Publ. Math. Debrecen
70/3-4 (2007), 271–290

Gröbner bases for complete ℓ-wide families

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Abstract. Let $n > 0$, k, ℓ be integers with $0 \leq \ell - 1 \leq k \leq n$, and consider the complete ℓ-*wide* family

$$\mathcal{F}^{k,\ell} = \{F \subseteq [n] : k - \ell < |F| \leq k\}.$$

We describe (reduced) Gröbner bases of the ideal of polynomials, over an arbitrary field $\mathbb{F}$, which vanish on the characteristic vectors of the elements of $\mathcal{F}^{k,\ell}$.

As an application, we obtain results on certain inclusion matrices related to $\mathcal{F}^{k,\ell}$. We show that if $0 \leq m \leq \min(k, n - k + \ell - 1)$ then

$$\operatorname{rank}_{\mathbb{F}} \; I\left(\mathcal{F}^{k,\ell}, \binom{[n]}{\leq m}\right) = \sum_{i=\max(0,m-\ell+1)}^{m} \binom{n}{i}, \tag{1}$$

where $\mathbb{F}$ is an arbitrary field. We prove also a special case of a conjecture of Frankl related to the determination of the maximum number of subsets of $[n]$ with no shattered set of size t and with no chain of size $\ell + 1$. The paper extends the results obtained for the case of uniform families (the case $\ell = 1$) in [11].

Mathematics Subject Classification: 13P10, 05E99.
Key words and phrases: Gröbner bases, ℓ-wide families, shattered set, inclusion matrix, Hilbert function.
Research supported in part by OTKA grants T42706, T42481, NK63066, and the Center for Applied Mathematics and Computational Physics of the BUTE. Part of this work was done during the Special Semester on Gröbner Bases, 2006, organized by RICAM, Austrian Academy of Sciences, and RISC, Johannes Kepler University, Linz, Austria.

1. Introduction

Throughout the paper n, ℓ are positive integers, k is a nonnegative integer such that $0 \le \ell - 1 \le k \le n$.

Let $[n]$ stand for the set $\{1, 2, \ldots, n\}$. The family of all subsets of $[n]$ is denoted by $2^{[n]}$. For an integer $0 \le d \le n$ we denote by $\binom{[n]}{d}$ the family of all d element subsets of $[n]$, and $\binom{[n]}{\le d} = \binom{[n]}{0} \cup \cdots \cup \binom{[n]}{d}$ the subsets of size at most d.

Let $\mathcal{F}^{k,\ell}$ denote the *complete ℓ-wide* family

$$\mathcal{F}^{k,\ell} = \{F \subseteq [n] : k - \ell < |F| \le k\}.$$

A set family $\mathcal{F} \subseteq 2^{[n]}$ is *ℓ-wide* if $\mathcal{F} \subseteq \mathcal{F}^{k,\ell}$ for a suitable k. Following [2], we recall the notion of order shattering.

A set

$$T = \{s_1 < s_2 < \cdots < s_d\} \subseteq [n]$$

is *order shattered* by the family $\mathcal{F} \subseteq 2^{[n]}$ if the following holds: in the case $T = \emptyset$ the family $\mathcal{F}$ has to contain a set; when $|T| > 0$, then there are 2^d sets in $\mathcal{F}$ that can be divided into two families $\mathcal{F}_0$ and $\mathcal{F}_1$ such that $s_d \notin F$ for all $F \in \mathcal{F}_0$, $s_d \in F$ for all $F \in \mathcal{F}_1$, and both $\mathcal{F}_0, \mathcal{F}_1$ order shatter the set $T \setminus \{s_d\}$, furthermore $Q \cap F_0 = Q \cap F_1$ holds for $Q = \{s_d + 1, s_d + 2, \ldots, n\}$ and all $F_0 \in \mathcal{F}_0$, $F_1 \in \mathcal{F}_1$.

Let

$$\operatorname{osh}(\mathcal{F}) = \{T \subseteq [n] : \mathcal{F} \text{ order shatters } T\}.$$

Notice that $\operatorname{osh}(\mathcal{F})$ is a down-set, i.e., if $A \in \operatorname{osh}(\mathcal{F})$ and $B \subseteq A$, then $B \in \operatorname{osh}(\mathcal{F})$. In [2] it was established that $|\operatorname{osh}(\mathcal{F})| = |\mathcal{F}|$ for every $\mathcal{F}$, and for $0 \le d \le n/2$ we have

$$\operatorname{osh}\left(\binom{[n]}{d}\right) = \big\{\{s_1 < \cdots < s_j\} \subset [n] : j \le d \text{ and } s_i \ge 2i \text{ for } 1 \le i \le j\big\}. \quad (2)$$

It is immediate that for a nonempty family $\mathcal{F} \subseteq 2^{[n]}$ we have

$$\operatorname{osh}(\operatorname{co}(\mathcal{F})) = \operatorname{osh}(\mathcal{F}), \quad (3)$$

where

$$\operatorname{co}(\mathcal{F}) = \{[n] \setminus F : F \in \mathcal{F}\}.$$

Let $\mathbb{F}$ be a field. We denote by $\mathbb{F}[x_1, \ldots, x_n]$ the ring of polynomials in variables $x_1, \ldots, x_n$ over $\mathbb{F}$. We write $\mathbb{F}[x_1, \ldots, x_n]_{\le s}$ for the vector space of all polynomials over $\mathbb{F}$ with degree at most s.

For a subset $F \subseteq [n]$ we write $x_F = \prod_{j\in F} x_j$, and $x^F = \prod_{j\in F}(x_j - 1)$. In particular, $x_\emptyset = x^\emptyset = 1$.

Let $v_F \in \{0,1\}^n$ denote the characteristic vector of a set $F \subseteq [n]$. For a family of subsets $\mathcal{F} \subseteq 2^{[n]}$, let $V(\mathcal{F}) = \{v_F : F \in \mathcal{F}\} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n$. A polynomial $f \in S = \mathbb{F}[x_1, \dots, x_n]$ can be considered as a function from $V(\mathcal{F})$ to $\mathbb{F}$ in the straightforward way.

For the study of polynomial functions on $V(\mathcal{F})$, it is useful to consider the ideal $I(V(\mathcal{F}))$:

$$I(V(\mathcal{F})) := \{f \in S : f(v) = 0 \text{ whenever } v \in V(\mathcal{F})\}. \tag{4}$$

In fact, substitution gives rise to an $\mathbb{F}$-homomorphism from S to the ring of $\mathbb{F}$-valued functions on $V(\mathcal{F})$. This map is seen to be surjective by an easy interpolation argument, and the kernel is exactly $I(V(\mathcal{F}))$. This way one can identify $S/I(V(\mathcal{F}))$ with the space of $\mathbb{F}$-valued functions on $V(\mathcal{F})$. In particular, $\dim_{\mathbb{F}} S/I(V(\mathcal{F})) = |\mathcal{F}|$.

Consider a family $\mathcal{F}$ of subsets of $[n]$. We say that $\mathcal{F}$ *shatters* T if

$$\{E \cap T : E \in \mathcal{F}\} = 2^T. \tag{5}$$

Then define

$$\mathrm{sh}(\mathcal{F}) = \{T \subseteq [n] : \mathcal{F} \text{ shatters } T\}. \tag{6}$$

From the definition we see that $\mathrm{osh}(\mathcal{F}) \subseteq \mathrm{sh}(\mathcal{F})$.

