Publ. Math. Debrecen
70/3-4 (2007), 291–305

On the 4-dimensional hyperbolic hypercube mosaic

By LÁSZLÓ NÉMETH (Sopron)

Abstract. We construct successively belts in the 4-dimensional hyperbolic regular hypercube mosaic beginning with a vertex, then considering all elements of the mosaic adjacent to it, then all elements of the mosaic adjacent to but different from those already considered, etc. In this article we determine the limits of the ratios of the volumes of the consecutive belts for the hyperbolic mosaic $\{4, 3, 3, 5\}$ and its dual mosaic.

Introduction

In the 4-dimensional hyperbolic space there is a hypercube mosaic ([1]). Its Schäfli's symbol is $\{4, 3, 3, 5\}$. In this article we examine this mosaic and its dual mosaic $\{5, 3, 3, 4\}$.

For the mosaic $\{4, 3, 3, 5\}$, let us fix a point P, as a vertex of the mosaic and create belts around it. The belt 1 consists of the fundamental domains (hypercubes) of the mosaic containing P. (The belt 0 is the point P.) The belt 2 consists of the fundamental domains having at least one common vertex with the belt 1. If the belt i is known, let the belt $(i+1)$ consist of the fundamental domains that have common vertices with the belt i, but do not have with the belt $(i-1)$. Let Π_i denote the polyhedron determined by the outer boundary of the belt i $(\Pi_0 = P)$. Π_i is the union of all the belts j, where $j = 0 \dots i$.

Let V_i denote the volume of the belt i, and F_i^k $(0 \leq k < 4)$, denote the the sum of the k-dimensional volumes of the k-dimensional faces of the mosaic on the surface of Π_i. (The 0-dimensional volume of a vertex is 1.) Furthermore, let $S_i = \sum_{j=0}^{i} V_j$, it is the volume of Π_i.

Mathematics Subject Classification: 52C22, 52B10, 05B45.
Key words and phrases: 4-dimensional hyperbolic space, hypercube mosaic.

In this article we are going to prove the following theorems.

Theorem 1. *For* $0 \le k \le 3$ *we have* $\lim_{i\to\infty} \frac{V_{i+1}}{V_i} = \lim_{i\to\infty} \frac{S_{i+1}}{S_i} = \lim_{i\to\infty} \frac{F^k_{i+1}}{F^k_i} \approx 2381.8$ *(it can be called* crystal-growing ratio*) and* $\lim_{i\to\infty} \frac{V_i}{S_i} \approx 0.9996$ *in case of the mosaic* $\{4,3,3,5\}$.

The mosaic $\mathcal{M}$ is the *reciprocal* (or *dual*) mosaic of the mosaic $\mathcal{N}$, if the elements of $\mathcal{M}$ are the Dirichlet–Voronoi cells of the vertices of $\mathcal{N}$. Coxeter [1] showed that the mosaic $\{s,r,q,p\}$ is the dual of the mosaic $\{p,q,r,s\}$. Thus the mosaic $\{5,3,3,4\}$ is the dual of $\{4,3,3,5\}$, and its elements are 120-cells.

In case of the dual mosaic let the belt 0 be the D-V cell of the vertex P in the original mosaic. Its centre is P. We denote it by Π^*_0. Let the belt 1 be formed by the elements of the dual mosaic different from Π^*_0, but having common vertices with Π^*_0. Their centres coincide with the vertices of the original mosaic on the surface of Π_1, as they are their D-V cells in the original mosaic. We define the further belts similarly as in the case of the original mosaic, and Π^*_i as the union of the 0-th,..., i-th belts. Also V_i, F^k_i $(0 \le k \le 3)$ and S_i are defined analogously as for the original mosaic.

Theorem 2. *For the mosaic* $\{4,3,3,5\}$ *and its dual mosaic* $\{5,3,3,4\}$, *for* $0 \le k \le 3$, *each of the limits* $\lim_{i\to\infty} \frac{V_i}{S_i}$, $\lim_{i\to\infty} \frac{V_{i+1}}{V_i}$, $\lim_{i\to\infty} \frac{S_{i+1}}{S_i}$ *and* $\lim_{i\to\infty} \frac{F^k_{i+1}}{F^k_i}$ *are equal.*

Kárteszi [5], Horváth [3] and Vermes [9] dealt with similar problems in the hyperbolic plane, Zeitler [11] solved the hyperbolic cube mosaic $\{4,3,5\}$ case and the author [6] determined the limits of V_{i+1}/V_i, S_{i+1}/S_i, V_i/S_i for eight mosaics in the three-dimensional hyperbolic space with unbounded fundamental domains.

We classify the vertices of a mosaic incident to the surface of Π_i (they are not necessarily also vertices of Π_i) depending on their edge-distance from the belt $(i-1)$ (also from Π_{i-1}). The *edge-distance* of two vertices of a mosaic is l if the two vertices are connected to each other with l edges of the mosaic, but $(l-1)$ edges do not yet connect them. The edge-distance of the vertex Q of the mosaic from Π_{i-1} is l if there exists a vertex of the mosaic on the surface of Π_{i-1} so that its edge-distance from Q is l and the edge-distances of the other vertices on the surface of Π_{i-1} from Q are not smaller.

We denote the vertices of the mosaic on the surface of Π_i whose edge-distances from Π_{i-1} are 1 by A_i, and by B_i, C_i or D_i if their edge-distances are 2, 3 or 4 edges, respectively. If we can not determine the edge-distance of a vertex from Π_{i-1} during the actual examination (or it is not important), then we denote it by W_i. Let a_i, b_i, c_i, d_i denote the numbers of the vertices of type A_i, B_i, C_i, D_i on the surface of Π_i, respectively. Let r_i be the number of the hypercubes in the belt i. Further, let f_i^k $(0 \leq k \leq 3)$ denote the number of the k-dimensional faces (cubes, squares, edges, vertices) of the mosaic on the surface of Π_i.

