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On normal approximations to strongly mixing random fields

By VIDMANTAS BENTKUS (Vilnius) and JONAS KAZYS SUNKLODAS (Vilnius)

Abstract. Let ξ_n be a strongly mixing sequence of real random variables such that $\mathbb{E}\xi_n = 0$. Write $S_n = \xi_1 + \cdots + \xi_n$ and consider the normalized sums $Z_n = S_n/B_n$, where $B_n^2 = \mathbb{E}S_n^2$. Assume that a thrice differentiable function $h : \mathbb{R} \to \mathbb{R}$ satisfies $\sup_{x\in\mathbb{R}} |h'''(x)| < \infty$. We obtain optimal (in a sense) bounds for $\Delta_n = |\mathbb{E}h(Z_n) - \mathbb{E}h(N)|$, where N is a standard normal random variable. Namely, we show that $\Delta_n = O(n^{-1/2})$, provided that the random variables ξ_n are bounded by a constant, $B_n^2 \geq c_0 n$, where c_0 is a positive constant, and that the strong mixing coefficients $\alpha(r)$ satisfy $\sum_{r=1}^{\infty} r\alpha(r) < \infty$. The results extend to the case of random fields $\{\xi_a, a \in \mathbb{Z}^d\}$. To prove the results we apply a new method.

1. Introduction

Let $\{\xi_a, a \in \mathbb{Z}^d\}$ be a random field of real random variables (r.v.'s) ξ_a indexed by points $a \in \mathbb{Z}^d$ of the standard lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ ($\mathbb{R}$ is the real line). For a subset $U \subset \mathbb{Z}^d$, let $\mathcal{F}_U$ be the minimal σ-algebra such that all r.v.'s ξ_a with $a \in U$ are measurable.

Introduce the distance

$$d(U, V) = \inf\{\|a - b\| : a \in U,\ b \in V\}$$

between the subsets $U, V \subset \mathbb{Z}^d$, where $\|a\| = \max\{|a_i| : 1 \leq i \leq d\}$, for $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$. Write $|U| = \operatorname{card} U$ for the number of elements in the set U.

Henceforth we assume that for $U, V \subset \mathbb{Z}^d$

$$\sup\left\{|\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_U,\ B \in \mathcal{F}_V\right\} \leq f(|U| + |V|)\alpha(d(U, V)) \qquad (1)$$

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with a nonnegative nondecreasing function $f(n)$ of integer argument $n \geq 2$ and nonnegative function $\alpha(r)$ of integer argument $r \geq 1$ such that $\alpha(r) \to 0$ as $r \to \infty$.

If condition (1) is satisfied, we say that the random field $\{\xi_a,\, a \in \mathbb{Z}^d\}$ satisfies the strong mixing condition.

In what follows, N is a standard normal r.v.

Write

$$S_V = \sum_{a \in V} \xi_a, \quad B_V^2 = \mathbb{E} S_V^2, \quad Z_V = S_V / B_V$$

for a fixed nonempty set $V \subset \mathbb{Z}^d$ with $|V| < \infty$ and $B_V > 0$, and, for the functions $h : \mathbb{R} \to \mathbb{R}$,

$$\Delta_V = |\mathbb{E} h(Z_V) - \mathbb{E} h(N)|.$$

Introduce the boundedness condition

$$\mathbb{P}\{|\xi_a| \leq M\} = 1 \quad \text{for all } a \in V \tag{2}$$

with a non-random constant $M > 0$.

Write

$$\lambda_d = \sum_{r=1}^{\infty} r^{2d-1} \alpha(r) \quad \text{and} \quad \|h'''\|_\infty = \sup_{x \in \mathbb{R}} |h'''(x)|.$$

Here and in what follows, h', h'', and h''' denote the first, second, and third derivative of the function h, respectively.

Our main result is the following Berry–Esseen (or Lyapunov) type bound for Δ_V.

Theorem 1. *Assume that a real random field $\{\xi_a, a \in \mathbb{Z}^d\}$, $d \geq 1$, with $\mathbb{E}\xi_a = 0$ for all $a \in V$, satisfies the strong mixing condition* (1) *with $\lambda_d < \infty$ and boundedness condition* (2)*. Let h be a thrice differentiable function such that $\|h'''\|_\infty < \infty$. Then*

$$\Delta_V \leq C \|h'''\|_\infty f(|V|) M^3 |V| B_V^{-3}, \tag{3}$$

where $C = C(d, \lambda_d)$ is a positive finite factor depending only on the dimension d and λ_d.

Bound (3) is optimal in the sense that for dependent fields (with an infinite range of dependence) it provides convergence rates in the CLT of the same order as in the case of i.i.d. summands with finite third absolute moment, i.e., in the special case ($\sup_n f(n) < \infty$ in the definition of condition (1) and $B_V^2 \geq c_0 |V|$, where c_0 is a positive constant), the obtained upper bound of Δ_V is of order $O(|V|^{-1/2})$.

Note that the stationarity of an approached random field is not requested.

In the case $d = 1$ one can use in (3) $f \equiv 1$ and $V = \{1, \dots, n\}$ with $|V| = n$. Note that the strong mixing condition (1) for $d = 1$ is not the same as the strong mixing condition introduced by Rosenblatt [26] between "the past" and "the future" (because one can use the order on the real line).

Write $\operatorname{cov}(\xi, \eta) = \mathbb{E}\xi\eta - \mathbb{E}\xi\mathbb{E}\eta$ for real r.v.'s ξ and η.

Now we present an analogous version of Theorem 1, when the mixing coefficients are defined as follows: for $n \in \mathbb{N}$, $k, l \in \mathbb{N} \cup \{\infty\}$,

$$\begin{aligned}\alpha_{k,l}(n) = \sup\{|\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_U,\ B \in \mathcal{F}_V,\\ U, V \subset \mathbb{Z}^d,\ |U| \le k,\ |V| \le l,\ d(U, V) \ge n\},\end{aligned}$$

where $\mathbb{N} = \{1, 2, \dots\}$.

Denote

$$\Lambda_d = \sum_{r=1}^{\infty} r^{2d-1} \alpha_{2,\infty}(r),$$

where $\alpha_{k,\infty}(n) = \sup_l \alpha_{k,l}(n)$.

The following statement is valid.