For example, let $n = 4$ and $\mathcal{F} = \{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}\}$. Simple applications of the definitions give

$$\mathrm{sh}(\mathcal{F}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}, \quad \text{and}$$
$$\mathrm{osh}(\mathcal{F}) = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}.$$

Recall that a *chain* of size p in $2^{[n]}$ is a sequence $A_1, \dots, A_p$ of subsets of $[n]$ with $A_1 \subset \cdots \subset A_p$.

For families $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ the *inclusion matrix* $I(\mathcal{F}, \mathcal{G})$ is a (0,1) matrix of size $|\mathcal{F}| \times |\mathcal{G}|$ whose rows and columns are indexed by the elements of $\mathcal{F}$ and $\mathcal{G}$, respectively. The entry at position (F, G) is 1 if $G \subseteq F$ and 0 otherwise ($F \in \mathcal{F}$, $G \in \mathcal{G}$).

Inclusion matrices and their ranks are quite useful in the combinatorics of finite set families. In Chapter 7 of [4] there is an excellent treatment of this subject which highlights the importance of inclusion matrices $I(\mathcal{F}, \mathcal{G})$ with $\mathcal{G} = \binom{[n]}{m}$ and $\mathcal{G} = \binom{[n]}{\leq m}$. By extending the rank formula available for uniform families (i.e. the special case $\ell = 1$) we prove the following result.

Theorem 1.1. *Let $n > 0$, ℓ, k, m be integers. Let $0 \le \ell - 1 \le k \le n$, and $0 \le m \le \min(k, n-k+\ell-1)$. Let $\mathbb{F}$ be an arbitrary field. Then we have*

$$\operatorname{rank}_{\mathbb{F}} \ I\left(\mathcal{F}^{k,\ell}, \binom{[n]}{\le m}\right) = \sum_{i=\max(0,m-\ell+1)}^{m} \binom{n}{i}. \tag{7}$$

As a surprising application, we prove, using the notion of order-shattering, the following special case of a conjecture by Frankl, [8]: Let $g(n,t,d)$ denote the maximum number of subsets of $[n]$ with no shattered set of size t and no chain of size $d+1$. In [8] Frankl proposed the following conjecture.

Conjecture 1 (Frankl [8])**.** *Assume that $2t \le n+d$. Then*

$$g(n,t,d) \le \sum_{i=\max(0,t-d)}^{t-1} \binom{n}{i}. \tag{8}$$

Clearly, if $\mathcal{F}$ is an ℓ-wide family of sets, then $\mathcal{F}$ does not contain any chain of size $\ell+1$. We prove the following special case of this conjecture.

Theorem 1.2. *Suppose that $2t \le n+\ell$ and let $\mathcal{F} \subseteq 2^{[n]}$ be an ℓ-wide family with no shattered set of size t. Then*

$$|\mathcal{F}| \le \sum_{i=\max(0,t-\ell)}^{t-1} \binom{n}{i}.$$

In Section 2 we collected preliminaries about Gröbner bases and polynomials. In Section 3 we state our main result. We give an explicit description of the reduced Gröbner basis and the initial ideals for the ideals $I(V(\mathcal{F}^{k,\ell}))$ (Theorem 3.1 and its Corollaries). Our results extend those of [11] obtained for the case $\ell = 1$. The Gröbner bases turn out to be largely independent of the monomial order and the field. In [6] Bernasconi and Egidi pointed out the importance of knowing Gröbner bases of $I(V(\mathcal{F}))$, where $\mathcal{F}$ is a symmetric set family. Our result gives this information for the case of complete ℓ-wide families.

In Section 4 we generalize (2) and determine the sets order shattered by *ℓ-wide* families. Then, building on this result, in Section 5 we prove our main results. In Section 6 we determine the Hilbert function of $S/I(V(\mathcal{F}^{k,\ell}))$ (see the next section for the definition), and obtain a special case of a conjecture by Frankl, [8].

2. Preliminaries

2.1. Gröbner bases and standard monomials. We recall now some basic facts concerning Gröbner bases in polynomial rings. A total order $\prec$ on the monomials composed from variables $x_1, x_2, \ldots, x_m$ is a *term order*, if 1 is the minimal element of $\prec$, and $uw \prec vw$ holds for any monomials u, v, w with $u \prec v$. Two important term orders are the lexicographic order $\prec_l$ and the deglex order $\prec_{dl}$. We have

$$x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \prec_l x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}$$

iff $i_k < j_k$ holds for the smallest index k such that $i_k \neq j_k$. As for deglex, we have $u \prec_{dl} v$ iff either $\deg u < \deg v$, or $\deg u = \deg v$, and $u \prec_l v$.

The *leading monomial* $\mathrm{lm}(f)$ of a nonzero polynomial $f \in S$ is the largest (with respect to $\prec$) monomial which appears with nonzero coefficient in f when expressed as an $\mathbb{F}$-linear combination of monomials.

Let I be an ideal of S. A finite subset $G \subseteq I$ is a *Gröbner basis* of I if for every $f \in I$ there exists a $g \in G$ such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(f)$. Using that $\prec$ is a well founded order, it follows that G is actually a basis of I, i.e. G generates I as an ideal of S. It is known (cf. [7, Chapter 1, Corollary 3.12] or [1, Corollary 1.6.5, Theorem 1.9.1]) that every nonzero ideal I of S has a Gröbner basis with respect to $\prec$.

A monomial $w \in S$ is called a *standard monomial for I* if it is not a leading monomial of any $f \in I$. Let $\mathrm{sm}(\prec, I, \mathbb{F})$ stand for the set of all standard monomials of I with respect to the term-order $\prec$ over $\mathbb{F}$. It follows from the definition and existence of Gröbner bases (see [7, Chapter 1, Section 4]) that for a nonzero ideal I the set $\mathrm{sm}(\prec, I, \mathbb{F})$ is a basis of the $\mathbb{F}$-vector-space S/I. In fact, every $g \in S$ can be written uniquely as $g = h + f$ where $f \in I$ and h is a unique $\mathbb{F}$-linear combination of monomials from $\mathrm{sm}(\prec, I, \mathbb{F})$. If $\mathcal{F} \subseteq 2^{[n]}$, then $x_i^2 - x_i \in I(V(\mathcal{F}))$, hence x_i^2 is a leading term for $I(V(\mathcal{F}))$. It follows that the standard monomials for this ideal are all square-free, i.e. of form x_G for $G \subseteq [n]$. We put

$$\mathrm{Sm}(\prec, \mathcal{F}, \mathbb{F}) = \{G \subseteq [n] : x_G \in \mathrm{sm}(\prec, I(V(\mathcal{F})), \mathbb{F})\} \subseteq 2^{[n]}.$$

It is immediate that $\mathrm{Sm}(\prec, \mathcal{F}, \mathbb{F})$ is a downward closed set system. By the discussion after (4) the standard monomials for $I(V(\mathcal{F}))$ form a basis of the functions from $V(\mathcal{F})$ to $\mathbb{F}$, hence

$$|\,\mathrm{Sm}(\prec, \mathcal{F}, \mathbb{F})| = |\mathcal{F}|. \qquad (9)$$

In Theorem 4.3 of [2] the following was proved

Theorem 2.1. *Let $\mathcal{F} \subseteq 2^{[n]}$ be a nonempty family and let $\mathbb{F}$ be a field. Then $\mathcal{M} = \{x_F : F \in \mathrm{osh}\,(\mathcal{F})\}$ is the set of standard monomials with respect to the lexicographic order $\prec_l$ of the ideal of all polynomials $f \in \mathbb{F}[x_1, \ldots, x_n]$ which vanish on $V(\mathcal{F}) \subseteq \mathbb{F}^n$.*

A Gröbner basis $\{f_1, \ldots, f_m\}$ of I is *reduced* if the coefficient of $\mathrm{lm}(f_i)$ is 1, and no nonzero monomial in f_i is divisible by any $\mathrm{lm}(f_j)$, $j \neq i$. By a theorem of Buchberger ([1, Theorem 1.8.7]), for a fixed term order $\prec$, any nonzero ideal of S has a unique reduced Gröbner basis.

