Publ. Math. Debrecen
70/3-4 (2007), 307–318

On a class of conformally invariant horizontal endomorphisms

By M. M. REZAII (Tehran), M. BARZEGARI (Tehran) and B. NAJAFI (Tehran)

Abstract. We construct a class of conformally invariant horizontal endomorphisms (or Ehresmann connections) on a Finsler manifold which contains the Wagner endomorphisms studied in detail in [12].

1. Introduction

The set of all horizontal endomorphisms over a manifold M constitutes an affine space modelled on the real vector space $\Psi^1(TM)$ of semibasic vector 1-forms on TM. Motivated by a result of Sz. Szakál and J. Szilasi ([7], Proposition 2.7), in this paper we study the orbit of the canonical horizontal endomorphism (the so-called Barthel endomorphism) of a Finsler manifold under the action of a subspace of $\Psi^1(TM)$ depending on the Finsler structure. The elements of this orbit will be mentioned as *L-horizontal endomorphisms* ($L \in \Psi^1(TM)$). We shall point out that Wagner endomorphisms studied in [11], [12] and [7] can be obtained as *special L-horizontal endomorphisms.* Our main result states that *the set of all conservative L-horizontal endomorphisms on a Finsler manifold is conformally closed.*

Conventions. (i) We work on an n-dimensional connected smooth manifold M whose topology is Hausdorff and has a countable base. $C^\infty(M)$ denotes the ring of smooth real-valued functions on M, $\mathcal{X}(M)$ stands for the $C^\infty(M)$-module of (smooth) vector fields on M. $\Omega(M) := \bigoplus_{i=0}^n \Omega^k(M)$ is the graded algebra of differential forms on M, with multiplication given by the wedge product. The

---

Mathematics Subject Classification: 53C60.
Key words and phrases: Finsler manifolds, conformal change, horizontal endomorphisms, Ehresmann connections.

symbols d, i_X, $\mathcal{L}_X$ $(X \in \mathcal{X}(M))$ denote the exterior derivative, the substitution operator and the Lie derivative.

(ii) TM is the $2n$-dimensional tangent manifold of M, $\mathring{T}M \subset TM$ is the open submanifold of the non-zero tangent vectors to M. f^v and f^c stand for the vertical and the complete lift of a smooth function on M into TM.

2. Preliminaries

2.1. For any vector field X on M there exist unique vector fields X^v, X^c on TM such that

$$X^v f^c = (Xf)^v, \quad X^c f^c = (Xf)^c \quad (f \in C^\infty(M)). \tag{1}$$

X^v is the *vertical lift*, X^c is the *complete lift* of X. The $C^\infty(TM)$-module of vertical vector fields on TM will be denoted by $\mathfrak{X}^v(TM)$. The *Liouville vector field* $C \in \mathfrak{X}^v(TM)$ is generated by the flow of positive dilatation $\delta_t : v \in TM \longmapsto \delta_t(v) := e^t v \in TM$ $(t \in \mathbb{R})$. A function $F \in C^\infty(TM)$, a vector field $\xi \in \mathfrak{X}(\mathring{T}M)$, and a differential form $\alpha \in \Omega(\mathring{T}M)$ are called *r-homogeneous* $(r \in \mathbb{Z})$, if the relations

$$CF = rF, \quad [C, \xi] = (r-1)\xi, \quad \mathcal{L}_C \alpha = r\alpha$$

hold, respectively. Notice that

$$[C, X^v] = -X^v, \quad [C, X^v] = 0 \quad (X \in \mathcal{X}(M)), \tag{2a–b}$$

so X^v is 0-homogeneous, X^c is 1-homogeneous vector field on TM.

2.2. By a *vector k-form* on TM we mean a skew-symmetric $C^\infty(TM)$-multilinear map $K : (\mathfrak{X}(TM))^k \to \mathfrak{X}(TM)$ if $k \in \{1, \ldots, 2n\}$, and a vector field on TM, if $k = 0$. In particular, a vector 1-form on TM is just a type $(1,1)$ tensor field. The $C^\infty(TM)$-module of vector k-forms on TM will be denoted by $\Psi^k(TM)$. There is a unique vector 1-form $J \in \Psi^1(TM)$ such that

$$JX^v = 0, \quad JX^c = X^v \quad (X \in \mathcal{X}(M)). \tag{3a–b}$$

J is called the *vertical endomorphism*. Clearly, J is of rank n and $J^2 = 0$. A differential form $\alpha \in \Omega^k(TM)$ is *semibasic*, if $i_{J\xi}\alpha = 0$; a vector form $K \in \Psi^k(TM)$ is *semibasic*, if $i_{J\xi}K = 0$ and $J \circ K = 0$ $(k \geq 1$, $\xi \in \mathfrak{X}(TM))$.