Theorem 2. *Assume that a real random field $\{\xi_a, a \in \mathbb{Z}^d\}$, $d \ge 1$, with $\mathbb{E}\xi_a = 0$ for all $a \in V$, satisfies the condition $\Lambda_d < \infty$ and boundedness condition* (2). *Let h be a thrice differentiable function such that $\|h'''\|_\infty < \infty$. Then*

$$\Delta_V \le C \|h'''\|_\infty M^3 |V| B_V^{-3}, \tag{4}$$

where $C = C(d, \Lambda_d)$ is a positive finite factor depending only on the dimension d and Λ_d.

There is a rich literature to normal approximations of sums of weekly dependent sequences and fields, see papers [8], [14], [21], [24], [25], [28]–[36] and books [7], [13], and [18]. On the central limit theorem for random fields, see papers [3], [9], [19], [20], [22], [23]. About the mixing properties of random fields, see the books [11] and [13]. Equivalent mixing conditions for stationary random fields have been discussed in [4]. Let us discuss here only the results where bounds for errors of approximations are of the same order (as the sample size increases) as in the independent case. We omit the discussion where the dependence has a finite range, like for m-dependent sequences and fields. We only note that the case of

m-dependent random fields is well studied – see, for example, [5]–[7], [14], [16], [27], [30]–[33] and others.

For the sequence $\{\xi_i, i \geq 1\}$ with $\mathbb{E}\xi_i = 0$, let $\{\zeta_i, i \geq 1\}$ with $\mathbb{E}\zeta_i = 0$, be a Gaussian sequence such that the covariance structures of $\{\xi_i, i \geq 1\}$ and $\{\zeta_i, i \geq 1\}$ are identical, that is, $\mathbb{E}\xi_i\xi_{i+j} = \mathbb{E}\zeta_i\zeta_{i+j}$. As far as we are aware, Utev (1991) was the first to obtain the optimal bound

$$\left|\mathbb{E}h(n^{-1/2}(\xi_1 + \cdots + \xi_n)) - \mathbb{E}h(n^{-1/2}(\zeta_1 + \cdots + \zeta_n))\right| = O(n^{-1/2})$$

assuming a weak stationarity of $\{\xi_i, i \geq 1\}$ and imposing some moment assumptions. His conditions include smootheness of h, as well as conditions on Ibragimov's (or uniformly strong) mixing. He used a nice and original modification of the classical Bernstein method and that of convolutions. This is quite unexpected since Bernstein's method (versus the Stein method [28], [29]) is commonly considered as a method which cannot produce bounds even close to optimal ones.

Rio (1996) established an optimal bound for the Kolmogorov metric

$$\sup_{x \in \mathbb{R}} |\mathbb{P}\{n^{-1/2}(\xi_1 + \cdots + \xi_n) < x\} - \Phi(x)| = O(n^{-1/2})$$

assuming that a strictly stationary sequence $\{\xi_i, i \geq 1\}$ satisfies the φ-mixing condition with $\sum_{r=1}^{\infty} r\varphi(r) < \infty$ and boundedness condition (2) for $V = \{1, \ldots, n\}$, where $\varphi(r)$ stand for Ibragimov's mixing coefficients, and $\Phi(x)$ is the standard normal distribution function. To prove this result, a modified version of the classical composition (or Lindeberg, Bergstrem, etc.) method was applied.

To prove Theorem 1, we extend a method introduced in Bentkus (2003). The idea is to consider an appropriate curve joining Z_V and N in the space of r.v.'s, and to apply the Newton–Leibnitz formula along the curve. Using this method, no assumptions on independence or equidistribution are needed, however an additive structure of the object is desirable (e.g., Z_V is a sum). The method can be combined with the Fourier transform, i.e., it is applicable to characteristic functions. Some examples in the context of the CLT in $\mathbb{R}^d$ are provided in Bentkus (2003), the case of non-identically distributed summands is considered in Bentkus (2004). The present paper is just a pilot study of dependent sequences and fields. Our plan would be to obtain asymptotic (Edgeworth) expansions assuming sufficient smoothness, as well as to get Berry–Esseen bounds for Kolmogorov's and other metrics, each time looking for optimal combinations of smoothness, moment (or boundedness) and dependency conditions such that the bound is of the same order as in the independent case, as the sample size increases. We hope that one cannot replace the strong mixing condition in Theorem 1 by another (considerably) weaker dependency assumption.

2. Auxiliary Lemmas and Proposition 4

Let $\mathcal{G}$ and $\mathcal{H}$ be sub-σ-algebras of the σ-algebra $\mathcal{F}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote the strong mixing coefficient between $\mathcal{G}$ and $\mathcal{H}$ by

$$\alpha(\mathcal{G}, \mathcal{H}) = \sup\{|\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{G},\ B \in \mathcal{H}\}.$$

Lemma 3 (see [17, p. 388] and [15, p. 277])**.** *Suppose that ξ and η are real r.v.'s which are $\mathcal{G}$-and $\mathcal{H}$-measurable, respectively, and that $\mathbb{P}\{|\xi| \le C_1\} = \mathbb{P}\{|\eta| \le C_2\} = 1$. Then*

$$|\operatorname{cov}(\xi, \eta)| \le 4C_1C_2\alpha(\mathcal{G}, \mathcal{H}). \tag{5}$$

Proposition 4. *Let $\{A_a, a \in V\}$ and $\{Y_a, a \in V\}$ be two real random fields, defined on a fixed nonempty set V with $|V| < \infty$ of standard lattice $\mathbb{Z}^d$, $d \ge 1$, such that $Z_V = \sum_{a\in V} A_a$ and $Y_V = \sum_{a \in V} Y_a$ are independent, $\mathbb{E}Z_V = \mathbb{E}Y_V = 0$, $\mathbb{E}Z_V^2 = \mathbb{E}Y_V^2$, $\mathbb{E}|A_a|^3 < \infty$ and $\mathbb{E}|Y_a|^3 < \infty$ for all $a \in V$. Let the r.v. τ be uniformly distributed in the interval $[0,1]$ and independent of all the other r.v.'s. Let $h : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function such that $\|h'''\|_\infty < \infty$. Introduce the following notation:*

$$X_a = X_a(\gamma) = A_a \sin\gamma + Y_a \cos\gamma, \qquad T_V = T_V(\gamma) = \sum_{a \in V} X_a,$$

$$t_a^{(r)} = \sum_{\substack{b \in V: \|b-a\| = r,\\ a\text{-fixed},\, a \in V}} X_b \quad (t_a^{(0)} = X_a),$$

$$T_a^{(m)} = T_V - \sum_{r=0}^{m} t_a^{(r)}, \qquad u_a^{(r)} = X_a' t_a^{(r)} - \mathbb{E}X_a' t_a^{(r)},$$

where $0 \le \gamma \le \pi/2$. Here and in what follows, X_a' and T_V' denote the first order derivatives with respect to γ of X_a and T_V, respectively.

