Extrinsic spheres of a generalized Hopf manifold

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To the memory of Professor András Rapcsák

Abstract. In the present article we study the geometry of extrinsic spheres in locally conformal Kaehler manifolds with parallel Lee form (generalized Hopf manifolds). Our method consists in analysing the positions of the mean curvature vector field with respect to the three canonical distributions of a g.H.m.. Using two celebrated theorems of M. OBATA we determine sufficient conditions for an extrinsic sphere to be isometric with a standard one.

1. Introduction

Initially K. NOMIZU called *sphere* a totally umbilical submanifold with parallel mean curvature vector field in an arbitrary Riemannian manifold. Since this definition reflects an extrinsic property of the submanifold, the name *extrinsic sphere* was found more adequate.

One dimensional extrinsic spheres are called simply *circles*. They are the *geodesic* (or *curvature*) *circles* previously considered by S. LIE, G. DARBOUX and W. BLASCHKE. It was proved in [No-Ya] that an isometric submanifold of a Riemannian manifold $\tilde{M}$ is an extrinsic sphere if and only if every circle on it is also a circle in $\tilde{M}$.

A natural question when studying such objets is: when is an extrinsic sphere isometric with a standard sphere? Every classification attempts to answer it. For a Kaehlerian ambient space the classification was carried out in [Ch] while for a Sasakian ambient in [Ya-Ne-Ka] and [Ha].

In this paper we discuss extrinsic spheres in a particular class of Hermitian manifolds, namely the generalized Hopf manifolds (g.H.m.). The

second section presents the necessary background about g.H.m. and their submanifolds as well as other definitions we shall need later. In the third section we describe the canonical distributions of a g.H.m. The fourth section discusses the mean curvature vector of an extrinsic sphere in connection with these distributions while in the fifth section we present a classification of extrinsic spheres in generalized Hopf manifolds and provide a class of examples. Finally we discuss the case of a locally conformally flat ambient manifold.

Some of the results in this paper were announced in [Ia-Or-Ma].

2. Preliminaries

All manifolds and geometric objects on them are supposed differentiable of class $\mathcal{C}^\infty$. The notations are standard.

A *locally conformal Kaehler* manifold is a Hermitian manifold $(\tilde{M}, J, g)$ of complex dimension $\tilde{m}$ whose metric is conformally related to a Kaehlerian one in a neighborhood of each of its points (cf. [Val]). Denoting with Ω the fundamental 2-form on $\tilde{M}$ (defined by $\Omega(X, Y) = g(X, JY)$) we have an alternative definition in the following

Theorem 2.1. *A Hermitian manifold is a locally conformal Kaehler manifold if and only if the equation*

$$d\Omega = \omega \wedge \Omega$$

defines a closed global 1-form ω (called the Lee form).

We shall suppose ω without singularities; hence it is possible to consider the unitary 1-form $u = \omega/2c$, $2c = \|\omega\|$. We set $U = u^\#$, $V = -JU$, $v = u \circ J$.

If ω is parallel with respect to the Levi–Civita connection $\tilde{\nabla}$ of $\tilde{M}$ (or, equivalently, $c = \text{ct.}$ and $\tilde{\nabla}u = 0$) then $\tilde{M}$ is called a *generalized Hopf manifold* (g.H.m.). This name is motivated by the example of the complex Hopf manifold $S^1 \times S^{2n-1}$ (which, for topological reasons, is known not to admit Kaehlerian metrics) endowed with the Boothby metric (cf. [Val]) $ds^2 = (\sum dz^i \otimes d\bar{z}^i/|z|^2$. The structure of a compact g.H.m. whose distribution generated by the vector fields U and V is regular in the sense of R. PALAIS is best understood:

Theorem 2.2. *A compact, connected regular generalized Hopf manifold $\tilde{M}$ is a $T_{\mathbb{C}}^1$-principal analytic fibre bundle over a Hodge manifold (for a well determined 1-dimensional complex torus).*