2.3. We recall that if θ_r and θ_s are graded derivations of degree r and s, respectively, of a graded algebra, then their *graded commutator* is defined by

$$[\theta_r, \theta_s] := \theta_r \circ \theta_s - (-1)^{rs} \theta_s \circ \theta_r. \tag{4}$$

Then $[\theta_r, \theta_s]$ is a graded derivation of degree $r + s$. By the *Frölicher–Nijenhuis theory* of vector forms to any vector k-form $K \in \Psi^k(TM)$ two graded derivations of $\Omega(TM)$ are associated, denoted by i_K and d_K. i_K is of degree $k - 1$, d_K is of degree k, and the following rules are prescribed:

$$i_K \restriction C^\infty(TM) = 0; \quad i_K \circ \alpha = \alpha \circ K, \quad \text{if } \alpha \in \Omega^1(TM); \tag{5}$$

$$d_K := [i_K, d] \overset{(4)}{=} i_K \circ d - (-1)^{k-1} d \circ i_K. \tag{6}$$

Then, in particular, for all $F \in C^\infty(TM), K \in \Psi^k(TM)$ we have $d_K F = dF \circ K$. For vector 0-forms $\xi \in \Psi^0(TM) = \mathfrak{X}(TM)$, i.e., for vector fields on TM, i_ξ and d_ξ reduce to the usual substitution operator and Lie derivative, respectively. To any vector forms $K \in \Psi^k(TM)$, $L \in \Psi^\ell(TM)$ there is a unique vector $(k+l)$-form $[K, L] \in \Psi^{k+l}(TM)$, the *Frölicher–Nijenhuis bracket* of K and L such that

$$d_{[K,L]} = [d_K, d_L].$$

In this paper we are going to systematically use the Frölicher–Nijenhuis calculus of vector forms. A detailed account on the theoretical background can be found e.g. in monographs [5], [6], and (of course) in the original source [2]. A well applicable list of formulae is gathered together (among others) in the reference [7] by Sz. Szakál and J. Szilasi. Concerning the vertical endomorphism and the Liouville vector field we have

$$[J, C] = J, \quad [J, J] = 0. \tag{7a–b}$$

A vector form K on TM is called *homogeneous* of degree $r \in \mathbb{Z}$ if $[C, K] = (r - 1)K$. We note finally that the complete lift f^c of a function $f \in C^\infty(M)$ is 1-homogeneous and

$$d_J f^c = d(f^v) =: (df)^v; \tag{8}$$

see [8], Lemma 2.

2.4. By a *semispray* over M we mean a C^1 vector field $S : TM \to TTM$, smooth on $\overset{\circ}{T}M$, satisfying the condition $JS = C$. A 2-homogeneous semispray is called a *spray*. If S is a semispray over M and K is a vector 1-form on TM, then for any vector field ξ on TM we have

$$K[J\xi, S] = K\xi, \tag{9}$$

in particular

$$J[J\xi, S] = J\xi. \tag{10}$$

Indeed, by Proposition 1.7 of [3] the vector field $[J\xi, S] - \xi$ is always vertical.

Two sprays S and $\overline{S}$ over M are said to be (*pointwise*) *projectively related* if there is a smooth function P on $\overset{\circ}{T}M$ such that $\overline{S} = S + PC$ (over $\overset{\circ}{T}M$). Then the *projective factor* P is necessarily 1-homogeneous.

2.5. A vector 1-form $\mathbf{h} \in \Psi^1(TM)$, smooth – in general – only over $\overset{\circ}{T}M$ is said to be a *horizontal endomorphism* (or *Ehresmann connection*) over M if it is a projector (i.e., $\mathbf{h}^2 = \mathbf{h}$) and $\operatorname{Ker}\mathbf{h} = \mathfrak{X}^v(TM)$, or, equivalently, if $J \circ \mathbf{h} = J$ and $\mathbf{h} \circ J = 0$. $\mathbf{h}$ is called *homogeneous* if it is 1-homogeneous in the above sense, i.e. $[C, \mathbf{h}] = 0$. The (*weak*) *torsion* of $\mathbf{h}$ is the vector 2-form $\mathbf{t} := [J, \mathbf{h}]$. If S is a semispray over M, then $S_{\mathbf{h}} := \mathbf{h} \circ S$ is also a semispray, depending only on the Ehresmann connection. $S_{\mathbf{h}}$ is called the associated semispray to $\mathbf{h}$.