We define the sequence r_i^* as the number of the elements of the dual mosaic in the i-th belt, and f_i^{*k}, $0 \leq k < 4$, as the number of k-dimensional faces on the surface of Π_i^* of the elements of the dual mosaic, and by f_i^{*4} we mean r_i^*, similarly as in the case of the original mosaic.

As the mosaic $\{4,3,3,5\}$ is regular, its elements are regular and congruent. The nearest vertices to a common vertex (i.e., at edge distance 1) determine a regular polyhedron, and these are congruent, for all the vertices. We denote this *vertex figure* belonging to the vertex P by Ω_P, and those belonging to A_i, B_i, C_i or D_i by Ω_A, Ω_B, Ω_C or Ω_D, respectively.

1. The mosaic $\{4,3,3,5\}$

The 4-dimensional hyperbolic hypercube mosaic consists of hypercubes $\{4,3,3\}$. The vertex figures of the mosaic are 600-cells $\{3,3,5\}$. The cells of the 600-cell are tetrahedra ($\{3,3\}$) and the neighbouring vertices of a vertex of the 600-cell form an icosahedron ($\{3,5\}$). (In Figure 1 we can see the 2-dimensional graph of the 600-cell.) The numbers of the vertices, edges, (2-dimensional) faces and cells of the 600-cell are 120, 720, 1200 and 600, respectively.

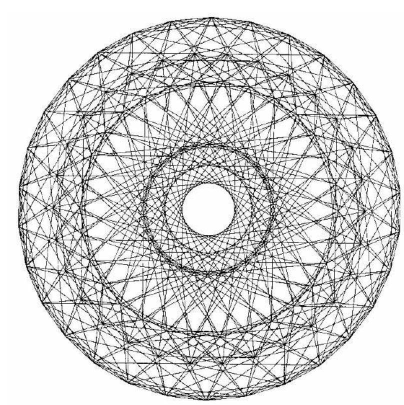

Figure 1

Examining the belt 1 in detail, we can see that on the surface of Π_1 there are four types of the vertices. The vertices A_1 are 1 edge away from P, and B_1, C_1 and D_1 are 2, 3 and 4 edges away from P, respectively. Thus the edge-distances of the vertices of the mosaic on the surface of Π_1 from P are at most 4 edges. (In Figure 2 we can see only a part — a vertex figure of one vertex — of the 600-cell, and the method how do we have to form the belt 1.) The vertices A_1 are the vertices of Ω_P. The vertices B_1 correspond to the edges of Ω_P. They are the vertices, diagonally opposite to P, of the squares with two edges leading from P to the endpoints of the edges of Ω_P. The vertices C_1 correspond to the (2-dimensional) faces of Ω_P, while the vertices D_1 correspond to the cells (3-dimensional faces) of Ω_P, in a similar sense. They are the vertices, diagonally opposite to P, of the cubes/hypercubes with three/four edges leading from P to the vertices of 2-dimensional/3-dimensional faces of Ω_P. We will say that the above A_1, B_1, C_1, D_1 *belong to* the vertices, edges, faces, cells of Ω_P, respectively (and, analogously, for $i > 1$, $A_1, \dots, D_1, \dots, A_i, \dots,$ D_i *belong to* vertices, $\dots$, cells of $\Omega_{A_{i-1}}, \dots, \Omega_{D_{i-1}}$'s; moreover, the hypercubes containing the cells also are called *belonging to* the cell). Now, we can state $a_1 = 120$, $b_1 = 720$, $c_1 = 1200$, $d_1 = 600$ and $r_1 = f_1^4 = 600$, $f_1^3 = 2400$, $f_1^2 = 7200$, $f_1^1 = 7440$, $f_1^0 = 2640$.

It is also true, that there are 20 adjacent hypercubes of the belt 1 with a common vertex A_1 (as there are 20 cells of Ω_P containing the same vertex A_1; another reason is that 20 hypercubes close around an edge PA_1). Similarly, there are 5, 2 or 1 adjacent hypercubes at a vertex B_1, C_1 or D_1, respectively, because 5, 2 and 1 cells of a 600-cell have a common edge, face and cell of the 600-cell, respectively.

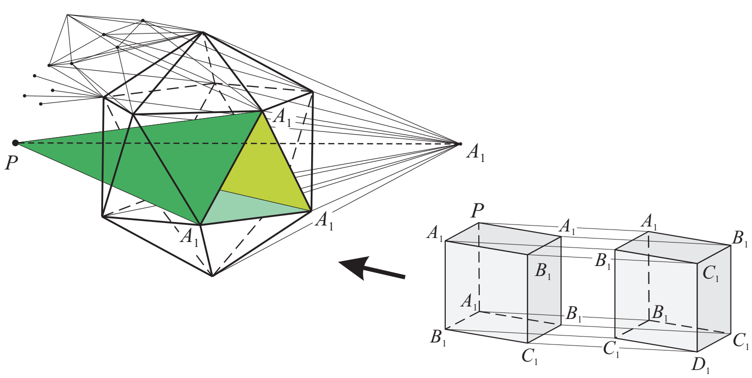

Figure 2

In the following we construct the belts.