The *initial ideal* $\mathrm{ini}(I)$ of an ideal I is the ideal in S generated by the monomials $\{\mathrm{lm}(f) : f \in I\}$.

Now we introduce the notion of reduction, which is ubiquitous in the computational applications of Gröbner bases. Let $\mathcal{G}$ be a set of polynomials in $\mathbb{F}[x_1, \ldots, x_n]$ and let $f \in \mathbb{F}[x_1, \ldots, x_n]$ be a fixed polynomial. Let $\prec$ be an arbitrary term-order. We can reduce f by the set $\mathcal{G}$ with respect to $\prec$. This gives a new polynomial $h \in \mathbb{F}[x_1, \ldots, x_n]$.

Here *reduction* means that we possibly repeatedly replace monomials in f by smaller ones (with respect to $\prec$) as follows: if w is a monomial occurring in f and $\mathrm{lm}(g)$ divides w for some $g \in \mathcal{G}$ (i.e. $w = \mathrm{lm}(g)u$ for some monomial u), then we replace w in f with $u(\mathrm{lm}(g) - g)$. Clearly the monomials in $u(\mathrm{lm}(g) - g)$ are $\prec$-smaller than w.

Later we shall use the following characterization of Gröbner bases (see Theorem 3.10 in Chapter 1 of [7], or Theorem 1.9.1 in [1]):

Theorem 2.2. *A nonempty finite set $\mathcal{G}$ of polynomials is a Gröbner basis of the ideal I generated by $\mathcal{G}$ iff every $f \in I$ reduces to zero with respect to $\mathcal{G}$.*

Let I be an ideal of $S = \mathbb{F}[x_1, \ldots, x_n]$. The *Hilbert function* of the algebra S/I is the sequence $h_{S/I}(0), h_{S/I}(1), \ldots$. Here $h_{S/I}(m)$ is the dimension over $\mathbb{F}$ of the quotient $\mathbb{F}[x_1, \ldots, x_n]_{\leq m}/(I \cap \mathbb{F}[x_1, \ldots, x_n]_{\leq m})$ (see [5, Section 9.3]).

In the case when $I = I(V(\mathcal{F}))$ for some set system $\mathcal{F} \subseteq 2^{[n]}$, the number $h_{\mathcal{F}}(m) := h_{S/I}(m)$ is the dimension of the space of functions from $V(\mathcal{F})$ to $\mathbb{F}$ which can be represented as polynomials of degree at most m.

In the combinatorial literature $h_{\mathcal{F}}(m)$ is usually given in terms of inclusion matrices. It is a simple matter to verify that

$$h_{\mathcal{F}}(m) = \mathrm{rank}_{\mathbb{F}}\ I\left(\mathcal{F}, \binom{[n]}{\leq m}\right). \tag{10}$$

On the other hand,

$$h_{\mathcal{F}}(m) = \left|\mathrm{Sm}(\prec, \mathcal{F}, \mathbb{F}) \cap \binom{[n]}{\leq m}\right| \tag{11}$$

where $\prec$ is an arbitrary degree-compatible term order, for instance deglex. This holds because any function from $V(\mathcal{F})$ to $\mathbb{F}$ can be $\mathcal{G}$-reduced to a linear combination of standard monomials, where $\mathcal{G}$ is a Gröbner basis of $I(V(\mathcal{F}))$. Moreover, in the presence of a degree-compatible order, reduction cannot increase the degree.

In Theorem 6.2, and Corollaries 5.7 and 4.2 we provide also the Hilbert function and the standard monomials for the ideals $I(V(\mathcal{F}^{k,\ell}))$.

2.2. The polynomials $f_{H,k}$. We introduce a family of polynomials with integral coefficients. These polynomials will be the "nontrivial" elements of the Gröbner bases of ℓ-wide families.

Let ℓ, t be positive integers. We define $\mathcal{H}(t,\ell)$ as the set of those subsets $H = \{s_1 < \cdots < s_t\}$ of $[n]$ for which t is the smallest index j with $s_j < 2j - \ell + 1$.

We remark that $\mathcal{H}(t,\ell) = \emptyset$ for $t < \ell$. Indeed, if $H \in \mathcal{H}(t,\ell)$, then $t \le s_t < 2t - \ell + 1$, and $\ell - 1 < t$. Also if $t > (n+\ell)/2$, then $\mathcal{H}(t,\ell) = \emptyset$ again, because then $s_{t-1} \ge 2(t-1) - \ell + 1 > n - 1$ would imply that $s_t > n$.

The elements of $\mathcal{H}(t,\ell)$ are t-subsets of $[n]$, and we have $H \in \mathcal{H}(t,\ell)$ iff $s_1 \ge 3 - \ell, s_2 \ge 5 - \ell, \ldots, s_{t-1} \ge 2t - \ell - 1$ and $s_t < 2t - \ell + 1$. It follows that $s_t = 2t - \ell$ (in the case $t = 1$ we have $\ell = 1$ as well). For $t > 1$ we have also $s_{t-1} = 2t - \ell - 1$.

As examples, for n large enough, we have $\mathcal{H}(2,2) = \{\{1,2\}\}$, $\mathcal{H}(3,2) = \{\{1,3,4\},\{2,3,4\}\}$, and $\mathcal{H}(4,2) = \{\{1,3,5,6\},\{1,4,5,6\},\{2,3,5,6\},\{2,4,5,6\},\{3,4,5,6\}\}$.

For a subset $J \subseteq [n]$ and an integer $0 \le i \le |J|$ we denote by $\sigma_{J,i}$ the i-th elementary symmetric polynomial of the variables x_j, $j \in J$:

$$\sigma_{J,i} := \sum_{T \subseteq J, |T| = i} x_T \in \mathbb{F}[x_1, \ldots, x_n].$$

In particular, $\sigma_{J,0} = 1$.

Now let $0 < \ell \le t < (n+\ell)/2$, $0 \le k \le n$ and $H \in \mathcal{H}(t,\ell)$. Then put $H' = H \cup \{2t - \ell + 1, 2t - \ell + 2, \ldots, n\}$.

We write

$$f_{H,k} = f_{H,k}(x_1, \ldots, x_n) := \sum_{j=0}^{t} (-1)^{t-j} \binom{k-j}{t-j} \sigma_{H',j}.$$

Note that $f_{H,k}$ depends on t and ℓ through H. Moreover, H uniquely determines t and ℓ. Specifically, we have $f_{\{1\},k} = x_1 + x_2 + \cdots + x_n - k$, and

$$f_{\{2,3\},k} = \sigma_{U,2} - (k-1)\sigma_{U,1} + \binom{k}{2},$$

where $U = \{2, 3, \ldots, n\}$.