A fundamental result due to M. Crampin and J. Grifone states that *any semispray S generates a horizontal endomorphism of zero weak torsion* by the formula

$$\mathbf{h} = \frac{1}{2}\,(1_{\mathfrak{X}(TM)} + [J, S]). \tag{11}$$

Its associated semispray is $S_{\mathbf{h}} = \frac{1}{2}\,(S + [C, S])$. If S is a spray then $S_{\mathbf{h}} = S$, and $\mathbf{h}$ is homogeneous. For a recent treatment of these facts we refer to [6].

3. Some calculus on Finsler manifolds

3.1. Let a function $E : TM \to \mathbb{R}$ be given. Assume:

(i) $E(v) > 0$ for all $v \in \overset{\circ}{T}M$, $E(0) = 0$;

(ii) E is of class C^1 on TM, smooth on $\overset{\circ}{T}M$;

(iii) E is (positive-)homogeneous of degree 2, i.e., $CE = 2E$;

(iv) the *fundamental 2-form* $\omega := d\,d_J E$ is non-degenerate.

Then (M, E) is said to be a *Finsler manifold* with energy E. Notice that ω is semibasic and we have the relations

$$i_J\omega = 0, \quad i_C\omega = d_J E, \quad \mathcal{L}_C\omega = \omega. \tag{12a–c}$$

Due to the non-degeneracy of ω, for any 1-form $\beta \in \Omega^1(TM)$ there is unique vector field $\beta^\#$ on TM (smooth, in general, only on $\overset{\circ}{T}M$) such that

$$i_{\beta^\#}\,\omega = \beta. \tag{13}$$

This map $\# : \beta \to \beta^\#$ is called the (Finslerian) *sharp operator*. In particular, the *gradient* of a function $f \in C^\infty(TM)$ is the vector field $\operatorname{grad} f := (df)^\#$.

Lemma 1. *If (M,E) is a Finsler manifold and β is a semibasic 1-form on TM, then*

$$\beta^\# \in \mathfrak{X}^v(TM), \quad \left[C, \beta^\#\right] = (\mathcal{L}_C\,\beta)^\# - \beta^\#. \tag{14a–b}$$

Proof. Using (1.4g) of [7], (12a) and (13) we get

$$i_{J\beta^\#}\omega = i_{\beta^\#} \circ i_J\omega - i_J \circ i_{\beta^\#}\omega = -i_J\beta = 0,$$

since β is semibasic. Thus $J\beta^\# = 0$, therefore $\beta^\#$ is vertical. As for (14b),

$$i_{[C,\beta^\#]}\,\omega = \mathcal{L}_C\, i_{\beta^\#}\,\omega - i_{\beta^\#}\,\mathcal{L}_C\,\omega \overset{(12c),(13)}{=} \mathcal{L}_C\,\beta - \beta,$$

hence $[C,\beta^\#] = (\mathcal{L}_C\,\beta)^\# - \beta^\#$. □

3.2. Following Grifone [3], by the *potential* of a semibasic form $K \in \Psi^k(TM)$ we mean the $(k-1)$-form $K^\circ := i_S K$, where S is any semispray over M ($k \geqq 1$). Clearly, K° is independent of the choice of S.

Lemma 2. *Let (M,E) be a Finsler manifold and L be an r-homogeneous semibasic vector 1-form on TM. Then $d_L E$, $(d_L E)^\#$ and L° are $(r{+}1)$-homogeneous, while $[J,(d_L E)^\#]$ is r-homogeneous.*

Proof. We have $[\mathcal{L}_C, d_L] = d_{[C,L]} = (r-1)d_L$, hence

$$\mathcal{L}_C\, d_L E = d_L \mathcal{L}_C\, E + (r-1)d_L\, E = (r+1)d_L\, E.$$

This proves the r-homogeneity of $d_L\, E$. Using this fact and (14b), we get

$$[C,(d_L E)^\#] = (\mathcal{L}_C d_L E)^\# - (d_L E)^\# = (r+1)(d_L E)^\# - (d_L E)^\# = r(d_L E)^\#,$$

as was to be shown. Since

$$[C,L]^\circ = [C,L]S = [C,L^\circ] - L[JS,S] \overset{(9)}{=} [C,L^\circ] - L^\circ$$

and, on the other side, $[C, L]^\circ = (r-1)L^\circ$, it follows that $[C, L^\circ] = r\,L^\circ$, i.e., L° is $(r+1)$-homogeneous.