In Figures 3, 4, 5 and 6 we can see the vertex figures. A vertex W_{i-1} ($i \geq 1$) and all the edges, faces, cells of Ω_A belonging to some A_{i-1}, which contain W_{i-1} and the hypercubes belonging to the cells also containing W_{i-1} are in the belt i. An edge (it is bold in Figure 4) and all the faces, cells of Ω_B belonging to some B_{i-1} and the hypercubes belonging to the cells which contain the edge are in the belt i. A face (it is shaded in Figure 5) and all the cells of Ω_C belonging to some C_{i-1} and the hypercubes belonging to the cells which contain the face are in the belt i. A cell (it is shaded in Figure 6) of Ω_D belonging to some D_{i-1} and the hypercube belonging to this cell are in the belt i.

As the edge-distance of any two vertices of a hypercube is not bigger than four edges, the edge-distance of any vertex on the outer boundary of the belt j (for any j) from the belt $(j-1)$ is one, two, three or four. Thus on the outer boundary of the belt j there are only vertices of type A_j, B_j, C_j and D_j.

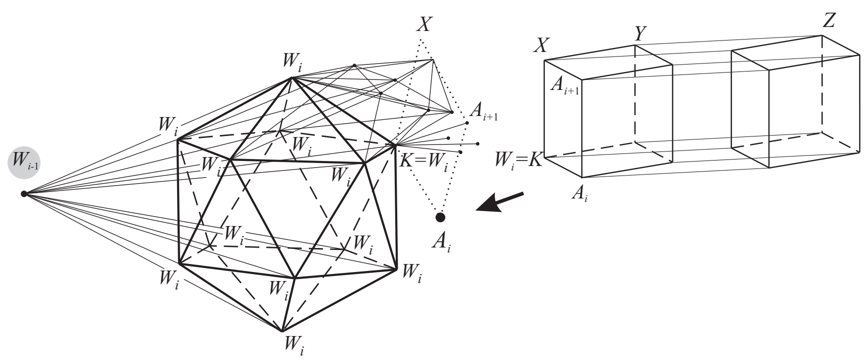

Figure 3

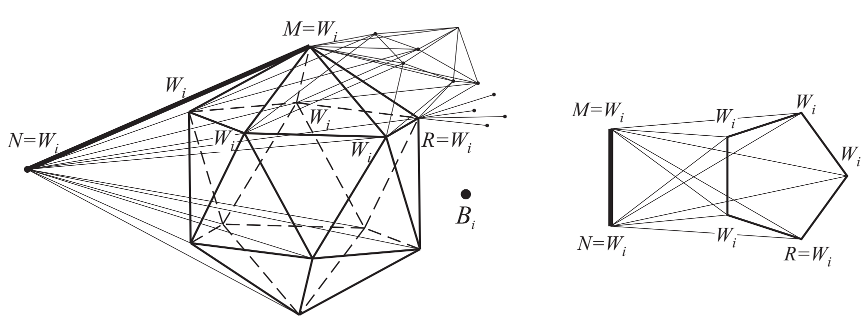

Figure 4

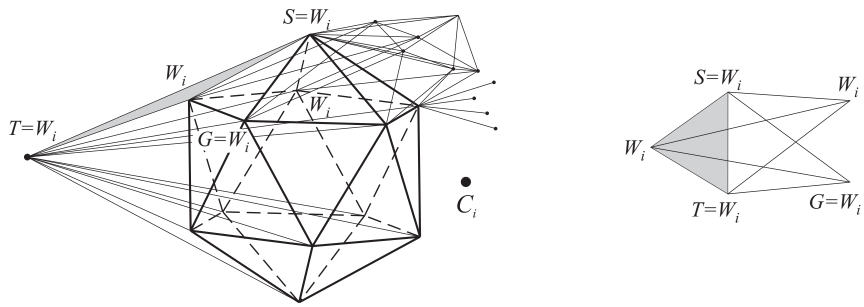

Figure 5

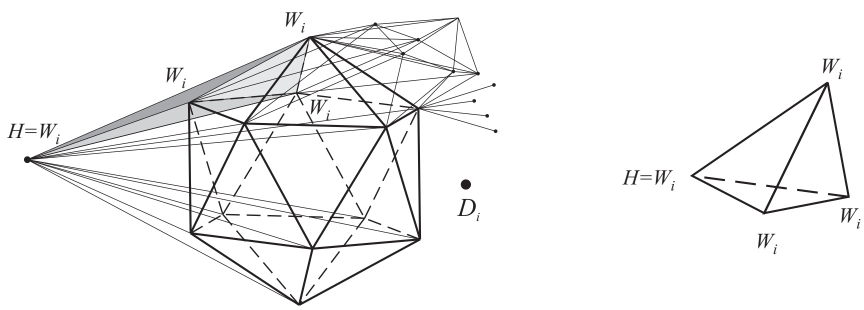

Figure 6

Lemma 1.1. $a_{i+1} = 107a_i + 113b_i + 115c_i + 116d_i \quad (i \geq 1)$.

Proof. We classify the vertices of Ω's for determining the number of the vertices A_{i+1}. As they are one edge-distant from the current vertex, the vertices A_{i+1} come from among them. For each A_i the Ω_A has a vertex W_{i-1} and 12 other vertices W_i (which have common edges with W_{i-1}) (Figure 3). They are in (or on) the belt i. Thus the remaining $120-13$ vertices are all A_{i+1} and they are one edge away from only the examined A_i. The number of all A_{i+1} belonging to all A_i is $107a_i$.

Similarly for the vertices B_i, there are 7 vertices of Ω_B in the belt i (Figure 4), so the remaining $120-7$ vertices are A_{i+1}. Their edge-distances are 1 from the current B_i, and only from this vertex. Their total number is $113b_i$. For C_i, Ω_C has 5 vertices in the belt i (Figure 5), so the remaining $120-5$ vertices are A_{i+1} and they belong to the current C_i. Their total number is $115c_i$. Similarly for D_i the number of all A_{i+1} is $116d_i$.