3. The main results

Recall that we have $n > 0$, and $0 \le \ell - 1 \le k \le n$. We denote by $I(k, \ell)$ the ideal $I(V(\mathcal{F}^{k,\ell}))$. The main contribution of the paper is an explicit description of the reduced Gröbner bases for the ideals $I(k, \ell)$. First we state the following

Theorem 3.1. *Let $n > 0$, k and ℓ be integers such that $0 < \ell - 1 \le k \le n$. Let $\mathbb{F}$ be a field, and $\prec$ be an arbitrary term order on the monomials of $S = \mathbb{F}[x_1, \dots, x_n]$ for which $x_n \prec x_{n-1} \prec \dots \prec x_1$. If $k < (n+\ell)/2$, then the following set $\mathcal{G}$ of polynomials is a Gröbner basis with respect to $\prec$ of the ideal $I(k, \ell)$ of S:*

$$\mathcal{G} = \{x_1^2 - x_1, \dots, x_n^2 - x_n\} \cup \left\{x_J : J \in \binom{[n]}{k+1}\right\}$$
$$\cup \{f_{H,k} : H \in \mathcal{H}(t, \ell) \text{ for some } t,\ \ell \le t \le k\}.$$

Similarly, if $k \ge (n + \ell)/2$, then the set $\mathcal{G}^*$ below is a Gröbner basis of $I(k, \ell)$:

$$\mathcal{G}^* = \{x_1^2 - x_1, \dots, x_n^2 - x_n\} \cup \left\{x^J : J \in \binom{[n]}{n-(k-\ell)}\right\}$$
$$\cup \{f_{H,k} : H \in \mathcal{H}(t, \ell) \text{ for some } t,\ \ell \le t \le n - (k - \ell + 1)\}.$$

Theorem 3.1 allows us to describe the initial ideals and reduced Gröbner bases of the ideals $I(k, \ell)$. Let n, $\mathbb{F}$ and $\prec$ be as in Theorem 3.1. Uniform families (i.e. $\ell = 1$) have been treated in [11]. Here we focus on the case $\ell > 1$. This differs slightly from the uniform case because there x_1 is a leading monomial.

For $0 < \ell \le k + 1$ let $\mathcal{B}(k, \ell)$ denote the collection of subsets $U \subseteq [n]$, where $U = \{u_1 < \dots < u_{k+1}\}$ and $u_j \ge 2j - \ell + 1$ holds for $j = 1, \dots, k$.

For example, if $n = 5$, $k = 2$ and $\ell = 2$, then

$$\mathcal{B}(2, 2) = \{\{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}.$$

Corollary 3.2. *Let $1 < \ell \le k + 1$. Assume that $k < (n + \ell)/2$. Then*

$$\{x_i^2 : i = 1, \dots, n\} \cup \{x_U : U \in \mathcal{B}(k, \ell)\}$$
$$\cup \{x_H : H \in \mathcal{H}(t, \ell) \text{ for some } t,\ \ell \le t \le k\}$$

minimally generates $\mathrm{ini}(I(k, \ell))$.

Next assume that $k \ge (n + \ell)/2$. Then

$$\{x_i^2 : i = 1, \dots, n\} \cup \{x_U : U \in \mathcal{B}(n - k + \ell - 1, \ell)\}$$
$$\cup \{x_H : H \in \mathcal{H}(t, \ell) \text{ for some } t,\ \ell \le t \le n - k + \ell - 1\}$$

minimally generates $\mathrm{ini}(I(k, \ell))$.

As an example, if $n = 5$, $k = 2$ and $\ell = 2$, then $k < (n + \ell)/2$, thus

$$\{x_i^2 : i = 1, \dots, 5\} \cup \{x_U : U \in \mathcal{B}(2,2)\} \cup \{x_H : H \in \mathcal{H}(2,2)\}$$

minimally generates $\mathrm{ini}(I(2,2))$.

It turns out that a subset of $\mathcal{G}$ ($\mathcal{G}^*$ resp.) is the reduced Gröbner basis of $I(k,\ell)$.

Corollary 3.3. *Let n, $\mathbb{F}$, and $\prec$ as in Theorem 3.1, and $1 < \ell \le k+1$. Assume that $k < (n+\ell)/2$. Then the following set is the reduced Gröbner basis with respect to $\prec$ of the ideal $I(k,\ell)$:*

$$\{x_1^2 - x_1, \dots, x_n^2 - x_n\} \cup \{x_J : J \in \mathcal{B}(k,\ell)\} \\ \cup \{f_{H,k} : H \in \mathcal{H}(t,\ell) \text{ for some } t,\ \ell \le t \le k\}.$$

In the case $k \ge (n+\ell)/2$ the following set is the reduced Gröbner basis with respect to $\prec$ of the ideal $I(k,\ell)$:

$$\{x_1^2 - x_1, \dots, x_n^2 - x_n\} \cup \{x^J : J \in \mathcal{B}(n-k+\ell-1,\ell)\} \\ \cup \{f_{H,k} : H \in \mathcal{H}(t,\ell) \text{ for some } t,\ \ell \le t \le n-k+\ell-1\}.$$

We continue the preceding example. Let $\mathbb{F}$ be an arbitrary field and $\prec$ be a term order on the monomials of $S = \mathbb{F}[x_1, \dots, x_n]$ for which $x_n \prec x_{n-1} \prec \dots \prec x_1$. Let $n = 5$, $k = 2$ and $\ell = 2$. Clearly $k = 2 < (n+\ell)/2 = 3.5$, hence

$$\{x_1^2 - x_1, \dots, x_5^2 - x_5\} \cup \{x^J : J \in \mathcal{B}(2,2)\} \cup \{\sigma_{[5],2} - \sigma_{[5],1} + 3\}$$

is the reduced Gröbner basis with respect to $\prec$ of the ideal $I(2,2)$.

The functions $f_{H,k}$ are quite important in our discussion. Next we provide an alternative description for them. Let $H \in \mathcal{H}(t,\ell)$ and $H' = H \cup \{2t-\ell+1, 2t-\ell+2, \dots, n\} \subseteq [n]$, where $0 < \ell \le t \le (n+\ell)/2$. Let

$$l_{H'} = l_{H'}(x_1, \dots, x_n) := \sum_{j \in H'} x_j.$$

We denote by $g_{H,k}$ the polynomial obtained from

$$\prod_{j=0}^{t-1} (l_{H'} - k + j)$$

by application of the relations $x_i^2 - x_i$. Obviously $g_{H,k}$ is a linear combination of square-free monomials.

Proposition 3.4. *Assume that* $\mathrm{char}\, \mathbb{F} = 0$, *or* $\mathrm{char}\, \mathbb{F} > t$. *Then we have*

$$f_{H,k} = \frac{1}{t!} \cdot g_{H,k}.$$

We prove these statements in Section 5.

4. Order shattering by ℓ-wide families

In this section we describe the sets order shattered by the ℓ-wide families $\mathcal{F}^{k,\ell}$. For $0 \le \ell - 1 \le k \le n$ we write

$$D(k,\ell) = \{\{g_1 < \cdots < g_t\} \subseteq [n] : t \le k \text{ and } g_j \ge 2j - \ell + 1 \text{ if } 1 \le j \le t\}.$$

In particular $\emptyset \in D(k,\ell)$.

As an example, let $n = 4$, $k = 2$ and $\ell = 2$. Then

$$D(2,2) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}.$$

If $n = 5$, $k = 2$ and $\ell = 1$, then

$$D(2,1) = \{\emptyset, \{2\}, \{3\}, \{4\}, \{5\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}.$$

The following theorem is the combinatorial core of our results; it is an extension of (2).

We shall distinguish two cases: $k < (n+\ell)/2$ and $k \ge (n+\ell)/2$. The second case can be reduced to the first one by (3).