On a g.H.m. the covariant derivative of the complex structure is given by the formula

$$(2.1) \qquad (\tilde{\nabla}_X J)Y = c\left\{-g(X,Y)V - \Omega(X,Y)U + v(Y)X - u(Y)JX\right\}$$

which easily implies

$$(2.2) \qquad \tilde{\nabla}_X V = c\left\{u(X)V - v(X)U - JX\right\}.$$

Let now M be an isometric submanifold of real dimension $m \geq 2$. We denote with ∇ (resp. $\nabla^{\perp}$) the metric connexion induced in the tangent (resp. normal) bundle of M and with h (resp. A_N) the second fundamental form (resp. the Weingarten operator in the normal direction N). We denote with g the metric induced on M.

The vector field $H = (1/m)$ trace $_g h \in \mathcal{C}^{\infty}(T^{\perp}M)$ (for a vector bundle $E \to M$ we denote with $\mathcal{C}^{\infty}(E)$ the module of its differentiable sections; we consider only vector bundles over M) is called the mean curvature vector field of the submanifold. If $h = g \otimes H$, M is called a totally umbilical submanifold; if, moreover, H is non-zero and parallel in the normal bundle then M is an extrinsic sphere. In this last case $k = \|H\|$ is a non-zero constant and, letting $H = k\mu$, the Gauss and Weingarten formulae read:

$$\tilde{\nabla}_X Y = \nabla_X Y + kg(X,Y)\mu$$
$$\tilde{\nabla}_X N = -kg(N,\mu) + \nabla_X^{\perp} N.$$

In particular we hawe

$$(2.3) \qquad \tilde{\nabla}_X \mu = -kX.$$

To answer the question stated in the Introduction we shall make essential use of two results of M. OBATA that we now recall:

Theorem 2.3 ([Ob1,2]). *Let M^m ($m \geq 2$) be a complete, connected and simply connected Riemannian manifold. Then M is isometric with a sphere of radius $1/\sqrt{k}$ if and only if there exists a nonconstant function f on M satisfying one the following equations:*

$$(1) \qquad \nabla df + kfg = 0$$

$$(2) \qquad \nabla\nabla df + k\sum_{\text{cicl}} 2df \otimes g + kdf \otimes g = 0.$$

To conclude the preliminaries we recall the definitions of two structures which will appear in our classification.

Firstly a (1,1) tensor field ϕ and a 1-form η on a Riemannian manifold (N, g) satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ g(\phi X, Y) + g(X, \phi Y) = 0, \ 2ag(\phi X, Y) = d\eta(X, Y)$$

where $\xi = \eta^{\#}$, define a metric (homothetic, if $a \neq 1$) contact structure. (cf. [Bl]). If the almost complex structure defined on $N \times \mathbb{R}$ by $J(X, b\frac{d}{dt}) = (\phi X - b\xi, \ \eta(X)\frac{d}{dt})$ is integrable then the structure is called Sasakian. The standard contact structure of an odd-dimensional sphere is a typical example.

On the other hand, a real submanifold M of a Hermitian manifold $(\tilde{M}, J, g)$ is a CR submanifold (cf. [Be]) if it is endowed with a pair of complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$, the first being holomorphic ($JX \in \mathcal{C}^{\infty}(\mathcal{D})$, $X \in \mathcal{C}^{\infty}(\mathcal{D})$) and the second totally real ($JX \in \mathcal{C}^{\infty}(T^{\perp}M)$, $X \in \mathcal{C}^{\infty}(\mathcal{D}^{\perp})$).