Summarizing the four statements we get the Lemma. □

Lemma 1.2. $b_{i+1} = 606a_i + 652b_i + 669c_i + 678d_i \quad (i \geq 1)$.

Proof. Now we classify the edges of the Ω's for determining the number of the vertices B_{i+1}. There are vertices B_{i+1} only along the edges whose neither endpoints are incident to the surface of Π_i. If one of them is on the surface then the vertex belonging to that edge is A_{i+1} or it is in the belt i. An example is in Figure 3 the vertex X. It is 2 edges away from A_i, but its edge-distance from $K = W_i$ (namely from Π_i) is 1. Thus the vertex X is A_{i+1} which was counted for K (in Lemma 1.1).

We are going to count the edges of the Ω's that have at least one vertex W_i and subtract their numbers from the number of all the edges, from 720.

In case of Ω_A for A_i the numbers of the edges joining to all the vertices W_i and W_{i-1} are 12. The number of all edges containing W_i is $13 \cdot 12$, but now 42 edges are counted doubly. Thus the number of the edges with one vertex W_i is $13 \cdot 12 - 42 = 114$. Similarly for Ω_B, Ω_C, Ω_D for the vertices B_i, C_i and D_i the numbers of the edges with one vertex some W_i are $7 \cdot 12 - 16 = 68$, $5 \cdot 12 - 9 = 51$ and $4 \cdot 12 - 6 = 42$, respectively.

Summarizing the results, we get that $b_{i+1} = (720 - 114)a_i + (720 - 68)b_i + (720 - 51)c_i + (720 - 42)d_i$. $\square$

Lemma 1.3. $c_{i+1} = 970a_i + 1055b_i + 1088c_i + 1106d_i \quad (i \geq 1)$.

Proof. Now we classify the 2-dimensional faces of the Ω's for determining the number of the vertices C_{i+1}. There are vertices C_{i+1} only along the faces whose neither vertices are incident to the surface of Π_i. If a face has got at least one vertex W_i, then the vertex of the hypercube, which is 3 edges away from the examined centre of Ω, is 2 edges away from the vertex W_i. So it is not a vertex C_{i+1}. An example is in Figure 3 the vertex Y. It is 3 edges away from A_i, but from $K = W_i$ it is 2 edges away. The vertex Y is not an A_{i+1}.

We are going to examine the faces with at least one vertex W_i and subtract their numbers from the number of the faces.

In case of a 600-cell 30 faces join to a vertex and 5 faces to an edge. So, $12 \cdot 30$ faces (with multiplicity) contain the vertices W_i of an Ω_A. But we counted doubly (or triply) the faces with two (or three) W_i, so from the sum we subtract $30 \cdot 5$. Since the number of the faces with three W_i are subtracted doubly, now we add their number, 20. Thus, the number of the faces containing at least one W_i is $12 \cdot 30 - 30 \cdot 5 + 20 = 230$. (It includes the number of faces with a vertex W_{i-1} too.) Similarly, we get the numbers of the faces with W_i for the Ω_B, Ω_C, Ω_D's, they are $7 \cdot 30 - 16 \cdot 5 + 15 = 145$, $5 \cdot 30 - 9 \cdot 5 + 7 = 112$ and $4 \cdot 30 - 6 \cdot 5 + 4 = 94$.

Summarizing we get $c_{i+1} = (1200-230)a_i + (1200-145)b_i + (1200-112)c_i + (1200-94)d_i$. $\square$

Lemma 1.4. $d_{i+1} = 470a_i + 515b_i + 533c_i + 543d_i \quad (i \geq 1)$.

Proof. We classify the cells of the Ω's for determining the number of the vertices D_{i+1}. There are vertices D_{i+1} only along the cells whose neither vertices are incident to the surface of Π_i. An example is in Figure 3 the vertex Z. It is 4 edges away from A_i, but its edge-distance from $K = W_i$ is 3. The vertex Z is not an A_{i+1}.

First of all we count the numbers of the cells containing one, two, three or four vertices W_i, and then we subtract their number from the number of the cells, i.e., from 600. There are 20, 5 or 2 cells containing a common vertex, edge or face, respectively. For Ω_A the numbers of the cells containing at least one, two or three W_i's (there is not a cell containing four W_i's) are $12 \cdot 20$, $30 \cdot 5$ and $20 \cdot 2$. Thus the number of the cells is $12 \cdot 20 - 30 \cdot 5 + 20 \cdot 2 = 130$. (It includes the cells with a vertex W_{i-1}, too.) Similarly, the numbers of cells with W_i for the Ω_B's, Ω_C's, Ω_D's are $7 \cdot 20 - 16 \cdot 5 + 15 \cdot 2 - 5 = 85$, $5 \cdot 20 - 9 \cdot 5 + 7 \cdot 2 - 2 = 67$ and $4 \cdot 20 - 6 \cdot 5 + 4 \cdot 2 - 1 = 57$.

Summarizing we get $c_{i+1} = (600-130)a_i + (600-85)b_i + (600-67)c_i + (600-57)d_i$. $\square$

The numbers of the hypercubes of the belt i can be determined by the help of the numbers of the vertices of the belt i, of the different types $(A_i, \dots, D_i)$. So the sequences r_{i+1} and f_{i+1}^k $(0 \leq k \leq 3)$ will turn out to be homogeneous linear functions of a_i b_i c_i d_i, cf. the following Lemmas 1.5, 1.6.

Lemma 1.5. $r_{i+1} = 510a_i + \frac{2185}{4}b_i + \frac{1119}{2}c_i + \frac{1133}{2}d_i \quad (i \geq 1)$.