Theorem 4.1. (a) *Let* $0 \le k < (n+\ell)/2$. *Then*

$$\mathrm{osh}(\mathcal{F}^{k,\ell}) = D(k,\ell)$$

(b) *If* $k \ge (n+\ell)/2$, *then*

$$\mathrm{osh}(\mathcal{F}^{k,\ell}) = D(n-k+\ell-1,\ell).$$

Proof. Part (b) follows from part (a) by complementarity. Indeed, $\mathrm{co}(\mathcal{F}^{k,\ell}) = \mathcal{F}^{n-(k-\ell+1),\ell}$, hence by (3)

$$\mathrm{osh}(\mathcal{F}^{k,\ell}) = \mathrm{osh}\left(\mathrm{co}(\mathcal{F}^{k,\ell})\right) = \mathrm{osh}(\mathcal{F}^{n-k+\ell-1,\ell}).$$

On the other hand, $n-(k-\ell+1) < (n+\ell)/2$ follows from $n \le 2k-\ell$, and hence case (a) applies for the ℓ-wide family $\mathcal{F}^{n-(k-\ell+1),\ell}$.

We turn to the proof of case (a) now. First we verify that $\mathrm{osh}(\mathcal{F}^{k,\ell}) \subseteq D(k,\ell)$. For this it is enough to check that if $D \notin D(k,\ell)$, then $D \notin \mathrm{osh}(\mathcal{F}^{k,\ell})$.

Assume that $D = \{d_1 < \cdots < d_s\} \notin D(k,\ell)$. Then either $s > k$ or there is an index $i \le s \le k$ such that $d_i < 2i - \ell + 1$. In the first case $D \notin \mathrm{osh}(\mathcal{F}^{k,\ell})$, because $\mathrm{osh}(\mathcal{F}^{k,\ell}) \subseteq \mathrm{osh}\left(\binom{[n]}{\le k}\right) = \binom{[n]}{\le k}$. In the second case let t be the smallest such index i. Let $H := \{d_1, \ldots, d_t\} \subseteq D$. Then obviously $\ell \le t \le k$, hence $H \in \mathcal{H}(t, \ell)$. Also, $D \in \mathrm{osh}(\mathcal{F}^{k,\ell})$ would imply that $H \in \mathrm{osh}(\mathcal{F}^{k,\ell})$, because $\mathrm{osh}(\mathcal{F}^{k,\ell})$ is downward closed. From the latter fact we derive a contradiction below.

Indeed, suppose that there exists $H \in \mathcal{H}(t, \ell) \cap \mathrm{osh}(\mathcal{F}^{k,\ell})$, where $0 < \ell \le t \le k < (n + \ell)/2$. Let $H = \{h_1 < \cdots < h_t\}$, then $h_t = 2t - \ell$ and $h_i \ge 2i - \ell + 1$ for each $1 \le i \le t - 1$. Since $H \in \mathrm{osh}(\mathcal{F}^{k,\ell})$, the definition of order-shattering gives us two subsets $F_0, F_1 \in \mathcal{F}^{k,\ell}$ such that $F_0 \cap T = F_1 \cap T$, where $T = \{h_t + 1, \ldots, n\}$, and $F_0 \cap H = \emptyset$, $F_1 \cap H = H$.

Let $P := [n] \setminus (T \cup H)$. Clearly $|P| = t - \ell$. By definition

$$F_0 = (F_0 \cap T) \cup (F_0 \cap H) \cup (F_0 \cap P)$$

is a decomposition into disjoint sets. Let $r := |F_0 \cap T|$. Then

$$|F_0| = |F_0 \cap T| + |F_0 \cap H| + |F_0 \cap P| \le r + t - \ell,$$

because $|F_0 \cap P| \le |P| = t - \ell$. Similarly

$$|F_1| = |F_1 \cap T| + |F_1 \cap H| + |F_1 \cap P| \ge r + t,$$

because $|F_1 \cap H| = |H| = t$. Hence $|F_1| - |F_0| \ge \ell$, which contradicts to $F_0, F_1 \in \mathcal{F}^{k,\ell}$.

Now we prove that $D(k, \ell) \subseteq \mathrm{osh}(\mathcal{F}^{k,\ell})$. Let $D = \{d_1 < \cdots < d_t\} \in D(k, \ell)$. This means that $d_i \ge 2i - \ell + 1$ for $1 \le i \le t$.

We extend the base set $[n]$ by $\ell - 1$ new elements: let

$$X = \{-\ell + 2, -\ell + 3, \ldots, 0, \ldots, n\}.$$

We intend to establish that $D \in \mathrm{osh}\binom{X}{k}$. We note first that in the definition of order-shattering only the ordering of the elements of the ground set is what matters, not the elements themselves. From $0 \le k < (n + \ell)/2$ we have $0 \le k \le |X|/2$. In view of (2), $D \in \mathrm{osh}\binom{X}{k}$ holds iff there are at least $2i - 1$ elements of X which are smaller than d_i. This is indeed true because $D \in D(k, \ell)$.

Let $F \in \binom{X}{k}$ be an arbitrary subset. We define $F' := F \cap [n]$. Then we have $F' \in \mathcal{F}^{k,\ell}$, because we dropped at most $\ell - 1$ elements from F, and $|F| = k$. Let $\mathcal{G} = \{F' : F \in \binom{X}{k}\}$. Then $\mathcal{G} \subseteq \mathcal{F}^{k,\ell}$, and hence $D \in \mathrm{osh}\binom{X}{k} \cap 2^{[n]} = \mathrm{osh}(\mathcal{G}) \subseteq \mathrm{osh}(\mathcal{F}^{k,\ell})$. This completes the proof. □

For $k < (n + \ell)/2$ we denote by $\mathcal{M}(k, \ell)$ the set of all monomials x_G such that $G \in D(k, \ell)$. For $k \ge (n + \ell)/2$ we set $\mathcal{M}(k, \ell)$ to be $\mathcal{M}(n - (k - \ell + 1), \ell)$.

Theorems 2.1 and 4.1 give us the lexicographic standard monomials of ℓ-wide families.

Corollary 4.2. *Let $n > 0$, k, ℓ be integers, $0 \le \ell - 1 \le k \le n$. Then $\mathcal{M}(k, \ell)$ is the set of standard monomials for $I(k, \ell)$ with respect to the lexicographic order $\prec_l$. In particular,*

$$|\mathcal{M}(k, \ell)| = \sum_{i=k-\ell+1}^{k} \binom{n}{i} \tag{12}$$

and $\mathcal{M}(k, \ell)$ constitutes an $\mathbb{F}$ basis of the space of functions from $V(\mathcal{F}^{k,\ell})$ to $\mathbb{F}$.

Remark. One can obtain (12) by lattice path counting techniques (see [12] for an excellent account on those methods). This provides an alternative way to prove Theorem 4.1.

5. Proofs of the main results

We extend the argument of [11] from the case $\ell = 1$ to general ℓ-wide families. For the reader's convenience we include here proofs of some auxiliary facts which have been given in [11].