Then

$$\mathbb{E}h(Z_V) - \mathbb{E}h(Y_V) = \int_0^{\pi/2} \mathbb{E}T_V' h'(T_V)d\gamma, \tag{6}$$

and the following expansion is valid for $\mathbb{E}T_V'h'(T_V)$:

$$\mathbb{E}T_V'h'(T_V) = \Sigma_1 + \cdots + \Sigma_6, \tag{7}$$

where

$$\Sigma_1 = \sum_{a\in V}\sum_{r\ge 1}\sum_{q=r+1}^{2r} \mathbb{E}u_a^{(r)} t_a^{(q)} h'''(T_a^{(q)} + \tau t_a^{(q)}),$$

$$\Sigma_2 = \sum_{a\in V}\sum_{r\ge 1}\sum_{q\ge 2r+1} \mathbb{E}u_a^{(r)} t_a^{(q)} h'''(T_a^{(q)} + \tau t_a^{(q)}),$$

$$\Sigma_3 = \sum_{a\in V}\sum_{r\geq 1}\mathbb{E}u_a^{(0)}t_a^{(r)}h'''(T_a^{(r)}+\tau t_a^{(r)}),$$
$$\Sigma_4 = \sum_{a\in V}\sum_{r\geq 1}\mathbb{E}X_a'(t_a^{(r)})^2(1-\tau)h'''(T_a^{(r)}+\tau t_a^{(r)}),$$
$$\Sigma_5 = \sum_{a\in V}\mathbb{E}[X_a'X_a(1-\tau)-\mathbb{E}X_a'X_a]X_a h'''(T_a^{(0)}+\tau X_a),$$
$$\Sigma_6 = -\sum_{a\in V}\sum_{r\geq 1}\mathbb{E}X_a't_a^{(r)}\sum_{q=0}^{r}\mathbb{E}t_a^{(q)}h'''(T_a^{(q)}+\tau t_a^{(q)}).$$

Proof. The fundamental theorem of integral calculus yields

$$h(Z_V)-h(Y_V)=\int_0^{\pi/2}T_V'(\gamma)h'(T_V(\gamma))\mathrm{d}\gamma.$$

Here and in the sequel, we omit the argument γ in which it appears. Consider the integrand function $T_V'h'(T_V)$.

The Taylor expansion yields

$$\begin{aligned} T_V'h'(T_V) &= \sum_{a\in V}X_a'h'(T_a^{(0)}+X_a)\\ &= \sum_{a\in V}X_a'[h'(T_a^{(0)})+X_a h''(T_a^{(0)})+X_a^2\mathbb{E}_\tau(1-\tau)h'''(T_a^{(0)}+\tau X_a)], \end{aligned} \tag{8}$$

where $\mathbb{E}_\tau$ is the expectation taken with respect to the r.v. τ.

In what follows, the index a ($a\in V$) is fixed.

Expansion of $X_a'X_a h''(T_a^{(0)})$.

We rewrite the term $X_a'X_a h''(T_a^{(0)})$ in such a way:

$$X_a'X_a h''(T_a^{(0)}) = u_a^{(0)}h''(T_a^{(0)}) + h''(T_a^{(0)})\mathbb{E}X_a'X_a.$$

Note that $T_a^{(m-1)} = T_a^{(m)} + t_a^{(m)}$ for all $m = 0,1,2,\dots$ ($T_a^{(-1)} = T_V$). We shall use this fact without mentioning it in the sequel. Therefore, using the Taylor expansion, we eliminate the r.v. $t_a^{(1)}$ from the sum $T_a^{(0)}$ in the expression $u_a^{(0)}h''(T_a^{(0)})$. From the remaining sum $T_a^{(1)}$, we eliminate the r.v. $t_a^{(2)}$ and so on, until zero appears instead of $T_a^{(0)}$. Step by step we obtain that

$$u_a^{(0)}h''(T_a^{(0)}) = u_a^{(0)}h''(0) + \sum_{r\geq 1}u_a^{(0)}t_a^{(r)}\mathbb{E}_\tau h'''(T_a^{(r)}+\tau t_a^{(r)}). \tag{9}$$

Now, adding and subtracting we pack away, one by one, the r.v.'s $t_a^{(r)}$, $t_a^{(r-1)}, \dots$ into the sum $T_a^{(r)}$ up to the sum T_V in the term $h''(T_a^{(r)})$. Thus, using the Taylor expansion, we get that for all $r \geq 0$

$$h''(T_a^{(r)}) = h''(T_V) - \sum_{q=0}^{r} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}). \tag{10}$$

It follows from (9) and (10) with $r = 0$ that

$$X'_a X_a h''(T_a^{(0)}) = u_a^{(0)} h''(0) + \sum_{r \geq 1} u_a^{(0)} t_a^{(r)} \mathbb{E}_\tau h'''(T_a^{(r)} + \tau t_a^{(r)}) + h''(T_V) \mathbb{E} X'_a X_a - X_a \mathbb{E}_\tau h'''(T_a^{(0)} + \tau X_a) \mathbb{E} X'_a X_a. \tag{11}$$

Expansion of $X'_a h'(T_a^{(0)})$.

