_{-1} corresponding to ϕ^{-1} in the same way as G' corresponding to ϕ was defined in Step 4, we have $G'_{-1} = (G')^{-1}$.*

The proof is similar to the proof of Step 6 in [GyM], but with $\tilde{\mathfrak{X}}$ instead of $\tilde{X}$.

Step 7. *If $\{x_1,y_1\},\{x_2,y_2\} \in \tilde{\mathfrak{X}}$ and $\{x_1,y_1\} \cap \{x_2,y_2\} \neq \emptyset$, then we have*

$$G'(\{x_1,y_1\}) \cap G'(\{x_2,y_2\}) \neq \emptyset.$$

Further, if $\{x_1,y_1\},\{x_2,y_2\} \in \tilde{\mathfrak{X}}$ have exactly one element in common, then the same holds for $G'(\{x_1,y_1\})$ and $G'(\{x_2,y_2\})$.

Let $\{x,y_1\},\{x,y_2\} \in \tilde{\mathfrak{X}}$ with $y_1 \neq y_2$, and suppose that

$$G'(\{x,y_1\}) \cap G'(\{x,y_2\}) = \emptyset.$$

Then we may assume that $\infty \notin G'(\{x,y_2\})$. Let $K \subseteq X \setminus G'(\{x,y_1\})$ be compact such that $G'(\{x,y_2\}) \subseteq K^{\circ}$. Then it follows from the surjectivity of ϕ that there exist functions $f_1, f_2 \in C_0(X)$ with the following properties. The support of $\phi(f_2)$ is a subset of K, the range of $\phi(f_1)$ is included in $[0,1]$, the range of $\phi(f_2)$ is included in $[-1/2, 1/2]$, $\phi(f_1)$ is $1/2$ on the set K,

$$\phi(f_1) \in \mathcal{S}_s(G'(\{x, y_1\})), \qquad \phi(f_2) \in \mathcal{S}_s(G'(\{x, y_2\}))$$

and, finally,

$$\operatorname{diam}(\phi(f_1)(X)) = \operatorname{diam}(\phi(f_2)(X)) = 1.$$

Now $f_1, f_2 \in C_0(X)$ are functions with diameter 1 and by Step 5 we infer that $f_1 \in \mathcal{S}_s(\{x, y_1\})$, $f_2 \in \mathcal{S}_s(\{x, y_2\})$. Now we arrive at a contradiction as in the proof of Step 5 in [GyM].

The second statement of Step 7 follows now from Step 2.

Step 8. *Let $\{x_1, y_1\}, \{x_2, y_2\} \in \tilde{\mathcal{X}}$. Then $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$ if and only if $G'(\{x_1, y_1\}) \cap G'(\{x_2, y_2\}) = \emptyset$.*

The sufficiency follows from Step 7. By Steps 6 and 7 the necessity is obvious.

Step 9. *Let $x \in \mathcal{X}$. There exists a unique element $g(x) \in \mathcal{X}$ such that $g(x) \in G'(\{x, y\})$ for every $x, y \in \mathcal{X}, x \neq y$. The function $g : \mathcal{X} \to \mathcal{X}$ is bijective and $\{g(x), g(y)\} = G'(\{x, y\})$ ($\{x, y\} \in \tilde{\mathcal{X}}$).*

The proof is similar to the proof of Step 7 in [GyM], it suffices to take $\mathcal{X}$ and $\tilde{\mathcal{X}}$ instead of X and $\tilde{X}$, respectively.

Step 10. *There exists a complex number τ of modulus 1 such that, for every $\{x, y\} \in \tilde{\mathcal{X}}$, $0 \neq u \in \mathbb{C}$ and $f \in \mathcal{T}(x, y, u)$, we have $\phi(f) \in \mathcal{T}(g(x), g(y), \tau u)$.*

Similarly as in the proof of Step 8 in [GyM], but using $\tilde{\mathcal{X}}$ instead of $\tilde{X}$, we obtain that for every $\{x, y\} \in \tilde{\mathcal{X}}$ there exists a complex number $\tau(\{x, y\})$ of modulus 1 such that the implication

$$f \in \mathcal{T}(x, y, u) \Longrightarrow \phi(f) \in \mathcal{T}(g(x), g(y), \tau(\{x, y\})u) \tag{4}$$

holds for every $u \in \mathbb{C}$. It remains to show that τ does not depend on its variable $\{x, y\}$.

Now, we obtain as in the proof of Step 8 in [GyM] that τ is a constant function on $\tilde{X}$. Let this constant be denoted by the same symbol τ.