Finally, using the graded Jacobi identity, the $(r+1)$-homogeneity of $(d_L E)^\#$ and (7a), we get

$$\begin{aligned}[C,[J,(d_L E)^\#]] &= -[J,[(d_L E)^\#, C]] - [(d_L E)^\#,[C,J]] \\ &= (r+1)[J,(d_L E)^\#] + [(d_L E)^\#, J] = r[J,(d_L E)^\#],\end{aligned}$$

which proves the last claim. □

3.3. To conclude this section, we recall the *fundamental lemma of Finsler geometry* due to J. Grifone [3], see also [6]. Let (M, E) be a Finsler manifold. If

$$S_0 := -(dE)^\# \quad \text{over} \quad \overset{\circ}{T}M, \quad S_0(0) := 0$$

then S_0 is a spray over M, called the *canonical spray* of (M, E). S_0 generates a homogeneous horizontal endomorphism $\mathbf{h}_0$ according to (11), called the *canonical horizontal endomorphism* or the *Barthel endomorphism* of (M, E). $\mathbf{h}_0$ is conservative in the sense that $d_{\mathbf{h}_0} E = 0$.

4. *L*-horizontal endomorphisms on a Finsler manifold

Keeping the notation introduced in Section 3, throughout in the following we work on a Finsler manifold (M, E).

4.1. Sz. Szakál and J. Szilasi have shown in [7] that any homogeneous, conservative horizontal endomorphism $\mathbf{h}$ over M can be expressed as follows:

$$\mathbf{h} := \mathbf{h}_0 + \frac{1}{2}\,\mathbf{t}^\circ + \frac{1}{2}\,[J,(d_{\mathbf{t}^\circ} E)^\#]. \tag{15}$$

Next we consider a quite natural generalization.

Lemma 3 and definition. If L is a semibasic vector 1-form on TM and

$$\mathbf{h}_L := \mathbf{h}_0 + L + [J,(d_L E)^\#], \tag{16}$$

then $\mathbf{h}_L$ is also a horizontal endomorphism, called an *L-horizontal endomorphism* on (M, E). In particular, for any homogeneous, conservative horizontal endomorphism $\mathbf{h}$ we have $\mathbf{h} = \mathbf{h}_{\frac{1}{2}\mathbf{t}^\circ}$.

PROOF. Indeed, the term $L + [J, (d_L E)^{\#}]$ in (16) is semibasic, therefore $J \circ \mathbf{h}_L = J \circ \mathbf{h}_0 = J$, $\mathbf{h}_L \circ J = \mathbf{h}_0 \circ J = 0$. The relation $\mathbf{h} = \mathbf{h}_{\frac{1}{2}\mathbf{t}^\circ}$ is just a reformulation of (15). □

4.2. We gather together and prove some basic properties of L-horizontal endomorphisms.

Proposition 1. *Let L be a semibasic vector 1-form on TM.*

(i) *If L is 1-homogeneous, then $\mathbf{h}_L$ is homogeneous.*

(ii) *The associated semispray S_L to $\mathbf{h}_L$ is related to the canonical spray of (M, E) by*

$$S_L = S_0 + L^\circ + (\mathcal{L}_C d_L E)^{\#}.$$

If, in particular, L is 1-homogeneous, then S_L is a spray.

(iii) *The weak torsion of S_L is $\mathbf{t}_L = [J, L]$.*

(iv) *If L is 1-homogeneous and S_L is projectively related to S_0, then the projective factor is $\frac{3}{2}\frac{L^\circ E}{E} \upharpoonright \overset{\circ}{T}M$.*