Further, using the Taylor expansion, we eliminate the r.v. $t_a^{(1)}$ from the sum $T_a^{(0)}$ in the expression $X'_a h'(T_a^{(0)})$. From the remaining sum $T_a^{(1)}$, we eliminate the r.v. $t_a^{(2)}$ and so on, until zero appears instead of $T_a^{(0)}$. Step by step we obtain that

$$X'_a h'(T_a^{(0)}) = X'_a h'(0) + \sum_{r \geq 1} X'_a t_a^{(r)} h''(T_a^{(r)}) + \sum_{r \geq 1} X'_a (t_a^{(r)})^2 \mathbb{E}_\tau (1-\tau) h'''(T_a^{(r)} + \tau t_a^{(r)}). \tag{12}$$

Next we rewrite the term $X'_a t_a^{(r)} h''(T_a^{(r)})$ in (12) (for all $r \geq 1$) in such a way:

$$X'_a t_a^{(r)} h''(T_a^{(r)}) = u_a^{(r)} h''(T_a^{(r)}) + h''(T_a^{(r)}) \mathbb{E} X'_a t_a^{(r)}.$$

In the sum $T_a^{(r)}$ which appears in the term $u_a^{(r)} h''(T_a^{(r)})$, we increase a "hole" up to the "hole" in the sum $T_a^{(2r)}$ using the Taylor expansion. Later on, we eliminate, one by one, the r.v.'s $t_a^{(2r+1)}, t_a^{(2r+2)}, \dots$, from the sum $T_a^{(2r)}$ until zero appears instead of $T_a^{(2r)}$. Then, using (10) we get that for all $r \geq 1$

$$X'_a t_a^{(r)} h''(T_a^{(r)}) = u_a^{(r)} h''(0) + \sum_{q=r+1}^{2r} u_a^{(r)} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}) + \sum_{q \geq 2r+1} u_a^{(r)} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}) + h''(T_V) \mathbb{E} X'_a t_a^{(r)} - \sum_{q=0}^{r} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}) \mathbb{E} X'_a t_a^{(r)}. \tag{13}$$

Substituting (13) into (12) we obtain that

$$
\begin{aligned}
X_a' h'(T_a^{(0)}) &= X_a' h'(0) + \sum_{r\ge 1} u_a^{(r)} h''(0) + \sum_{r\ge 1} \sum_{q=r+1}^{2r} u_a^{(r)} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}) \\
&+ \sum_{r\ge 1} \sum_{q\ge 2r+1} u_a^{(r)} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}) \\
&+ h''(T_V) \sum_{r\ge 1} \mathbb{E} X_a' t_a^{(r)} - \sum_{r\ge 1} \sum_{q=0}^{r} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}) \mathbb{E} X_a' t_a^{(r)} \\
&+ \sum_{r\ge 1} X_a' (t_a^{(r)})^2 \mathbb{E}_\tau (1-\tau) h'''(T_a^{(r)} + \tau t_a^{(r)}). \qquad (14)
\end{aligned}
$$

It follows from (8), (14), and (11) that

$$
\begin{aligned}
T_V' h'(T_V) &= h'(0) T_V' + h''(0) \sum_{a\in V} \sum_{r\ge 0} u_a^{(r)} \\
&+ \sum_{a\in V} \sum_{r\ge 1} \sum_{q=r+1}^{2r} u_a^{(r)} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}) \\
&+ \sum_{a\in V} \sum_{r\ge 1} \sum_{q\ge 2r+1} u_a^{(r)} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}) + h''(T_V) \mathbb{E} T_V' T_V \\
&- \sum_{a\in V} \sum_{r\ge 1} \sum_{q=0}^{r} t_a^{(q)} \mathbb{E}_\tau h'''(T_a^{(q)} + \tau t_a^{(q)}) \mathbb{E} X_a' t_a^{(r)} \\
&+ \sum_{a\in V} \sum_{r\ge 1} X_a' (t_a^{(r)})^2 \mathbb{E}_\tau (1-\tau) h'''(T_a^{(r)} + \tau t_a^{(r)}) \\
&+ \sum_{a\in V} \sum_{r\ge 1} u_a^{(0)} t_a^{(r)} \mathbb{E}_\tau h'''(T_a^{(r)} + \tau t_a^{(r)}) - \sum_{a\in V} X_a \mathbb{E}_\tau h'''(T_a^{(0)} + \tau X_a) \mathbb{E} X_a' X_a \\
&+ \sum_{a\in V} X_a' X_a^2 \mathbb{E}_\tau (1-\tau) h'''(T_a^{(0)} + \tau X_a). \qquad (15)
\end{aligned}
$$

Since the r.v. Z_V is independent of Y_V, $\mathbb{E} Z_V = \mathbb{E} Y_V = 0$ and $\mathbb{E} Z_V^2 = \mathbb{E} Y_V^2$, one has that

$$
\mathbb{E} T_V' = 0 \quad \text{and} \quad \mathbb{E} T_V' T_V = 0. \qquad (16)
$$

From (15) and (16) it follows the proof of the basic identity (6)–(7). □

In what follows, we denote the centered r.v. as $\widehat{\xi} = \xi - \mathbb{E}\xi$.

Lemma 5. *Let* $\{Y_a, a \in V\}$ *be independent r.v.'s and* $\{Y_a, a \in V\}$ *be independent of the random field* $\{A_a, a \in V\}$, $\mathbb{E} A_a = \mathbb{E} Y_a = 0$, $\mathbb{E}|A_a|^3 < \infty$ *and*

$\mathbb{E}|Y_a|^3 < \infty$ for all $a \in V$, where V is a fixed nonempty set V with $|V| < \infty$ of standard lattice $\mathbb{Z}^d$, $d \geq 1$. Denote

$$X_a = pA_a + qY_a, \quad t_a^{(r)} = \sum_{\substack{b\in V:\|b-a\|=r,\\ a\text{-fixed},\ a\in V}} X_b \quad (t_a^{(0)} = X_a), \quad u_a^{(r)} = X_a' t_a^{(r)} - \mathbb{E}X_a' t_a^{(r)},$$

where $p = \sin\gamma$, $q = \cos\gamma$, $0 \leq \gamma \leq \pi/2$. Here and in what follows, X_a' denote the first order derivative with respect to γ of X_a.

The following connections are valid:

If $a \neq b$, then

$$X_a' X_b^2 = p^2 q A_a A_b^2 + 2pq^2 A_a A_b Y_b + q^3 A_a Y_b^2 - p^3 Y_a A_b^2 - 2p^2 q Y_a A_b Y_b - pq^2 Y_a Y_b^2. \tag{17}$$

If $a \neq b$, $a \neq c$, $b \neq c$, then

$$X_a' X_b X_c = p^2 q A_a A_b A_c + pq^2 A_a (A_b Y_c + Y_b A_c) + q^3 A_a Y_b Y_c - p^3 Y_a A_b A_c - p^2 q Y_a (A_b Y_c + Y_b A_c) - pq^2 Y_a Y_b Y_c. \tag{18}$$

If $a \neq b$, then

$$u_a^{(0)} X_b = p^2 q \widehat{A_a^2} A_b - p^2 q \widehat{Y_a^2} A_b + pq^2 \widehat{A_a^2} Y_b - pq^2 \widehat{Y_a^2} Y_b + p(q^2 - p^2) A_a Y_a A_b + q(q^2 - p^2) A_a Y_a Y_b. \tag{19}$$