3. Vector distributions on a generalized Hopf manifold

Let $\mathcal{D}^1$ be the 1-dimensional differentiable distribution generated by the Lie vector field U (which, by assumption, has no singularities). It is parallel with respect to the Levi–Civita connection $\tilde{\nabla}$ of $\tilde{M}$. Let $\mathcal{D}^2$ be the distribution generated by the vector field V and $\mathcal{D}^3$ the Kaehlerian distribution (studied mainly in [Ia-Or]) whose sections are orthogonal to both U and V. One thus obtains an orthogonal decomposition $TM = \mathcal{D}^1 \oplus \mathcal{D}^2 \oplus \mathcal{D}^3$. Let $\mathcal{D}$ be the distribution $\mathcal{D}^2 \oplus \mathcal{D}^3$.

It is known (cf. [Ch-Pi], [Val,2]) that $\mathcal{D}$ is integrable and totally geodesic. This is a consequence of the parallelism of U.

For each $X \in \mathcal{C}^{\infty}(\mathcal{D})$ we put

(3.1) $$JX = \phi X + \eta(X)U$$

where $\phi \in \mathcal{C}^{\infty}(\mathrm{Hom}(\mathcal{D}, \mathcal{D}))$ and $\eta \in \mathcal{C}^{\infty}(\mathcal{D}^*)$.
It is easily seen that $\eta = u \circ J$ and that

(3.2) $$\phi^2 = -I + v \otimes V \, .$$

Let ∇' be the connexion induced by $\tilde{\nabla}$ in the subbundle $\mathcal{D}$. One has

Proposition 3.1. *The following formulas hold on a generalized Hopf manifold:*

(3.3)
$$(\nabla'_X \phi)Y = c\left\{v(X)Y - g(X,Y)V\right\}$$

(3.4)
$$\nabla'_X V = c\phi X.$$

Proof. Using (2.1) and (3.1) we find

$$\nabla'_X \phi Y + X(\eta(Y))U + \eta(Y)\nabla_X U - \phi\nabla'_X Y - \eta(\nabla'_X Y)U =$$
$$= c\left\{g(X,Y)JU + u(JX)Y\right\}$$

for $X, Y \in \mathcal{C}^\infty(\mathcal{D})$. Comparing the components in $\mathcal{C}^\infty(\mathcal{D})$ and in $\mathcal{C}^\infty(\mathcal{D}^1)$ on the two sides of the previous equality one directly obtains (3.3) and

$$X(\eta(Y)) - \eta(\nabla'_X Y) = -c\Omega(X,Y),$$

a relation which may be written

$$X(g(V,Y) - g(\nabla'_X Y, V) = -c\Omega(X,Y).$$

This, combined with (3.1), leads to:

$$g(\nabla'_X V, Y) = cg(\phi X, Y)$$

for each $Y \in \mathcal{C}^\infty(\mathcal{D})$ and the proof is complete.

In a similar manner one verifies that ϕ is skewsymmetric (with respect to g) and satisfies the formula

$$g(\phi X, \phi Y) = -g(X,Y) + v(X)v(Y).$$

We may conclude:

Corollary 3.2 [Val]. *An integral manifold of maximal dimension of the distribution $\mathcal{D}$ bears a homothetic Sasakian structure.*

4. Mean curvature vector of an extrinsic sphere.
Properties related to the canonical distributions

Let M be an extrinsic sphere of the generalized Hopf manifold $\tilde{M}$, with mean curvature vector field H. Let us put

(4.1)
$$J\mu = \mu' + \mu''$$

where μ' (resp. μ'') is the tangent (resp. normal) part of $J\mu$. Similarly, we denote by U' (U''), V' (V'') the tangent (normal to M and μ) component of U, V. Then we have

$$(4.2) \qquad\qquad U = U' + U'' + g(U, \mu)\mu$$

$$(4.3) \qquad\qquad V = V' + V'' + g(V, \mu)\mu \,.$$

If $U \in \mathcal{C}^\infty(T^\perp M)$ (note that this also implies $\mathcal{C}^\infty(TM) \subset \mathcal{C}^\infty(\mathcal{D})$) we define $\phi' \in \mathcal{C}^\infty(\mathrm{Hom}(\mathcal{D}, \mathcal{D}))$ and $\phi'' \in \mathcal{C}^\infty(\mathrm{Hom}(\mathcal{D}, T^\perp M))$ by means of the decomposition formula