Proof. We again classify the cells of the Ω's for determining the number of the hypercubes in the belt $(i+1)$. The hypercubes belonging to the cells whose all the four vertices are W_i (or W_{i-1}) are in the belt i, the others are the only new hypercubes (in the belt $(i+1)$). The new hypercubes with vertices W_i are not only connected as new hypercubes to the centre of the considered Ω but also to other vertices.

In case of a cell with three vertices W_i, the hypercube belongs to eight vertices on the outer boundary of the belt i at the same time, to the eight vertices of the cube generated by the three vertices W_i and the centre of Ω. So the number of the hypercubes like this is divided by eight because of the multiplicity. In case of a cell with three/two/one vertices W_i, the hypercube belongs to eight/four/two vertices on the outer boundary of the belt i at the same time. These eight/four/two vertices of the cube/square/edge are generated by the three/two/one vertices W_i and the centre of Ω. So the number of the hypercubes like this is divided by eight/four/two because of the multiplicity. The hypercubes connected to cells with no vertices W_i belong only to the centre of the considered Ω as new hypercubes.

An Ω_A has twenty cells with three vertices W_i and one vertex W_{i-1}. The vertex W_{i-1} is the common vertex of them (see Figure 3). Therefore the hypercubes along these cells are in the belt $(i+1)$. The Ω_A has also twenty cells with three vertices W_i, because an icosahedron has twenty faces and two cells of Ω_A connect to every face. The Ω_A has thirty cells with two vertices W_i, because an icosahedron has thirty edges and five cells of Ω_A connect to every edge, but four of them have three W_i's. The Ω_A has sixty cells with only one vertex W_i, because an icosahedron has twelve vertices and twenty cells of Ω_A connect to every vertex, but fifteen of them have three or two vertices W_i. The number of the examined cells is $20+20+30+60=130$, so the number of the remaining cells is $600-130$. The hypercubes along these cells belong only to the considered vertices A_i. Thus the number of the hypercubes belonging to an A_i is $20\cdot 0+\frac{20\cdot(2-1)}{8}+\frac{30\cdot(5-4)}{4}+\frac{12\cdot(20-15)}{2}+(600-130)=510$.

An Ω_B has five cells with only vertices W_i. They are around the edge MN (see Figure 4). There are ten cells, which have only three vertices W_i. (They connect to the edge of the pentagon with the vertices W_i and vertex M or N.) The Ω_B has twenty cells with only two vertices W_i. There are two cells along any edge of the pentagon and one along any edge like NR. Among the cells with vertices W_i we only do not count the cells with only one vertex W_i. They are the cells with one vertex of the pentagon. The vertex R belongs to twenty cells of Ω_B, but ten of them have three or two vertices W_i. So, there are $5\cdot 10$ cells with only one vertex W_i. The remaining cells are connected to the vertex B_i. Thus the number of the hypercubes belonging to a B_i is $5\cdot 0+\frac{5+5}{8}+\frac{5\cdot 2+10\cdot 1}{4}+\frac{5\cdot 10}{2}+(600-85)=\frac{2185}{4}$.

Similarly, in cases of the vertices C_i, or D_i, the number of the hypercubes are $2\cdot 0+\frac{6}{8}+\frac{15}{4}+\frac{2\cdot 10+3\cdot 8}{2}+(600-67)=\frac{1119}{2}$, or $1\cdot 0+\frac{4}{8}+\frac{6\cdot 2}{4}+\frac{4\cdot 10}{2}+(600-57)=\frac{1133}{2}$, respectively. Summarizing we get the lemma. $\square$

Lemma 1.6. *$f^3_{i+1}=\frac{3975}{2}a_i+\frac{8585}{4}b_i+\frac{8825}{4}c_i+\frac{4477}{2}d_i$, $f^2_{i+1}=\frac{11925}{2}a_i+\frac{25755}{4}b_i+\frac{26475}{4}c_i+\frac{13431}{2}d_i$, $f^1_{i+1}=6128a_i+\frac{13255}{2}b_i+\frac{13635}{2}c_i+6920d_i$ and $f^0_{i+1}=2153a_i+2335b_i+2405c_i+2443d_i$ $(i\geq 1)$.*

Proof. For determining the number of the 3-dimensional faces (cubes) on the outer boundary the belt i we classify the 2-dimensional faces of the Ω's. A 3-dimensional face determined by the centre of the considered Ω and three vertices

W_i is on the outer boundary of the belt i, and it is a cube on the outer boundary of the belt i, which is counted at each of its eight vertices. So, we divided by eight the number of the cubes like this because of the multiplicity. The other 3-dimensional faces are not on the surface of the belt i. There are twenty, ten, six or four 2-dimensional faces of Ω_A, Ω_B, Ω_C or Ω_D, respectively (see fig. 3-6). Naturally, the 3-dimensional faces determined by the vertex W_{i-1} of Ω_A, by the edge of MN of Ω_B and by the triangle TSW_i of Ω_C are not on the outer boundary of the belt i. They are in the belt i.

Thus $f_i^3 = \frac{20}{8}a_i + \frac{10}{8}b_i + \frac{6}{8}c_i + \frac{4}{8}d_i$, so $f_{i+1}^3 = \frac{20}{8}(107a_i + 113b_i + 115c_i + 116d_i) + \frac{10}{8}(107a_i + 113b_i + 115c_i + 116d_i) + \frac{6}{8}(107a_i + 113b_i + 115c_i + 116d_i) + \frac{4}{8}(107a_i + 113b_i + 115c_i + 116d_i) = \frac{3975}{2}a_i + \frac{8585}{4}b_i + \frac{8825}{4}c_i + \frac{4477}{2}d_i$.