For all $r \geq 1$:

$$u_a^{(r)} = pq \sum_{\substack{b\in V:\|b-a\|=r,\\ a\text{-fixed},\ a\in V}} \widehat{A_a A_b} + q^2 A_a \sum_{\substack{b\in V:\|b-a\|=r,\\ a\text{-fixed},\ a\in V}} Y_b - p^2 Y_a \sum_{\substack{b\in V:\|b-a\|=r,\\ a\text{-fixed},\ a\in V}} A_b - pq Y_a \sum_{\substack{b\in V:\|b-a\|=r,\\ a\text{-fixed},\ a\in V}} Y_b, \tag{20}$$

$$X_a' t_a^{(r)} = pq A_a \sum_{\substack{b\in V:\|b-a\|=r,\\ a\text{-fixed},\ a\in V}} A_b + q^2 A_a \sum_{\substack{b\in V:\|b-a\|=r,\\ a\text{-fixed},\ a\in V}} Y_b - p^2 Y_a \sum_{\substack{b\in V:\|b-a\|=r,\\ a\text{-fixed},\ a\in V}} A_b - pq Y_a \sum_{\substack{b\in V:\|b-a\|=r,\\ a\text{-fixed},\ a\in V}} Y_b. \tag{21}$$

$$|\mathbb{E}X_a' X_b| \leq \frac{1}{2} |\mathbb{E}A_a A_b|, \quad a \neq b, \tag{22}$$

$$|\mathbb{E}X_a'X_a| \le \frac{1}{2}|\mathbb{E}A_a^2 - \mathbb{E}Y_a^2|, \tag{23}$$

$$\mathbb{E}|X_a'|X_a^2 \le \frac{1}{2}\mathbb{E}|A_a|^3 + \mathbb{E}|A_a|\mathbb{E}Y_a^2 + \mathbb{E}|Y_a|\mathbb{E}A_a^2 + \frac{1}{2}\mathbb{E}|Y_a|^3. \tag{24}$$

The proof of Lemma 5 is elementary, so we omit it.

Lemma 6. *Let $\{\xi_a, a \in \mathbb{Z}^d\}$, $d \ge 1$, be a real random field satisfying condition* (1), $\Lambda_* = \sum_{b\in\mathbb{Z}^d\setminus\{0\}} \alpha(\|b\|) < \infty$, $\mathbb{E}\xi_a = 0$ *and* $\mathbb{P}\{|\xi_a| \le M\} = 1$ *for all* $a \in V$. *Then*

$$B_V^2 \le (1 + 4f(2)\Lambda_*)M^2|V|. \tag{25}$$

The proof of Lemma 6 follows from (5) and (1).

3. Proof of Theorem 1

To estimate the difference $\mathbb{E}h(Z_V) - \mathbb{E}h(N)$, we use Proposition 4. Let A_a and Y_a in Proposition 4 be $A_a = \xi_a/B_V$, $Y_a = \eta_a/\sqrt{|V|}$ where $\{\eta_a, a \in V\}$ are independent standard normal r.v.'s, and $\{\eta_a, a \in V\}$ are independent of the random field $\{\xi_a, a \in V\}$. It is assumed that $B_V > 0$. It is evident that $\sum_{a\in V} Y_a = Y_V$ is also a standard normal r.v.

First of all note that, if $Y_a \sim \mathcal{N}(0, \frac{1}{|V|})$ (i.e., if the r.v. Y_a is normally distributed with $\mathbb{E}Y_a = 0$ and $\mathbb{E}Y_a^2 = \frac{1}{|V|}$), then it is easy to see that

$$\mathbb{E}|Y_a| = C_1\frac{1}{|V|^{1/2}}, \quad \mathbb{E}|Y_a|^3 = C_3\frac{1}{|V|^{3/2}}, \tag{26}$$

where $C_1 = \sqrt{2/\pi}$, $C_3 = \sqrt{8/\pi}$.

If $\tau \sim U[0,1]$ (i.e., the r.v. τ is uniformly distributed in the interval $[0,1]$), then

$$\mathbb{E}|1-\tau| = \frac{1}{2}. \tag{27}$$

Moreover, for any fixed $a \in \mathbb{Z}^d$, $d \ge 1$,

$$|\{b \in \mathbb{Z}^d : \|a-b\| = r\}| \le 2d(2r+1)^{d-1}. \tag{28}$$

We shall use relations (26)–(28) without mentioning them in the sequel.

We now estimate the terms on the right-hand side of (7).

Since for all $b \in V$

$$\mathbb{E}|X_b| \le \frac{M}{B_V} + C_1\frac{1}{|V|^{1/2}}, \tag{29}$$

one has that, for all $a \in V$ and $q \geq 0$,

$$\mathbb{E}|t_a^{(q)}| \leq 2d(2q+1)^{d-1}\left(\frac{M}{B_V} + C_1\frac{1}{|V|^{1/2}}\right). \tag{30}$$

Using (22), (5) from Lemma 3, and (1), we obtain that, for all $a \in V$ and $r \geq 1$,

$$|\mathbb{E}X'_a t_a^{(r)}| \leq 4df(2)\frac{M^2}{B_V^2}(2r+1)^{d-1}\alpha(r). \tag{31}$$

From (31) and (30) it follows that

$$\begin{aligned}|\Sigma_6| &\leq \|h'''\|_\infty \sum_{a\in V}\sum_{r\geq 1}|\mathbb{E}X'_a t_a^{(r)}|\sum_{q=0}^{r}\mathbb{E}|t_a^{(q)}| \\ &\leq \|h'''\|_\infty 16d^2 12^{d-1}f(2)\left(1 + C_1\frac{B_V}{|V|^{1/2}M}\right)\frac{|V|M^3}{B_V^3}\sum_{r\geq 1}r^{2d-1}\alpha(r). \end{aligned} \tag{32}$$

Since the r.v. τ is independent of all the other r.v.'s, $\mathbb{P}\{|\xi_a| \leq M\} = 1$ for all $a \in V$, using (24), (29), and (23) we obtain

$$\begin{aligned}|\Sigma_5| &\leq \|h'''\|_\infty \sum_{a\in V}(\mathbb{E}|X'_a|X_a^2\mathbb{E}|1-\tau| + |\mathbb{E}X'_a X_a|\mathbb{E}|X_a|) \\ &\leq \|h'''\|_\infty\left[\frac{3}{4} + C_1\frac{B_V}{|V|^{1/2}M} + \frac{B_V^2}{|V|M^2} + \left(\frac{1}{2}C_1 + \frac{1}{4}C_3\right)\frac{B_V^3}{|V|^{3/2}M^3}\right] \\ &\quad\times\frac{|V|M^3}{B_V^3}. \end{aligned} \tag{33}$$