$$(4.4) \qquad\qquad \phi X = \phi' X + \phi'' X - g(X, \mu')\mu \,.$$

Lemma 4.1. *Let M be an extrinsic sphere orthogonal to the Lee vector field U of the generalized Hopf manifold $\tilde{M}$. If H is also orthogonal to U then*

$$(4.5) \qquad (\nabla'_X \phi')Y = c\{g(V', Y)X - g(X, Y)V'\}+$$
$$+ k\{g(X, Y)\mu' - g(Y, \mu')X\}$$

$$(4.6) \qquad (\nabla'_X \phi'')Y \stackrel{\mathrm{def}}{=} \nabla_X^\perp \phi'' Y - \phi'' \nabla_X Y = g(X, Y)\{k\mu'' - cU''\} \,.$$

PROOF. The proof consists in a direct computation. Because $\mathcal{D}$ is a totally geodesic foliation we have

$$(\nabla'_X \phi)Y = \nabla'_X \phi Y - \phi\nabla'_X Y = \nabla'_X \phi' Y + \nabla'_X \phi'' Y - Xg(Y, \mu')\mu+$$
$$+ kg(Y, \mu')X - \phi\{\nabla_X Y + kg(X, Y)\mu\} =$$
$$= \nabla_X \phi' Y + kg(X, \phi' Y)\mu + \nabla_X^\perp \phi'' Y - Xg(Y, \mu')\mu + kg(Y, \mu')X-$$
$$- \phi\nabla_X Y - \phi''\nabla_X Y + g(\nabla_X Y, \mu')\mu - kg(X, Y)\mu' - kg(X, Y)\mu'' \,.$$

The conclusion now follows from Proposition 3.1 and by comparing the tangent and the normal components on the two sides of the above equation.

We may now pass to the examination of the three mutually exclusive positions of the mean curvature vector with respect to the canonical distributions, namely:
  a) H is not orthogonal to $\mathcal{D}^1$
  b) H is orthogonal to $\mathcal{D}^1$ but not to $\mathcal{D}^2$
  c) $H \in \mathcal{C}^\infty(\mathcal{D}^3)$.
    Let us first note that H cannot be a section of $\mathcal{D}^1$. Indeed, this will be a consequence of

Proposition 4.2. *If the function $f = g(U, \mu)$ is constant on M then it is identically zero.*

PROOF. We have successively:

$$Xg(U, \mu) = g(\tilde{\nabla}_X U, \mu) + g(U, \tilde{\nabla}_X \mu) = -kg(U', X).$$

If f is constant then $g(U', X) = 0$ so that $U' = 0$. On the other hand, covariant differentiation of (4.2) implies $\nabla_X U' = kg(U, \mu)X$. As $U' = 0$, we conclude that $U \perp \mu$ and the proof is complete.

Now, if $U = U'$, then $H \perp U$ so $H \in \mathcal{C}^\infty(\mathcal{D}^1)$; thus H is not orthogonal to $\mathcal{D}^1$ and from the above proof we derive the

Corollary 4.3. *There is no extrinsic sphere tangent to the Lee vector field U of a generalized Hopf manifold.*

As a direct consequence of formulae (2.3) and (3.4) we obtain

Lemma 4.4. *For an extrinsic sphere of a generalized Hopf manifold the following formula holds*

$$Xg(V, \mu) = -kg(V, X) - cg(X, \mu') \quad X \in \mathcal{C}^\infty(\mathcal{D}).$$

Lemma 4.5. *Suppose H is orthogonal to $\mathcal{D}^1$ and $f = g(V, \mu)$ is constant on M. Then*

i) $\nabla_X U' = kg(\mu, V') + c\phi' X$
ii) $\nabla_X^\perp V'' = c\phi'' X$
iii) $g(V, \mu) \equiv 0$ *on M.*

PROOF. If f is a constant, from Lemma 4.4 we conclude that μ' and V' are colinear and $\mu' = -(k/c)V'$. Then the three relations are derived from the covariant differentiation of (4.2), (4.3) and of the Gauss and Weingarten formulae.