Similarly, for determining the number of the 2-dimensional/1-dimensional faces on the outer boundary of the belt i we classify the 1-dimensional/0-dimensional faces of the Ω's. A 2-dimensional/1-dimensional face determined by the centre of the considered Ω and two/one vertices W_i are on the outer boundary of the belt i. Furthermore, they are 2-dimensional/1-dimensional faces on the outer boundary of the belt i, in case of the four/two vertices of the square/edge determined by the the centre of the considered Ω and the vertices W_i, too. So, we divide by four/two the number of the 2-dimensional/1-dimensional faces similarly. Thus $f_i^2 = \frac{30}{4}a_i + \frac{15}{4}b_i + \frac{9}{4}c_i + \frac{6}{4}d_i$, $f_i^1 = \frac{12}{2}a_i + \frac{7}{2}b_i + \frac{5}{2}c_i + \frac{4}{2}d_i$. (Moreover $f_i^2 = 3f_i^3$.) It proves the lemma.

Furthermore $f_{i+1}^0 = a_{i+1} + b_{i+1} + c_{i+1} + d_{i+1} = 2153a_i + 2335b_i + 2405c_i + 2443d_i$. □

Remark 1. Obviously, r_i and f_i^k $(i \geq 1, 0 \leq k \leq 3)$ are integers and a_1, b_1, c_1 and d_1 are dividable by eight.

2. The proof of Theorem 1

Let v denote the volume of a hypercube of the mosaic. Then $\lim\limits_{i\to\infty} \frac{V_{i+1}}{V_i} = \lim\limits_{i\to\infty} \frac{v \cdot r_{i+1}}{v \cdot r_i} = \lim\limits_{i\to\infty} \frac{r_{i+1}}{r_i}$ and $\lim\limits_{i\to\infty} \frac{V_i}{S_i} = \lim\limits_{i\to\infty} \frac{v \cdot r_i}{v \cdot \sum_{j=0}^{i} r_j} = \lim\limits_{i\to\infty} \frac{r_i}{\sum_{j=0}^{i} r_j}$ $(i \geq 1)$. Thus we can calculate considering the numbers of the hypercubes instead of the volumes.

From the previous lemmas we get the following linear recursion for the sequences a_i, b_i, c_i, d_i $(i \geq 1)$:

$$a_{i+1} = 107a_i + 113b_i + 115c_i + 116d_i$$

$$b_{i+1} = 606a_i + 652b_i + 669c_i + 678d_i$$

$$c_{i+1} = 970a_i + 1055b_i + 1088c_i + 1106d_i$$
$$d_{i+1} = 470a_i + 515b_i + 533c_i + 543d_i \tag{1}$$

and the sequence r_i can be given by the help of a_i, b_i, c_i and d_i the following way:

$$r_{i+1} \quad = \quad \alpha_1 a_i + \alpha_2 b_i + \alpha_3 c_i + \alpha_4 d_i, \tag{2}$$

where $i \geq 1$. In a shorter form

$$w_{i+1} = \mathbf{M} w_i, \tag{3}$$
$$r_{i+1} = \alpha^T w_i, \tag{4}$$

where

$$\mathbf{M} = \begin{pmatrix} 107 & 113 & 115 & 116 \\ 606 & 652 & 669 & 678 \\ 970 & 1055 & 1088 & 1106 \\ 470 & 515 & 533 & 543 \end{pmatrix}, \tag{5}$$

and $w_j = [a_j \ \ b_j \ \ c_j \ \ d_j]^T$, $\alpha = [\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \ \ \alpha_4]^T = [510 \, \frac{2185}{4} \, \frac{1119}{2} \, \frac{1133}{2}]^T$.

As a well-known consequence of the Jordan normal form of matrices (a simple proof cf. in [6], Lemma 1.11), we have

$$r_i = g_1 z_1^i + g_2 z_2^i + g_3 z_3^i + g_4 z_4^i, \tag{6}$$

where the z_j's are the eigenvalues of the matrix $\mathbf{M}$, provided that the z_j's are different and all are different from 0. Now $z_1 \approx 2381.8277$, $z_2 \approx 8.0476$, $z_3 \approx 0.1243$ and $z_4 \approx 0.0004$. They are all exactly real, since else there would be a conjugate complex pair among them, hence of the same absolute value. From §1 we have $(a_1, b_1, c_1, d_1) = (120, 720, 1200, 600)$.

By (1) we obtain in turn

	a_j	b_j	c_j	d_j
$j = 2$	301800	1751760	2845200	1392600
$j = 3$	718981080	4172659920	6776646000	3316675800
$j = 4$	1712490229320	9938559168720	16140802146000	7899748243800

These imply by Lemma 1.5 $r_2 = 1465800$, $r_3 = 3491614200$, $r_4 = 8316426109800$, $r_5 = 19808294143666200$. Putting these values to (6), for $2 \leq i \leq 5$, we obtain a system of four linear equations for $g_1, \dots, g_4$. The (unique) solutions are $g_1 \approx 615.5$, $g_2 \approx -13.8$, $g_3 \approx -14903.1$, $g_4 \approx 4343347.8$. We observe that $g_1 \neq 0$.