We now estimate $|\Sigma_4|$. We get

$$\begin{aligned}|\Sigma_4| &\leq \sum_{a\in V}\sum_{r\geq 1}\sum_{b\in V:\|b-a\|=r}|\mathbb{E}X'_a X_b^2(1-\tau)h'''(T_a^{(r)} + \tau t_a^{(r)})| \\ &\quad + \sum_{a\in V}\sum_{r\geq 1}\sum_{\substack{b\in V:\|b-a\|=r,\\ c\in V:\|c-a\|=r,\\ b\neq c}}|\mathbb{E}X'_a X_b X_c(1-\tau)h'''(T_a^{(r)} + \tau t_a^{(r)})|. \end{aligned} \tag{34}$$

Using the expression of $X'_a X_b^2$ from (17), taking into account that $h'''(T_a^{(r)} + \tau t_a^{(r)})$ does not contain Y_a, and $\{Y_a, a \in V\}$ are independent of the random field $\{A_a, a \in V\}$ and the r.v. τ, we obtain that, for all $a, b \in V : \|b-a\| = r \geq 1$,

$$\begin{aligned}&|\mathbb{E}X'_a X_b^2(1-\tau)h'''(T_a^{(r)} + \tau t_a^{(r)})| \leq \frac{1}{2}|\mathbb{E}A_a A_b^2(1-\tau)h'''(T_a^{(r)} + \tau t_a^{(r)})| \\ &+|\mathbb{E}A_a A_b Y_b(1-\tau)h'''(T_a^{(r)} + \tau t_a^{(r)})| + |\mathbb{E}A_a Y_b^2(1-\tau)h'''(T_a^{(r)} + \tau t_a^{(r)})|. \end{aligned} \tag{35}$$

Note that

$$T_a^{(r)} + \tau t_a^{(r)} = \sum_{c\in V:\|c-a\|>r} X_c + \tau \sum_{c\in V:\|c-a\|=r} X_c.$$

Denote the random vector $\mathcal{Y}$ and vector y as

$$\mathcal{Y} = (\tau, Y_c, c \in V : \|c-a\| \geq r), \quad y = (t, y_c, c \in V : \|c-a\| \geq r).$$

In the sequel, $\mathbb{P}_\xi$ is the distribution of a random vector ξ. Denote

$$s(r) = \sum_{c\in V:\|c-a\|\geq r} 1 = |V| - \sum_{c\in V:\|c-a\|\leq r-1} 1.$$

Then, applying (5) of Lemma 3 and (1), we get that, for all $a, b \in V$: $\|b-a\| = r \geq 1$,

$$\begin{aligned}
&|\mathbb{E}A_a A_b^2(1-\tau)h'''(T_a^{(r)} + \tau t_a^{(r)})| \\
&\quad = \Bigg| \int_{\mathbb{R}^{s(r)+1}} (1-t)\mathbb{E}A_a A_b^2 h''' \Bigg(\sin\gamma \Bigg(\sum_{c\in V:\|c-a\|>r} A_c + t \sum_{c\in V:\|c-a\|=r} A_c \Bigg) \\
&\qquad + \cos\gamma \Bigg(\sum_{c\in V:\|c-a\|>r} y_c + t \sum_{c\in V:\|c-a\|=r} y_c \Bigg) \Bigg) \mathbb{P}_\mathcal{Y}(\mathrm{d}y) \Bigg| \\
&\quad \leq 4\|h'''\|_\infty \frac{M^3}{B_V^3} f(s(r)+1)\alpha(r) \int_{\mathbb{R}^{s(r)+1}} |1-t| \mathbb{P}_\mathcal{Y}(\mathrm{d}y) \\
&\quad = \|h'''\|_\infty 2 \frac{M^3}{B_V^3} f(s(r)+1)\alpha(r). \qquad (36)
\end{aligned}$$

Similarly to (36), we get that, for all $a, b \in V : \|b-a\| = r \geq 1$,

$$\begin{aligned}
&|\mathbb{E}A_a A_b Y_b(1-\tau)h'''(T_a^{(r)} + \tau t_a^{(r)})| \\
&\quad \leq 4\|h'''\|_\infty \frac{M^2}{B_V^2} f(s(r)+1)\alpha(r) \int_{\mathbb{R}^{s(r)+1}} |y_b||1-t| \mathbb{P}_\mathcal{Y}(\mathrm{d}y) \\
&\quad = \|h'''\|_\infty 2C_1 \frac{M^2}{|V|^{1/2}B_V^2} f(s(r)+1)\alpha(r), \qquad (37)
\end{aligned}$$

$$\begin{aligned}
&|\mathbb{E}A_a Y_b^2(1-\tau)h'''(T_a^{(r)} + \tau t_a^{(r)})| \\
&\quad \leq 4\|h'''\|_\infty \frac{M}{B_V} f(s(r)+1)\alpha(r) \int_{\mathbb{R}^{s(r)+1}} y_b^2|1-t| \mathbb{P}_\mathcal{Y}(\mathrm{d}y) \\
&\quad = \|h'''\|_\infty 2 \frac{M}{|V|B_V} f(s(r)+1)\alpha(r). \qquad (38)
\end{aligned}$$

Substituting (36)–(38) into (35), we get that for all $a,b \in V : \|b-a\| = r \geq 1$

$$
\begin{aligned}
&|\mathbb{E}X_a' X_b^2 (1-\tau) h'''(T_a^{(r)} + \tau t_a^{(r)})| \\
&\qquad \leq \|h'''\|_\infty \left(1 + 2C_1 \frac{B_V}{|V|^{1/2}M} + 2\frac{B_V^2}{|V|M^2}\right) \frac{M^3}{B_V^3} f(s(r)+1)\alpha(r). \quad (39)
\end{aligned}
$$