Let $\mathbb{F}$ be a fixed field and let $\prec$ be an arbitrary term order on the monomials of $S = \mathbb{F}[x_1, \dots, x_n]$ for which $x_n \prec x_{n-1} \prec \dots \prec x_1$. Let $0 \le \ell - 1 \le k \le n$. For $t > 0$ and $G \subseteq [n]$ we put

$$\mathcal{F}^{k,t}(G) = \{D \subseteq [n] : k - t < |D \cap G| \le k\}.$$

Theorem 5.1. *Assume that $0 < \ell \le t \le (n + \ell)/2$, $H \in \mathcal{H}(t, \ell)$, and $0 \le k \le n$. Then $f_{H,k} \in I(V(\mathcal{F}^{k,t}(H')))$.*

Proof. We recall first that $H' = H \cup \{2t - \ell + 1, 2t - \ell + 2, \dots, n\} \subseteq [n]$. Let $D \in \mathcal{F}^{k,t}(H')$ and let $v = v_D$ be the characteristic vector of D. By definition

$$|D \cap H'| \in \{k, k - 1, \dots, k - t + 1\}. \tag{13}$$

Now

$$f_{H,k}(v) = \sum_{i=0}^{t} (-1)^{t-i} \binom{k-i}{t-i} \sigma_{H',i}(v) = \sum_{i=0}^{t} (-1)^{t-i} \binom{k-i}{t-i} \binom{|D \cap H'|}{i}.$$

We use the following identity

$$\binom{x - k + t - 1}{t} = \sum_{i=0}^{t} (-1)^{t-i} \binom{x}{i} \binom{k-i}{t-i}, \tag{14}$$

which holds for every $x \in \mathbb{C}$, $k \in \mathbb{Z}$ and $t \in \mathbb{Z}^+$. From (14) we infer that

$$f_{H,k}(v) = \binom{|D \cap H'| - k + t - 1}{t}, \tag{15}$$

which is indeed 0 because of (13). It remains to prove (14). We consider first the Vandermonde identity ([10], pp. 169–170)

$$\binom{x+s}{t} = \sum_{i=0}^{t} \binom{x}{i}\binom{s}{t-i}, \tag{16}$$

which holds for all $x, s \in \mathbb{C}$ and $t \in \mathbb{Z}^+$. By negating the upper variable s on the right-hand side we obtain

$$\binom{x+s}{t} = \sum_{i=0}^{t} \binom{x}{i} (-1)^{t-i} \binom{t-s-i-1}{t-i}.$$

Finally the substitution $s = t - k - 1$ gives (14). □

Lemma 5.2. *Assume that $0 \le k \le n$, $0 < \ell \le t \le \min(k, n-k+\ell-1)$, and $H \in \mathcal{H}(t, \ell)$. Then $f_{H,k}$ can be written as a linear combination of square-free monomials*

$$f_{H,k} = \sum_{U \subseteq H',\ |U| \le t} \alpha_U x_U, \tag{17}$$

where $\alpha_U \in \mathbb{F}$. The leading monomial of $f_{H,k}$ with respect to $\prec$ is x_H and the leading coefficient is $\alpha_H = 1$. Also we have $x_H \notin \mathcal{M}(k, \ell)$, but $x_U \in \mathcal{M}(k, \ell)$ for $U \subseteq H'$, $|U| \le t$, $U \ne H$. The latter monomials x_U are precisely the non-leading monomials in (17).

Proof. First suppose that $k < (n + \ell)/2$.

The statement about the form (17) follows from the fact that the (elementary) symmetric polynomials $\sigma_{H',i}$ $(0 \le i \le t)$ are linear combinations of monomials x_U with $U \subseteq H'$ and $|U| \le t$. Let $U = \{u_1 < \cdots < u_j\}$ be any such subset and write $H = \{s_1 < \cdots < s_t\}$. By the definition of H' we have $s_i \le u_i$ for $i = 1, \ldots, j$, hence $x_U \preceq x_{s_1} \cdots x_{s_j} \preceq x_H$. Also, the coefficient of x_H in $f_{H,k}$ is $(-1)^{t-t}\binom{k-t}{t-t} = 1$. These imply that x_H is the leading monomial of $f_{H,k}$.

It is immediate that $x_H \notin \mathcal{M}(k, \ell)$ because $s_t = 2t - \ell$. Next suppose that $U \subseteq H'$, $|U| = j \le t$, and $U \ne H$. If $j < t$, then $2i - \ell + 1 \le s_i \le u_i$ for $1 \le i \le j$, and $j \le k$ imply $x_U \in \mathcal{M}(k, \ell)$. If $|U| = t$, then from $U \ne H$ we infer additionally that $u_t > s_t = 2t - \ell$, giving that $x_U \in \mathcal{M}(k, \ell)$.

In the case $k \ge (n + \ell)/2$ we can give a similar proof, using the relations $\mathcal{M}(n - (k - \ell + 1), \ell) = \mathcal{M}(k, \ell)$ and $t \le n - k + \ell - 1$. This concludes the proof. □

Corollary 5.3. *Assume that* $0 \le k \le n$, $0 < \ell \le t \le \min(k, n-k+\ell-1)$, *and* $H \in \mathcal{H}(t,\ell)$. *Then* $x_H \notin \mathrm{Sm}(\prec, \mathcal{F}^{k,t}(H'), \mathbb{F})$.

Proof. This is obvious from Theorem 5.1 and Lemma 5.2, because x_H is the leading term of $f_{H,k}$, a polynomial vanishing on $V(\mathcal{F}^{k,t}(H'))$. □

Lemma 5.4. *Let* $0 < \ell \le t \le (n+\ell)/2$, $0 \le k \le n$, *and* $H \in \mathcal{H}(t,\ell)$. *Then* $\mathcal{F}^{k,\ell} \subseteq \mathcal{F}^{k,t}(H')$.

Proof. Let $D \in \mathcal{F}^{k,\ell}$. We know that $H' = H \cup \{2t-\ell+1, 2t-\ell+2, \dots, n\}$ has $n-t+\ell$ elements, hence the size of $[n] \setminus H'$ is $t-\ell$. Using also, that $k-\ell+1 \le |D|$, we obtain

$$k-t+1 \le |D| - |[n] \setminus H'| \le |D \cap H'| \le |D| \le k,$$

hence $D \in \mathcal{F}^{k,t}(H')$. □

Corollary 5.5. *Let* $0 < \ell \le t \le (n+\ell)/2$, $0 \le k \le n$, *and* $H \in \mathcal{H}(t,\ell)$. *Then* $f_{H,k} \in I(k,\ell)$.

Proof. We know that $f_{H,k} \in I(V(\mathcal{F}^{k,t}(H')))$ by Theorem 5.1 and $\mathcal{F}^{k,\ell} \subseteq \mathcal{F}^{k,t}(H')$ by Lemma 5.4. We conclude that $f_{H,k} \in I(V(\mathcal{F}^{k,t}(H'))) \subseteq I(k,\ell)$. □

Proof of Theorem 3.1. Let k, $0 < n$ and ℓ be integers, such that $0 \le \ell - 1 \le k \le n$. We consider first the case $k < (n+\ell)/2$. It is immediate that $x_i^2 - x_i \in I(k,\ell)$ and $x_J \in I(k,\ell)$ if $|J| = k+1$, hence by Corollary 5.5, $\mathcal{G} \subseteq I(k,\ell)$.

To show that $\mathcal{G}$ is a Gröbner basis of $I(k,\ell)$ we use the characterization of Gröbner bases from Theorem 2.2. Let I be the ideal of S generated by $\mathcal{G}$.

Lemma 5.6. $I = I(k,\ell)$.

Proof. It is immediate that $I \subseteq I(k,\ell)$, because $\mathcal{G} \subseteq I(k,\ell)$.

Now let $f \in S$ be an arbitrary polynomial. Obviously we can reduce by $\mathcal{G}$ any monomial of f which is divisible by x_i^2 for some i. We can thus assume that f contains only square-free monomials x_U, $U \subseteq [n]$. We can also eliminate those x_U for which $|U| > k$.