Finally, a similar computation yields

Lemma 4.6. *If H is orthogonal to $\mathcal{D}^1 \oplus \mathcal{D}^2$ and $Y \in \mathcal{C}^\infty(\mathcal{D})$ then*

$$(\nabla_X \phi')Y = (c + k^2/c)\{g(V', Y)X - g(X, Y)V'\}.$$

5. Main results

We now examine the three positions of the mean curvature vector field of an extrinsic sphere M with respect to the canonical distributions and, in each case, we find a function satisfying one of the equations in Obata's theorem. When we cannot prove the nonconstancy of this function, and thus cannot derive the isometry with a standard sphere, we point out some additional structures, compatible with the metric, on the submanifold.

In the first case, a), when H is not orthogonal to $\mathcal{D}^1$, letting $f = g(U, \mu)$ we find

$$df(X) = X(g(U, \mu)) = -kg(U', X)$$
$$(\nabla df)(X, Y) = (\nabla_X df)(Y) = -k\{\nabla_X g(U', Y) - g(U', \nabla_X Y)\} =$$
$$= kg(\nabla_X U', Y) = -kg(A_{U''} X, Y) = -k^2 fg(X, Y),$$

thus f satisfies equation (1).

In the second case, b), when H is orthogonal to $\mathcal{D}^1$ but not to $\mathcal{D}^2$ we let $f = g(V, \mu)$ and, as above, we see that f satisfies equation (1). Taking into account Proposition 4.2 and Lemma 4.5 we may state

Theorem 5.1. *Let M be a connected, simply connected, complete extrinsic sphere of a generalized Hopf manifold. If its mean curvature vector field is not a section of the Kaehlerian distribution $\mathcal{D}^3$, then M is isometric with a standard sphere of radius $1/\sqrt{k}$.*

The situation is much more complicated when H assumes its last position, c), i.e. $H \in \mathcal{C}^\infty(\mathcal{D}^3)$. We first show that $f = g(V', V')$ verifies equation (2) if U is normal to M. Indeed:

$$df(X) = 2g(\nabla_X V', V') = 2cg(\phi' X, V') = -2cg(X, \phi' V')$$
$$(\nabla df)(X, Y) = X(df(Y)) - df(\nabla_X Y) = -cXg(Y, \phi' V') +$$
$$+ 2cg(\nabla_X Y, \phi' V') = -2cg(Y, \nabla_X(\phi' V')).$$

Taking into account Lemma 4.6 we derive:

$$(5.1) \qquad \nabla_X(\phi' V') = (\nabla_X \phi')V' + \phi' \nabla_X V' =$$
$$= (c^2 + k/c)fX - g(X, V')V' + c(\phi')^2 X$$
$$(\nabla df)(X, Y) = 2(c^2 + k^2)g(X, V')g(Y, V') - fg(X, Y) +$$
$$+ c^2 g(\phi' X, \phi' Y)$$
$$(\nabla\nabla df)(X, Y, Z) = -(c^2 + k^2)\{2df(X)g(Y, Z)$$
$$+ df(Y)g(X, Z) + df(Z)g(X, Y)\}.$$

Thus we may state

Proposition 5.2. *Let M be a complete, connected and simply connected extrinsic sphere of a generalized Hopf manifold. If H is a section of the distribution $\mathcal{D}^3$, U is orthogonal to M and $f = g(V', V')$ is nonconstant, then M is isometric with a standard sphere of radius $1/\sqrt{c^2 + k^2}$.*