It only remains to estimate an analogous term in (34), where instead of $X_a' X_b^2$ we take $X_a' X_b X_c$. Using the expression of $X_a' X_b X_c$ from (18), taking into account that $h'''(T_a^{(r)} + \tau t_a^{(r)})$ does not contain Y_a, and $\{Y_a, a \in V\}$ are independent of the random field $\{A_a, a \in V\}$ and the r.v. τ, we obtain in the same way that, for all $a,b,c \in V : \|b-a\| = r \geq 1$, $\|c-a\| = r \geq 1$, $b \neq c$,

$$
\begin{aligned}
&\big|\mathbb{E}X_a' X_b X_c (1-\tau) h'''\big(T_a^{(r)} + \tau t_a^{(r)}\big)\big| \leq \frac{1}{2}\big|\mathbb{E}A_a A_b A_c (1-\tau) h'''\big(T_a^{(r)} + \tau t_a^{(r)}\big)\big| \\
&\quad + \frac{1}{2}\big|\mathbb{E}A_a A_b Y_c (1-\tau) h'''\big(T_a^{(r)} + \tau t_a^{(r)}\big)\big| + \frac{1}{2}\big|\mathbb{E}A_a Y_b A_c (1-\tau) h'''\big(T_a^{(r)} + \tau t_a^{(r)}\big)\big| \\
&\quad + \big|\mathbb{E}A_a Y_b Y_c (1-\tau) h'''\big(T_a^{(r)} + \tau t_a^{(r)}\big)\big| \\
&\leq 2\|h'''\|_\infty \frac{M^3}{B_V^3} f(s(r)+1)\alpha(r)\mathbb{E}|1-\tau| \\
&\quad + 2\|h'''\|_\infty \frac{M^2}{B_V^2} f(s(r)+1)\alpha(r)\mathbb{E}|Y_c|\mathbb{E}|1-\tau| \\
&\quad + 2\|h'''\|_\infty \frac{M^2}{B_V^2} f(s(r)+1)\alpha(r)\mathbb{E}|Y_b|\mathbb{E}|1-\tau| \\
&\quad + 4\|h'''\|_\infty \frac{M}{B_V} f(s(r)+1)\alpha(r)\mathbb{E}|Y_b|\mathbb{E}|Y_c|\mathbb{E}|1-\tau| \\
&\leq \|h'''\|_\infty \left(1 + 2C_1 \frac{B_V}{|V|^{1/2}M} + 2C_1^2 \frac{B_V^2}{|V|M^2}\right) \frac{M^3}{B_V^3} f(s(r)+1)\alpha(r). \quad (40)
\end{aligned}
$$

Substituting (39) and (40) into (34), we have

$$
\begin{aligned}
|\Sigma_4| \leq \|h'''\|_\infty 8d^2 3^{2(d-1)} &\left(1 + 2C_1 \frac{B_V}{|V|^{1/2}M} + (1+C_1^2)\frac{B_V^2}{|V|M^2}\right) \\
&\times \frac{|V|M^3}{B_V^3} f(|V|) \sum_{r\geq 1} r^{2(d-1)}\alpha(r). \quad (41)
\end{aligned}
$$

In the same manner as (35) and (40), using (19), (5) of Lemma 3, and (1), we obtain that, for all $a,b \in V : \|b-a\| = r \geq 1$,

$$
\big|\mathbb{E}u_a^{(0)} X_b h'''(T_a^{(r)} + \tau t_a^{(r)})\big|
$$

$$\leq \frac{1}{2}\big|\mathbb{E}A_b\widehat{A_a^2}h'''(T_a^{(r)}+\tau t_a^{(r)})\big| + \frac{1}{2}\big|\mathbb{E}Y_b\widehat{A_a^2}h'''(T_a^{(r)}+\tau t_a^{(r)})\big|$$
$$\leq \|h'''\|_\infty 4\left(1+C_1\frac{B_V}{|V|^{1/2}M}\right)\frac{M^3}{B_V^3}f(s(r)+1)\alpha(r). \tag{42}$$

Therefore

$$|\Sigma_3| \leq \sum_{a\in V}\sum_{r\geq 1}\sum_{b\in V:\|b-a\|=r}|\mathbb{E}u_a^{(0)}X_bh'''(T_a^{(r)}+\tau t_a^{(r)})|$$
$$\leq \|h'''\|_\infty 4\left(1+C_1\frac{B_V}{|V|^{1/2}M}\right)\frac{|V|M^3}{B_V^3}\sum_{r\geq 1}f(s(r)+1)\alpha(r)\sum_{b\in V:\|b-a\|=r}1$$
$$\leq \|h'''\|_\infty 8d3^{d-1}\left(1+C_1\frac{B_V}{|V|^{1/2}M}\right)\frac{|V|M^3}{B_V^3}f(|V|)\sum_{r\geq 1}r^{d-1}\alpha(r). \tag{43}$$

We now estimate $|\Sigma_2|$. Analogously, using (20), we get that, for all $a\in V$, $r\geq 1$, $q\geq 2r+1$,

$$\big|\mathbb{E}u_a^{(r)}t_a^{(q)}h'''(T_a^{(q)}+\tau t_a^{(q)})\big|$$
$$\leq \frac{1}{2}\sum_{c\in V:\|c-a\|=q}\left|\mathbb{E}h'''(T_a^{(q)}+\tau t_a^{(q)})A_c\sum_{b\in V:\|b-a\|=r}\widehat{A_aA_b}\right|$$
$$+\frac{1}{2}\sum_{c\in V:\|c-a\|=q}\left|\mathbb{E}h'''(T_a^{(q)}+\tau t_a^{(q)})Y_c\sum_{b\in V:\|b-a\|=r}\widehat{A_aA_b}\right|. \tag{44}$$

Note that
$$\left|\sum_{b\in V:\|b-a\|=r}\widehat{A_aA_b}\right| \leq 4d\frac{M^2}{B_V^2}(2r+1)^{d-1}.$$

Other than the previous notation, in the sequel

$$\mathcal{Y}=(\tau, Y_e, e\in V:\|e-a\|\geq q),\quad y=(t, y_e, e\in V:\|e-a\|\geq q).$$