Let us suppose that X is σ-compact but not compact, $x \in X$ and $f \in \mathcal{T}_s(x, \infty, 1)$. Let $z_n \in X$ with $z_n \to \infty$ and $z_n \neq x$. Since X_0 is compact and $g : X_0 \to X_0$ is a bijection, we may assume that there exists $y \in X_0$ for which $g(z_n) \to g(y)$. It is easy to see that there exist $f_n \in \mathcal{T}(x, z_n, 1)$ such that $f_n \to f$. Hence $\phi(f_n) \in \mathcal{T}(g(x), g(z_n), \tau))$. Since X is not compact and ϕ is continuous, thus from $f, \phi(f) \in C_0(X)$, $\phi(f_n)(g(x)) - \phi(f_n)(g(z_n)) = \tau$, $g(z_n) \to g(y)$ and $f_n \to f$ we deduce that

$$\phi(f)(g(x)) - \phi(f)(g(y)) = \tau.$$

Thus from $|\tau| = 1 = |\tau(\{x, \infty\})|$ and $\phi(f) \in \mathcal{T}_s(g(x), g(\infty), \tau(\{x, \infty\}))$ we infer that $g(y) = g(\infty)$, so

$$\tau(x, \infty) = \phi(f)(g(x)) - \phi(f)(g(\infty)) = \phi(f)(g(x)) - \phi(g(y)) = \tau.$$

Hence $\tau : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ is the constant τ, and the assertion follows from (4).

Step 11. *For every $f \in C_0(X)$, the function $\phi(f) \circ g - \tau \cdot f$ is constant on $\mathcal{X}$.*

Let $\{x, y\} \in \tilde{\mathcal{X}}$ and $f \in \mathcal{T}(x, y, 1)$. Now, similarly as in the proof of Step 9 in [GyM], we can prove that

$$\phi(f)(g(z)) - \tau f(z) = \phi(f)(g(x)) - \tau f(x) \tag{5}$$

holds for every $z \in X$.

If $\mathcal{X} = X$, then we are ready. Let us suppose that X is σ-compact but not compact. In the proof of Step 10 we showed that then there exist $z_n \in X$, $z_n \to \infty$ such that $g(z_n) \to g(\infty)$. Thus from (5) we infer that

$$\phi(f)(g(\infty)) - \tau f(\infty) = \lim_{n\to\infty} (\phi(f)(g(z_n)) - \tau f(z_n)) = \phi(f)(g(x)) - \tau f(x),$$

which proves the statement of Step 11.

Now we can complete the proof of the Theorem as follows. By the linearity of ϕ, there exists a linear functional $t : C_0(X) \to \mathbb{C}$ such that

$$\phi(f) \circ g - \tau \cdot f = t(f)1 \qquad (f \in C_0(X)).$$

Since $g : \mathcal{X} \to \mathcal{X}$ is a bijection, with the notation $\varphi = g^{-1}$ we have

$$\phi(f) - \tau \cdot f \circ \varphi = t(f)1 \qquad (f \in C_0(X)). \tag{6}$$

It follows from (6) that $f \circ \varphi$ is continuous for every $f \in C_0(X)$. Using Uryson's lemma, we deduce that φ is continuous. If X is σ-compact, then $\mathcal{X}$ is compact, so φ is a continuous bijection between compact Hausdorff spaces, thus φ is a homeomorphism. Let us consider the case when X is not σ-compact. Let us suppose that $x_n \in X$ such that $x_n \to \infty$ and suppose on the contrary that $x_n \to y \in X$. Then there exists $y_0 \in X$ such that $\varphi(y_0) = y$. Now by (6) we have

$$\phi(f)(y_0) - \tau \cdot f(\varphi(y_0)) = \phi(f)(x_n) - \tau \cdot f(\varphi(x_n)) \to$$
$$\phi(f)(\infty) - \tau \cdot f(\varphi(y_0)) = -\tau \cdot f(\varphi(y_0)),$$

thus $\phi(f)(y_0) = 0$ for every $f \in C_0(X)$, which is a contradiction. Now, defining φ at the point ∞ as $\varphi(\infty) = \infty$, $\varphi : X_0 \to X_0$ is a continuous bijection between compact Hausdorff spaces, thus $\varphi : X \to X$ is a homeomorphism.

If X is compact, then we are ready, since the relation $t(1) \neq -\tau$ is obvious and $\mathcal{X} = X$.

If X is not σ-compact, then for any $z_n \in X$ with $z_n \to \infty$ we have $\varphi(z_n) \to \infty$, since $\varphi : X \to X$ is a homeomorphism. Thus, by (6), we have

$$t(f) = \phi(f)(z_n) - \tau \cdot f(\varphi(z_n)) \to \phi(f)(\infty) - \tau \cdot f(\infty) = 0$$

for every $f \in C_0(X)$, which completes the proof.

Finally, suppose that X is σ-compact but not compact. Then $\varphi : X_0 \to X_0$ is a homeomorphism and by (6) we deduce that

$$t(f) = \phi(f)(\infty) - \tau \cdot f(\varphi(\infty)) = -\tau f(\varphi(\infty))$$

for every $f \in C_0(X)$. The proof of the Theorem is now complete. □

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