Suppose now that x_U is a monomial ($|U| \le k$, $U = \{u_1 < u_2 < \cdots < u_j\}$) appearing in f which is not in $\mathcal{M}(k,\ell)$. Then there exists an index $i \le j$ such that $u_i < 2i - \ell + 1$. Let t be the smallest such index i and put $H := \{u_1, \dots, u_t\}$. Then necessarily $\ell \le t$ and $H \in \mathcal{H}(t,\ell)$ and x_H divides x_U by definition. By Lemma 5.2 x_H is the leading monomial of $f_{H,k} \in \mathcal{G}$, hence via $f_{H,k}$ we can reduce f further.

What we obtained is that any $f \in S$ can be $\mathcal{G}$-reduced to a linear combination

$$\sum_{w \in \mathcal{M}(k,\ell)} \alpha_w \cdot w, \tag{18}$$

where $\alpha_w \in \mathbb{F}$. From (18) and Corollary 4.2 we deduce that

$$\dim_{\mathbb{F}} S/I \leq |\mathcal{M}(k,\ell)| = \sum_{i=k-\ell+1}^{k} \binom{n}{i} = \dim_{\mathbb{F}} S/I(k,\ell), \tag{19}$$

which, together with $I \subseteq I(k,\ell)$, implies that $I = I(k,\ell)$. The Lemma is proved. $\square$

If we consider the reduced form (18) of an $f \in I(k,\ell)$, then every α_w is zero by Corollary 4.2, because $\mathcal{M}(k,\ell)$ is a basis of the space of functions from $V(\mathcal{F}^{k,\ell})$ to $\mathbb{F}$. This proves that $\mathcal{G}$ is a Gröbner basis of $I(k,\ell)$, when $k < \frac{n+\ell}{2}$.

Essentially the same argument works when $k \geq (n+\ell)/2$.

It is immediate that $x_i^2 - x_i \in I(k,\ell)$. We know that if $H, J \subseteq [n]$, then $x^J(v_H) = 0$ iff $H \cap J \neq \emptyset$. Now if $H \in \mathcal{F}^{k,\ell}$ and $J \in \binom{[n]}{n-(k-\ell)}$, then H and J must intersect because

$$|H| + |J| \geq k - \ell + 1 + n - k + \ell = n + 1.$$

Assume now that t is an integer, such that $\ell \leq t \leq n - k + \ell - 1$. Then $k \geq \frac{n+\ell}{2}$ implies that $n - k + \ell - 1 < \frac{n+\ell}{2}$, showing that $t < \frac{n+\ell}{2}$. Corollary 5.5 applies, giving that $f_{H,k} \in I(k,\ell)$, whenever $H \in \mathcal{H}(t,\ell)$. We have therefore $\mathcal{G}^* \subseteq I(k,\ell)$.

From here we can prove that the ideal I generated by $\mathcal{G}^*$ is $I(k,\ell)$ as in Lemma 5.6. For this, one observes first that the leading term of a polynomial x^J, $|J| = n - (k-\ell)$ is x_J and all other terms have degree at most $n - k + \ell - 1$. These polynomials and $x_i^2 - x_i$ allow us to reduce any polynomial f into one of degree at most $n - (k - \ell + 1)$.

Reduction via polynomials $f_{H,k}$, where $H \in \mathcal{H}(t,\ell)$ gives a linear combination of type (18). From (18) we obtain (19) and $I = I(k,\ell)$ as before.

Finally, if $f \in I(k,\ell)$ then every α_w is zero by Corollary 4.2, therefore $\mathcal{G}^*$ is a Gröbner basis of $I(k,\ell)$. $\square$

Corollary 5.7. *Let $\mathbb{F}$, $\prec$, k, n, and ℓ be as in Theorem 3.1. Then*

$$\mathrm{sm}(\prec, \mathcal{F}^{k,\ell}, \mathbb{F}) = \mathcal{M}(k,\ell).$$

Remark. This was established in Corollary 4.2 for the lexicographic order $\prec_l$. The point here is that the statement holds for *any* term order $\prec$ on the monomials for which $x_n \prec x_{n-1} \prec \cdots \prec x_1$.

Proof. The argument of Theorem 3.1 shows that any monomial $w \notin \mathcal{M}(k,\ell)$ can be $\mathcal{G}$-reduced to $\prec$-smaller monomials. This means that w is a leading monomial for $I(k,\ell)$, giving the containment $\subseteq$. Equality then follows because by (9) and Corollary 4.2 both sides have the same size $|\mathcal{F}^{k,\ell}|$. □

Proof of Corollary 3.2. We assume first that $k < (n+\ell)/2$. Let $\mathcal{W}$ denote the set of monomials given in the statement. Clearly we have $\mathcal{W} \subset \mathrm{ini}(I(k,\ell))$. First we show that $\mathcal{W}$ is a generating set. For this it suffices to verify, that any monomial $w \notin \mathcal{M}(k,\ell)$ is divisible by an element of $\mathcal{W}$. If w is not square-free, then it is divisible by x_i^2 for some $1 \le i \le n$. We can therefore assume that $w = x_U$ for some $U = \{u_1 < \cdots < u_j\} \subseteq [n]$ and either there exists an $0 < i \le k$ such that $u_i < 2i - \ell + 1$, or $j > k$. In the first case let t be the smallest index i with $u_i < 2i - \ell + 1$. Then for $H = \{u_1, \ldots, u_t\}$ we have $H \in \mathcal{H}(t,\ell)$, where $\ell \le t \le k$, hence $x_H \in \mathcal{W}$ and x_H divides w. In the case $j > k$ for $H = \{u_1, \ldots, u_{k+1}\}$ we have $H \in \mathcal{B}(k,\ell)$, hence $x_H \in \mathcal{W}$, moreover x_H again divides w.

Using, that $2 \le \ell \le t$, it is easy to verify, that there are no nontrivial divisibilities among the elements of $\mathcal{W}$. This settles the minimality.

The case $k \ge (n+\ell)/2$ is reduced to the preceding one by using the following obvious consequence of Corollary 5.7:

$$\mathrm{ini}(I(k,\ell)) = \mathrm{ini}(I(n-k+\ell-1,\ell)).$$ □

Proof of Corollary 3.3. It follows from Corollary 3.2 that the leading terms of the sets of polynomials given in the statement are minimal generating sets of the initial ideal of $I(k,\ell)$. We have proven for $f_{H,k}$ and $f_{H,n-(k-\ell+1)}$ in Lemma 5.2 that all other (i.e. non-leading) monomials in these polynomials are actually standard monomials for $I(k,\ell)$ (and for $I(n-(k-\ell+1),\ell)$ as well). Also, because of $\ell > 1$ we have $x_i \in \mathcal{M}(k,\ell)$ for $i = 1, \ldots, n$.

It remains to check that the non-leading monomials of the polynomial x^J, where $J \in \mathcal{B}(k,\ell)$, are elements of $\mathcal{M}(k,\ell)$. Let $J = \{j_1 < \cdots < j_{k+1}\} \in \mathcal{B}(k,\ell)$. Then the non-leading monomials of x^J are x_U, where $U = \{u_1 < \cdots < u_m\} \subseteq J$ with $m = |U| \le k$. Clearly $u_i \ge j_i \ge 2i - \ell + 1$ holds whenever $1 \le i \le m$, hence $x_U \in \mathcal{M}(k,\ell)$. The proof is complete. □

Proof of Proposition 3.4. Let $D \subseteq [n]$ be an arbitrary set, $v = v_D$ the characteristic vector of D. From the definition of $g_{H,k}$ and (15) it is apparent that

$$\frac{1}{t!}g_{H,k}(v) = \binom{|D \cap H'| - k + t - 1}{t} = f_{H,k}(v).$$

Both $g_{H,k}$ and $f_{H,k}$ are linear combinations of square-free monomials. Square-free monomials are precisely the standard monomials for $\mathcal{F}^{n,n+1} = 2^{[n]}$. By the uniqueness of the standard decomposition of a function we conclude that

$$\frac{1}{t!}g_{H,k} = f_{H,k}. \qquad \square$$

6. Some consequences

6.1. The Hilbert function of $\mathcal{F}^{k,\ell}$. Here we give the values $h_{\mathcal{F}^{k,\ell}}(m)$ of the Hilbert function of the complete ℓ-wide family $\mathcal{F}^{k,\ell}$. We need first a useful fact of combinatorial nature about $\mathrm{osh}(\mathcal{F}^{k,\ell})$. As before $n > 0$, ℓ, k are integers, and $0 \le \ell - 1 \le k \le n$.