Denote
$$s(t_a^{(r)}) = \sum_{b\in V:\|b-a\|=r}1.$$

Then, applying (5) and (1) we have, for all $a\in V$, $r\geq 1$, $q\geq 2r+1$,

$$\left|\mathbb{E}h'''(T_a^{(q)}+\tau t_a^{(q)})A_c\sum_{b\in V:\|b-a\|=r}\widehat{A_aA_b}\right|$$

$$= \Bigg| \int_{\mathbb{R}^{s(q)+1}} \mathbb{E}h'''\Bigg(\sin\gamma\Bigg(\sum_{e\in V:\|e-a\|>q} A_e + t \sum_{e\in V:\|e-a\|=q} A_e\Bigg)$$

$$+ \cos\gamma\Bigg(\sum_{e\in V:\|e-a\|>q} y_e + t \sum_{e\in V:\|e-a\|=q} y_e\Bigg)\Bigg) A_c$$

$$\times \sum_{b\in V:\|b-a\|=r} \widehat{A_a A_b} \mathbb{P}_{\mathcal{Y}}(\mathrm{d}y)\Bigg|$$

$$\le \|h'''\|_\infty 16d \frac{M^3}{B_V^3} f(s(q)+s(t_a^{(r)}))(2r+1)^{d-1}\alpha(q-r), \tag{45}$$

$$\Bigg| \mathbb{E}h'''(T_a^{(q)} + \tau t_a^{(q)}) Y_c \sum_{b\in V:\|b-a\|=r} \widehat{A_a A_b} \Bigg|$$

$$\le \|h'''\|_\infty 16d \frac{M^2}{B_V^2} f(s(q)+s(t_a^{(r)}))(2r+1)^{d-1}\alpha(q-r)\mathbb{E}|Y_c|$$

$$\le \|h'''\|_\infty 16dC_1 \frac{M^2}{|V|^{1/2}B_V^2} f(s(q)+s(t_a^{(r)}))(2r+1)^{d-1}\alpha(q-r). \tag{46}$$

Therefore, from (44)–(46) we get that

$$|\Sigma_2| \le \sum_{a\in V}\sum_{r\ge 1}\sum_{q\ge 2r+1} \Big| \mathbb{E}u_a^{(r)} t_a^{(q)} h'''(T_a^{(q)} + \tau t_a^{(q)})|$$

$$\le \|h'''\|_\infty 16d^2 9^{d-1} \left(1 + C_1 \frac{B_V}{|V|^{1/2}M}\right) \frac{|V|M^3}{B_V^3} f(|V|) \sum_{r\ge 1}\sum_{q\ge 2r+1} (rq)^{d-1}\alpha(q-r).$$

Since

$$\sum_{r\ge 1}\sum_{q\ge 2r+1} q^{2(d-1)}\alpha(q-r) < 4^{d-1}\sum_{r\ge 1} r^{2d-1}\alpha(r), \tag{47}$$

one has

$$|\Sigma_2| \le \|h'''\|_\infty 16d^2 18^{d-1} \left(1 + C_1 \frac{B_V}{|V|^{1/2}M}\right) \frac{|V|M^3}{B_V^3} f(|V|) \sum_{r\ge 1} r^{2d-1}\alpha(r). \tag{48}$$

It only remains now to estimate $|\Sigma_1|$. Recalling the definition of the r.v. $u_a^{(r)}$, we see that the largest "hole" of the two ones in the term $\mathbb{E}u_a^{(r)} t_a^{(q)} h'''(T_a^{(q)} + \tau t_a^{(q)})$ is between the r.v.'s $t_a^{(r)}$ and X_a'. Applying (21) we get that, for all $a \in V$, $r \ge 1$, $r+1 \le q \le 2r$,

$$\big|\mathbb{E}X_a' t_a^{(r)} t_a^{(q)} h'''(T_a^{(q)} + \tau t_a^{(q)})\big|$$

$$\le \frac{1}{2}\Bigg| \mathbb{E}h'''(T_a^{(q)} + \tau t_a^{(q)}) A_a \sum_{c\in V:\|c-a\|=q} A_c \sum_{b\in V:\|b-a\|=r} A_b \Bigg|$$

$$+\frac{1}{2}\bigg|\mathbb{E}h'''(T_a^{(q)}+\tau t_a^{(q)})A_a\sum_{c\in V:\|c-a\|=q}Y_c\sum_{b\in V:\|b-a\|=r}A_b\bigg|$$
$$\le \|h'''\|_\infty 8d^2(2q+1)^{d-1}(2r+1)^{d-1}\frac{M^3}{B_V^3}f(|V|)\alpha(r)$$
$$+\|h'''\|_\infty 4d(2r+1)^{d-1}\frac{M^2}{B_V^2}f(|V|)\alpha(r)\mathbb{E}\bigg|\sum_{c\in V:\|c-a\|=q}Y_c\bigg|$$
$$\le \|h'''\|_\infty 8d^2 15^{d-1}\left(1+C_1\frac{B_V}{|V|^{1/2}M}\right)\frac{M^3}{B_V^3}f(|V|)r^{2(d-1)}\alpha(r). \tag{49}$$

Using (30) and (31), we get that, for all $a\in V$, $r\ge 1$, $r+1\le q\le 2r$,

$$|\mathbb{E}t_a^{(q)}h'''(T_a^{(q)}+\tau t_a^{(q)})\mathbb{E}X_a't_a^{(r)}|\le \|h'''\|_\infty\mathbb{E}|t_a^{(q)}|\,|\mathbb{E}X_a't_a^{(r)}|$$
$$\le \|h'''\|_\infty 8d^2 15^{d-1}\left(1+C_1\frac{B_V}{|V|^{1/2}M}\right)\frac{M^3}{B_V^3}f(2)r^{2(d-1)}\alpha(r). \tag{50}$$

From (49) and (50) it follows the estimate of $|\Sigma_1|$:

$$|\Sigma_1|\le \|h'''\|_\infty 16d^2 15^{d-1}\left(1+C_1\frac{B_V}{|V|^{1/2}M}\right)\frac{|V|M^3}{B_V^3}f(|V|)\sum_{r\ge 1}r^{2d-1}\alpha(r). \tag{51}$$

Substituting (51), (48), (43), (41), (33), and (32) into (7) and using (6), we have (3).

Theorem 1 is proved.

The proof of Theorem 2 is analogous to that of Theorem 1, so we omit it.

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VIDMANTAS BENTKUS
INSTITUTE OF MATHEMATICS AND INFORMATICS
AKADEMIJOS 4
LT 08663 VILNIUS
LITHUANIA

E-mail: bentkus@ktl.mii.lt

JONAS KAZYS SUNKLODAS
INSTITUTE OF MATHEMATICS AND INFORMATICS
AKADEMIJOS 4
LT 08663 VILNIUS
LITHUANIA

E-mail: sunkl@ktl.mii.lt

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