What happens if, under the same conditions as above, $f = g(V', V')$ is constant on M? From $df = 0$ we derive $g(X, \phi'V') = 0$. Using (5.1) we obtain:

$$(5.2) \qquad (\phi')^2 X = -\frac{c^2}{c^2 + k^2}\{fX - g(X, V')V'\}\,.$$

On the other hand:

$$\phi V' = \phi'' V' - g(V', \mu')\mu = \phi'' V' + (k/c)f\mu$$

$$\|\phi'' V'\|^2 = \|\phi V'\|^2 - 2(k/c)fg(\phi V', \mu) + (k^2/c^2)f^2\,.$$

We separately evaluate the terms on the right hand side of the last equality:

$$g(\phi V', \mu) = -g(V', \phi\mu) = -g(V', \mu) = (k/c)f\mu$$

$$\|\phi V'\|^2 = -g(V', \phi^2 V')$$

$$\phi^2 V' = -V' + fV$$

$$-g(V', \phi^2 V') = -g(V', fV - V') = -f\|V'\|^2 + f = f - f^2\,.$$

Finally we get

$$(5.3) \qquad \|\phi'' V'\| = \left(1 - \frac{c^2 + k^2}{c^2}f\right) f \geq 0\,.$$

It is thus necessary that

$$f \in \left[0, \frac{c^2}{c^2 + k^2}\right]$$

and, consequently, we are led to analyse the following cases:

Case I: $f \in \left(0, \frac{c^2}{c^2+k^2}\right)$. We let $\phi^* = \frac{1}{((c^2+k^2)f)^{1/2}}\phi$, $\xi^* = (1/f)V'$, $\eta^*(X) = g(X, \xi^*)$. A direct checking shows that $(\phi^*, \xi^*, \eta^*, g)$ defines a *homothetic Sasakian structure* on M.

Case II: $f = 0$. Then $V' = 0$, thus U and V are both orthogonal to M. Then from (2.2) we obtain:

$$g(JX, Y) = g(\phi'X, Y) = (1/c)g(\tilde{\nabla}_X V, Y) - (1/c)g(A_V X, Y)\,.$$

As J is antisymmetric and A_V is symmetric we deduce $A_V = 0$ and $g(JX, Y) = 0$ that is $JX \in \mathcal{C}^\infty(T^\perp M)$. We conclude that M is a *totally real* submanifold of $\tilde{M}$. On the other hand $\phi'X \in \mathcal{C}^\infty(T^\perp M)$, hence $\dim M \leq (1/2)\dim \tilde{M} - 1$.

Case III: $f = c^2/(c^2 + k^2)$ Now $\phi''V' = 0$, and, because $H \in \mathcal{C}^\infty(\mathcal{D}^3)$, we also have $\phi'V' = (kc/(c^2 + k^2))\mu \in \mathcal{C}^\infty(T^\perp M)$. Let us denote by $T'M$ the subbundle of TM whose sections are normal to V'. For each section $X \in \mathcal{C}^\infty(T'M)$ (5.2) implies:

$$(\phi')^2 X = -((c^2 + k^2)/c^2)fX = -X$$
$$\|\phi'X\|^2 = \|X\|^2 = \|\phi X\|^2.$$

Thus $\phi' = \phi$ on sections of $T'M$. This means that U and V are orthogonal on the sections of $T'M$ so that the restriction of J to $T'M$ coincides with the action of ϕ'. Moreover $\phi V' = JV'$ hence $JV' \in \mathcal{C}^\infty(T'M)$. This in turn implies $TM = T'M \oplus \{V'\}$ so that M is a CR submanifold of odd dimension in $\tilde{M}$ (its totally real distribution is 1-dimensional).