Since $\lim\limits_{i\to\infty}\left(\frac{z_j}{z_1}\right)^i = 0 \quad (2 \le j \le 4)$, then

$$\lim_{i\to\infty}\frac{V_{i+1}}{V_i} = \lim_{i\to\infty}\frac{r_{i+1}}{r_i} = \lim_{i\to\infty}\frac{g_1z_1^{i+1}+g_2z_2^{i+1}+g_3z_3^{i+1}+g_4z_4^{i+1}}{g_1z_1^{i}+g_2z_2^{i}+g_3z_3^{i}+g_4z_4^{i}}$$

$$= \lim_{i\to\infty}\frac{g_1z_1+g_2z_2\left(\frac{z_2}{z_1}\right)^i+g_3z_3\left(\frac{z_3}{z_1}\right)^i+g_4z_4\left(\frac{z_4}{z_1}\right)^i}{g_1+g_2\left(\frac{z_2}{z_1}\right)^i+g_3\left(\frac{z_3}{z_1}\right)^i+g_4\left(\frac{z_4}{z_1}\right)^i} = z_1.$$

Further,

$$\lim_{i\to\infty}\frac{V_i}{S_i} = \lim_{i\to\infty}\frac{r_i}{\sum\limits_{j=0}^{i} r_j} = \lim_{i\to\infty}\frac{g_1z_1^i+g_2z_2^i+g_3z_3^i+g_4z_4^i}{g_1\sum\limits_{j=0}^{i}z_1^j+g_2\sum\limits_{j=0}^{i}z_2^j+g_3\sum\limits_{j=0}^{i}z_3^j+g_4\sum\limits_{j=0}^{i}z_4^j}$$

$$= \lim_{i\to\infty}\frac{g_1+g_2\left(\frac{z_2}{z_1}\right)^i+\dots+g_4\left(\frac{z_4}{z_1}\right)^i}{g_1\frac{z_1-\frac{1}{z_1^i}}{z_1-1}+g_2\frac{z_2\left(\frac{z_2}{z_1}\right)^i-\frac{1}{z_1^i}}{z_2-1}+\dots+g_4\frac{z_4\left(\frac{z_4}{z_1}\right)^i-\frac{1}{z_1^i}}{z_4-1}}$$

$$= \frac{z_1-1}{z_1} \approx 0.9996.$$

Moreover, if $z(k,i) = \frac{\frac{z_k^{i+1}-1}{z_k-1}}{\frac{z_1^{i+1}-1}{z_1-1}} \quad (1 < k \le 4)$, then $\lim\limits_{i\to\infty} z(k,i) = 0$ and

$$\lim_{i\to\infty}\frac{S_{i+1}}{S_i} = \lim_{i\to\infty}\frac{v\cdot\sum\limits_{j=0}^{i+1}r_j}{v\cdot\sum\limits_{j=0}^{i}r_j}$$

$$= \lim_{i\to\infty}\frac{g_1z_1\sum\limits_{j=0}^{i}z_1^j+g_2z_2\sum\limits_{j=0}^{i}z_2^j+g_3z_3\sum\limits_{j=0}^{i}z_3^j+g_4z_4\sum\limits_{j=0}^{i}z_4^j}{g_1\sum\limits_{j=0}^{i}z_1^j+g_2\sum\limits_{j=0}^{i}z_2^j+g_3\sum\limits_{j=0}^{i}z_3^j+g_4\sum\limits_{j=0}^{i}z_4^j}$$

$$= \frac{g_1z_1\frac{z_1^{i+1}-1}{z_1-1}+g_2z_2\frac{z_2^{i+1}-1}{z_2-1}+g_3z_3\frac{z_3^{i+1}-1}{z_3-1}+g_4z_4\frac{z_4^{i+1}-1}{z_4-1}}{g_1\frac{z_1^{i+1}-1}{z_1-1}+g_2\frac{z_2^{i+1}-1}{z_2-1}+g_3\frac{z_3^{i+1}-1}{z_3-1}+g_4\frac{z_4^{i+1}-1}{z_4-1}}$$

$$= \frac{g_1z_1+g_2z_2z(2,i)+g_3z_3z(3,i)+g_4z_4z(4,i)}{g_1+g_2z(2,i)+g_3z(3,i)+g_4z(4,i)}$$

$$= z_1 = \lim_{i\to\infty}\frac{V_{i+1}}{V_i}.$$

Now let us consider f_i^k for $0 \le k \le 3$. From the above values of $(a_1, \dots, d_1)$, $\dots$, $(a_4, \dots, d_4)$ we have

	f_2^k	f_3^k	f_4^k	f_5^k
$k = 0$	6291360	14984962800	35691599787840	85011241243420560
$k = 1$	17840160	42493162800	101211400319040	241068117493415760
$k = 2$	17323200	41262300000	98279700796800	234085314374992800
$k = 3$	5774400	13754100000	32759900265600	78028438124997600

Analogously to the case r_i, this leads to the analogue of (6), with

$$\begin{aligned}(g_1 \ne 0, g_2, g_3, g_4) &\approx (2641.4, 13.8, -63968.8, 18640343.4), &&\text{or}\\ &\approx (7490.2, -7.8, -181396.9, 52858803.6), &&\text{or}\\ &\approx (7273.3, -32.5, -176142.1, 51327690.2), &&\text{or}\\ &\approx (2424.4, -10.8, -58714, 17109230.1),\end{aligned}$$

for f_i^0, or f_i^1, or f_i^2, or f_i^3, respectively. The rest of the proof is word by word the same as for r_i. □

Remark 2. From the proof of Theorem 1 we get that we can choose a hypercube for Π_0 instead of P. Then for $r_i, f_i^3, \dots,$ or f_i^0, we have $g_1 \approx 9106.7 \ne 0$, $35880.7 \ne 0$, $107642.2 \ne 0$, $110853.2 \ne 0$, or $39091.7 \ne 0$, respectively, and the present limits are equal to those in the previous case $\Pi_0 = P$.