Theorem 6.1. *Suppose that* $0 \le i \le \min(k, n - k + \ell - 1)$. *Then*

$$\left|\mathrm{osh}(\mathcal{F}^{k,\ell}) \cap \binom{[n]}{i}\right| = \binom{n}{i} - \binom{n}{i-\ell}.$$

Remark. The binomial coefficient $\binom{n}{j}$ is understood to be 0 if $j < 0$.

Proof. We use the description of $\mathrm{osh}(\mathcal{F}^{k,\ell})$ given in Theorem 4.1. With $d = \min(k, n - k + \ell - 1)$ we have $d < \frac{n+\ell}{2}$ and $\mathrm{osh}(\mathcal{F}^{k,\ell}) = D(d, \ell)$. We consider first the case $0 \le i < \ell$. Then

$$\mathrm{osh}(\mathcal{F}^{k,\ell}) \cap \binom{[n]}{i} = D(d,\ell) \cap \binom{[n]}{i} = \binom{[n]}{i}, \tag{20}$$

giving the claim of the Theorem in this case. We turn now to the case $\ell \le i \le d$. The case (a) of Theorem 4.1 applies to $\mathcal{F}^{i,\ell}$, giving that $\mathrm{osh}(\mathcal{F}^{i,\ell}) = D(i, \ell)$. Directly from the definition we see that

$$D(d,\ell) \cap \binom{[n]}{i} = D(i,\ell) \cap \binom{[n]}{i},$$

hence

$$\mathrm{osh}(\mathcal{F}^{k,\ell}) \cap \binom{[n]}{i} = D(d,\ell) \cap \binom{[n]}{i} = D(i,\ell) \cap \binom{[n]}{i} = \mathrm{osh}(\mathcal{F}^{i,\ell}) \cap \binom{[n]}{i}. \tag{21}$$

We have also the following consequences of Theorem 4.1:

$$\mathrm{osh}(\mathcal{F}^{i-1,\ell}) \subseteq \mathrm{osh}(\mathcal{F}^{i,\ell}) \tag{22}$$

and

$$\mathrm{osh}(\mathcal{F}^{i,\ell}) \cap \binom{[n]}{i} = \mathrm{osh}(\mathcal{F}^{i,\ell}) \setminus \mathrm{osh}(\mathcal{F}^{i-1,\ell}). \tag{23}$$

Corollary 4.2, applied to both $\mathcal{F}^{i,\ell}$ and $\mathcal{F}^{i-1,\ell}$ gives that the size of the set in (23) is

$$\sum_{j=i-\ell+1}^{i} \binom{n}{j} - \sum_{s=i-\ell}^{i-1} \binom{n}{s} = \binom{n}{i} - \binom{n}{i-\ell}.$$

This, together with (21), proves the statement. □

We turn now to the computation of the Hilbert function $h_{\mathcal{F}^{k,\ell}}(m)$. By (10), h gives the rank of certain important inclusion matrices:

$$\mathrm{rank}_{\mathbb{F}}\ I\left(\mathcal{F}^{k,\ell}, \binom{[n]}{\le m}\right) = h_{\mathcal{F}^{k,\ell}}(m).$$

Theorem 6.2. *Let $n > 0$, ℓ, k, m be integers, Let $0 \le \ell - 1 \le k \le n$, and $0 \le m \le \min(k, n-k+\ell-1)$. Let $\mathbb{F}$ be an arbitrary field. Then we have*

$$h_{\mathcal{F}^{k,\ell}}(m) = \left|\mathrm{osh}(\mathcal{F}^{k,\ell}) \cap \binom{[n]}{\le m}\right| = \sum_{i=\max(0,m-\ell+1)}^{m} \binom{n}{i}. \tag{24}$$

Proof. For short we write $\mathcal{F} = \mathcal{F}^{k,\ell}$. From (11) we infer that

$$h_{\mathcal{F}}(m) = \left|\mathrm{Sm}(\prec_{dl}, \mathcal{F}, \mathbb{F}) \cap \binom{[n]}{\le m}\right|,$$

where $\prec_{dl}$ is the deglex order (or any other term order which refines the partial ordering by degree). From Corollary 5.7 and Theorem 4.1 we have

$$\mathrm{Sm}(\prec_{dl}, \mathcal{F}, \mathbb{F}) = D(t, \ell) = \mathrm{osh}(\mathcal{F}),$$

where $t = \min(k, n-k+\ell+1)$. We consider the following decomposition into a disjoint union

$$\mathrm{osh}(\mathcal{F}) \cap \binom{[n]}{\le m} = \cup_{j=0}^{m} \left(\mathrm{osh}(\mathcal{F}) \cap \binom{[n]}{j}\right).$$

Turning to the sizes of the above sets, Theorem 6.1 gives that

$$h_{\mathcal{F}}(m) = \left|\cup_{j=0}^{m} \left(\mathrm{osh}(\mathcal{F}) \cap \binom{[n]}{j}\right)\right| = \sum_{j=0}^{m} \left(\binom{n}{j} - \binom{n}{j-\ell}\right) = \sum_{j=r}^{m} \binom{n}{j},$$

where $r = \max(0, m-\ell+1)$. This finishes the proof. □

6.2. A special case of a conjecture by Frankl. Finally we prove Theorem 1.2. It turns out to be a simple consequence of Theorem 6.2.

Proof of Theorem 1.2. $\mathcal{F}$ is an ℓ-wide family, therefore we have $\mathcal{F} \subseteq \mathcal{F}^{k,\ell}$ for some k, ℓ, with $0 \le \ell - 1 \le k \le n$. Put $r = \min(k, n-k+\ell-1, t-1)$. Theorem 4.1, the assumption on t together with $\mathrm{osh}(\mathcal{F}) \subseteq \mathrm{sh}(\mathcal{F})$ imply that the family $\mathcal{F} \subseteq \mathcal{F}^{k,\ell}$ has no order-shattered sets of size $> r$. We have thus

$$\mathrm{osh}(\mathcal{F}) \subseteq \mathrm{osh}(\mathcal{F}^{k,\ell}) \cap \binom{[n]}{\le r}.$$

Theorem 6.2 implies that

$$|\,\mathrm{osh}(\mathcal{F})\,\Big| \le |\,\mathrm{osh}(\mathcal{F}^{k,\ell}) \cap \binom{[n]}{\le r}\Big| = \sum_{i=\max(0,r-\ell+1)}^{r} \binom{n}{i} \le \sum_{i=\max(0,t-\ell)}^{t-1} \binom{n}{i}.$$

The last inequality follows from $r \le t-1$ and $t \le \frac{n+\ell}{2}$. The proof is complete. $\square$

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(Received September 6, 2004; revised October 3, 2006)