To conclude, we have proved

Theorem 5.3. *Let M be an extrinsic sphere of a generalized Hopf manifold $\tilde{M}$. If its mean curvature vector field is a section of the distribution $\mathcal{D}^3$, if U is normal to M and if $f = g(V', V')$ is a constant function on M, then M may be:*

  i) *a totally real submanifold and $\dim M \leq (1/2)\dim \tilde{M} - 1$, or*
 ii) *a CR submanifold and $\dim M$ is odd, or*
iii) *a homothetic Sasakian manifold.*

We now recall from [Ia-Or] that an integral manifold of the distribution $\mathcal{D}^3$ (which is not completely integrable) is a totally real submanifold. Then point ii) admits the following refinement (cf. loc. cit.) which we quote for the sake of completeness:

Theorem 5.4. *Let M be a complete, connected and simply connected integral manifold of the Kaehlerian distribution $\mathcal{D}^3$ of a generalized Hopf manifold. If M has flat normal connection, and is an extrinsic sphere then M is isometric with a standard sphere.*

Example. Let N be a $2n - 1$ dimensional Riemannian manifold endowed with a Sasakian structure (ϕ, ξ, η, g_1). Let us denote by u the lenght element on $S^1(1)$. On the product manifold $\tilde{M} = N \times S^1$ one may define a Hermitian structure (J, g) as follows (cf. [Bl-Ou]):

$$J(X_1, X_2) = (\phi X_1 - u(X_2)\xi, \eta(X_1)u^{\#}) \quad X_1 \in \mathcal{C}^{\infty}(TN), \ X_2 = fu^{\#}$$

$$g = g_1 + u \otimes u \,.$$

Moreover, it can be checked directly that this structure is locally conformal Kaehler with parallel Lee form (which is precisely u, with obvious identifications). Let now M be an extrinsic sphere of N. As g is a product metric and N is totally geodesic in $\tilde{M}$ we conclude that M is an extrinsic sphere in $\tilde{M}$ too.

6. Locally conformally flat ambient

We now consider $\tilde{M}$ to be a *locally conformally flat* g.H.m., meaning that the local Kaehler metrics are flat. Equivalently, the *Weyl connection* associated to the Levi–Civita connection of g and to the Lee form is flat. The complex Hopf manifold is a typical example. In this situation the curvature tensor $\tilde{R}$ of g has a particularly nice form (cf. [Val]):

(6.1) $\quad \tilde{R}(X, Y)Z = c^2 \{ u(X)u(Z)Y - u(Y)u(Z)X - u(X)g(Y, Z)U +$
$$+u(Y)g(X, Z)U + g(Y, Z)X - g(X, Z)Y \} \,.$$

First of all we have:

Lemma 6.1. *Let M be a submanifold with parallel second fundamental form in a locally conformally flat g.H.m. Then the Lee vector field is everywhere tangent or everywhere normal to M.*

PROOF. The Codazzi equation and formula (6.1) lead to:

$$\operatorname{nor}\tilde{R}(X, Y)Z = c^2 \{ g(U', Y)g(X, Z) - g(U', X)g(Y, Z) \}U'' = 0 \,.$$

As $\dim M \geq 2$ we can choose $X = Z$, $X \perp Y$ and derive $g(U', Y)U'' = 0$. Thus $U'' = 0$ or $U' = 0$ and the proof is complete.

Taking into account Corollary 4.3 we obtain

Corollary 6.2. *An extrinsic sphere of a locally conformally flat g.H.m. is everywhere normal to the Lee vector field.*

As a consequence, the Gauss equation for an extrinsic sphere will be

$$R(X, Y; Z, W) = (c^2 + k^2)\{g(X, Y)g(Z, W) - g(X, Z)g(Y, W)\}.$$

On the other hand, the Ricci equation and formula (6.1) together imply $R^\perp = 0$. Summing up and taking into account the well-known Hadamard theorem we have proved

Theorem 6.3. *A complete, connected and simply connected extrinsic sphere of a locally conformally flat generalized Hopf manifold is isometric with a standard sphere of radius $1/\sqrt{c^2 + k^2}$. Moreover, its normal bundle is flat.*

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