3. Dual mosaics

3.1. The proof of Theorem 2. The belt $(i+1)$ $(i \ge 0)$ of the dual mosaic is formed by the D-V cells of the original mosaic, whose centres are the vertices on the surface of Π_{i+1}. They are the vertices A_{i+1}, B_{i+1}, C_{i+1} and D_{i+1}. (We will not need to classify the vertices of Π^*_{i+1}, and determine their respective numbers.) Thus $r^*_{i+1} = f^{*4}_{i+1} = a_{i+1} + b_{i+1} + c_{i+1} + d_{i+1} = f^0_{i+1}$ $(i \ge 0)$.

Two elements of the dual mosaic (D-V cells for the original mosaic) join to each 3-dimensional face of the dual mosaic (common face of the D-V cells for the original mosaic). Thus on the surface of Π^*_{i+1} there are as many 3-dimensional faces of the dual mosaic, as the number of the D-V cells of the original mosaic joining to Π^*_{i+1} with 3-dimensional faces. They are the D-V cells for the original mosaic with centres A_{i+2}. That is why $f^{*3}_{i+1} = a_{i+2}$ $(i \ge 0)$.

Similarly, the D-V cells with centres B_{i+2}, C_{i+2} and D_{i+2} join to the surface of Π^*_{i+1} with 2, 1, 0-dimensional faces. Thus $f^{*2}_{i+1} = b_{i+2}$, $f^{*1}_{i+1} = c_{i+2}$, $f^{*0}_{i+1} = d_{i+2}$ $(i \geq 0)$.

Thus we have determined the sequences r^*_{i+1} and f^{*k}_{i+1} $(0 \leq k \leq 3)$ by the sequences a_i, b_i, c_i and d_i, as certain linear combinations of these four sequences (the numbers $a_{i+1}, \dots, d_{i+1}$, or $a_{i+2}, \dots, d_{i+2}$ can be calculated from $a_i, \dots, d_i$ by applying (1) once or twice, respectively). These equalities are the analogues of Lemmas 1.5, 1.6 for the original mosaic. Observe that also now we have the same matrix $\mathbf{M}$ as in (5). Thus, rather than (6), we have the same formula, with its right hand side containing the same $z_1, \dots, z_4$, and with its left hand side replaced, in turn, by r^*_{i+1}, f^{*3}_{i+1}, $\dots$, or f^{*0}_{i+1}, respectively. Again calculating r^*_{i+1}, f^{*3}_{i+1}, $\dots$, f^{*0}_{i+1}, for $0 \leq i \leq 3$, and putting them to the analogue of (6), we can uniquely solve the respective systems of linear equations, for g_1, $\dots$, g_4. We obtain, in the above order, $(g_1, g_2, g_3, g_4) \approx (615.5, -13.8, -14903.1, 4343347.8)$, $(2424.4, -10.8, -58714, 17109230.1)$, $(7273.3, -32.5, -176142.1, 51327690.2)$, $(7490.3, -7.8, -181396.9, 52858803.6)$, or $(2641.4, 13.8, -63968.8, 18640343.4)$, respectively. Observe that in each case we have $g_1 \neq 0$. Thus, analogously to the proof of Theorem 1, the limits for $V_i, F^3_i, \dots, F^0_i$ for the dual mosaic, i.e., those for $r^*_i, f^{*3}_i, \dots, f^{*0}_i$, are equal to those for $r_i, f^3_i, \dots, f^0_i$, i.e., to those for $V_i, F^3_i, \dots, F^0_i$ for the original mosaic, namely to z_1. □

References

[1] H. S. M. Coxeter, Regular honeycombs in hyperbolic space, *Proc. Int. Congress of Math.* **III**, *Amsterdam* (1954), 155–169.

[2] L. Fejes Tóth, Regular Figures, *Akadémiai Kiadó*, *Budapest*, 1964.

[3] J. Horváth, Über die regulären Mosaiken der hyperbolishen Ebene, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **7** (1964), 49–53.

[4] H.-C. Im Hof, Napier cycles and hyperbolic Coxeter groups, *Bull. Soc. Math. Belg. Sér. A* **42** (1990), 523–545.

[5] F. Kárteszi, Eine Bemerkung über das Dreiecksnetz der hyperbolischen Ebene, *Publ. Math. Debrecen* **5** (1957), 142–146.

[6] L. Németh, Combinatorial examination of mosaics with asymptotic pyramids and their reciprocals in 3-dimensional hyperbolic space, *Studia Sci. Math. Hungar.* **43**,(2) (2006), 247–264.

[7] I. Vermes, I., On the covering of the hyperbolic plane with asymptotic polygons, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **20** (1971), 341–347 (in *Hungarian*).

[8] I. Vermes, Über die Parkettierungsmöglichkeit der hyperbolischen Ebene durch nicht-total asymptotische Vielecke, *Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle* **1**, *Halle* (1971), 9–13.

[9] I. Vermes, I., Über die Parkettierungsmöglichkeit des dreidimensionalen hyperbolischen Raumes durch kongruente Polyeder, *Studia Sci. Math. Hungar.* **7** (1972), 267–287.

[10] E. B. Vinberg and O. V. Shvartsman, Discrete groups of motions of spaces of constant curvature, Geometry II, Encyclopaedia of Math. Sci., *Springer-Verlag*, 1991.

[11] H. Zeitler, Über eine Parkettierung des dreidimensionalen hyperbolischen Raumes, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **12** (1969), 3–10.

LÁSZLÓ NÉMETH
INSTITUTE OF MATHEMATICS
UNIVERSITY OF WEST HUNGARY
ADY E. U. 5.
H-9400 SOPRON
HUNGARY

E-mail: lnemeth@emk.nyme.hu

(Received September 23, 2004; revised October 17, 2006)