PROOF. The first claim is immediate: the 1-homogeneity of L implies the 1-homogeneity of $[J, (d_{\mathbf{t}^\circ} E)^{\#}]$ by Lemma 2.
To verify (ii), let S be a semispray over M. Then

$$\begin{aligned} S_L := \mathbf{h}_L \circ S &= h_0 \circ S + L^\circ + [J, (d_L E)^{\#}]S \\ &= S_0 + L^\circ + [C, (d_L E)^{\#}] - J[S, (d_L E)^{\#}] \\ &\overset{(14b),(10)}{=} S_0 + L^\circ + (\mathcal{L}_C d_L E)^{\#} - (d_L E)^{\#} + (d_L E)^{\#} \\ &= S_0 + L^\circ + (\mathcal{L}_C d_L E)^{\#}, \end{aligned}$$

as desired.
The weak torsion of $\mathbf{h}_L$ is

$$\mathbf{t}_L := [J, \mathbf{h}_L] = \big[J, \mathbf{h}_0 + L + [J, (d_L E)^{\#}]\big] = [J, L] + \big[J, [J, (d_L E)^{\#}]\big].$$

Applying the graded Jacobi identity we easily get that the last term of the right-hand side vanishes; this proves (iii).

To prove (iv), let $S_L = S_0 + PC, P \in C^\infty(\overset{\circ}{T}M)$. Then $PC = L^\circ + 2(d_L E)^{\#}$ by the 2-homogeneity of $(d_L E)^{\#}$. Now we act on the fundamental 2-form ω by the substitution operators induced by PC and $L^\circ + 2(d_L E)^{\#}$, respectively. We find:

$$i_{PC}\omega = P i_C \omega \overset{(12b)}{=} P d_J E; \tag{17}$$

$$i_{L^\circ}\omega + 2 i_{(d_L E)^{\#}}\omega \overset{(13)}{=} i_{L^\circ}\omega + 2 d_L E. \tag{18}$$

Evaluating the right-hand sides of (17) and (18) on a semispray S we have

$$Pd_J E(S) = P(CE) = 2PE$$

and

$$\begin{aligned}(i_{L^\circ}\omega + 2d_L E)(S) &= \omega(L^\circ, S) + 2dE(L^\circ) = d(d_J E)(L^\circ, S) + 2dE(L^\circ)\\ &= L^\circ d_J E(S) - S d_J E(L^\circ) - d_J E([L^\circ, S]) + 2L^\circ E\\ &= 4L^\circ E - dE(J[L^\circ, S]) \overset{(10)}{=} 4L^\circ E - L^\circ E = 3L^\circ E.\end{aligned}$$

Thus $2PE = 3L^\circ E$, which concludes the proof. □

4.3. The next proposition will imply that the Wagner endomorphisms can be considered as special L-horizontal endomorphism.

Proposition 2. *Let f be a smooth function on M. If K is a semibasic vector 1-form and*

$$L := f^c K - df^v \otimes K^\circ, \tag{19}$$

then L is also a semibasic 1-form, and

$$\begin{aligned}\mathbf{h}_L = \mathbf{h}_0 + f^c K - df^v \otimes K^\circ + f^c[J, (d_K E)^\#] + df^v \otimes (d_K E)^\#\\ - K^\circ E[J, \operatorname{grad} f^v] - d_J(K^\circ E) \otimes \operatorname{grad} f^v.\end{aligned} \tag{20}$$

Proof. It is obvious that L is indeed semibasic. To verify (20), it is enough to check that under the choice (19), $(d_L E)^\# = f^c (d_K E)^\# - (K^\circ E)\operatorname{grad} f^v$. To see this, let X be any vector field on M. Then we get

$$\begin{aligned}i_{(d_L E)^\#}\omega(X^c) &= LX^C(E) = (f^c K(X^c) - df^v(X^c)K^\circ)E = f^c(KX^c)E\\ &\quad - (K^\circ E)df^v(X^c) = f^c d_K E(X^c) - (K^\circ E)i_{\operatorname{grad} f^v}\omega(X^c)\\ &= i_{f^c(d_K E)^\# - (K^\circ E)\operatorname{grad} f^v}\omega(X^c),\end{aligned}$$

which yields the desired relation. We may now apply some standard rules for calculation of the Frölicher–Nijenhuis theory and relation (8) to obtain (20). □

Corollary. *The class of the L-horizontal endomorphisms of a Finsler manifold contains the Wagner endomorphisms.*

Proof. As Vincze has shown in [12], the Wagner endomorphism $\overline{\mathbf{h}}$ associated to a smooth function f on M can be represented in the form

$$\overline{\mathbf{h}} = \mathbf{h}_0 + f^c J - E[J, \operatorname{grad} f^v] - d_J E \otimes \operatorname{grad} f^v.$$

Replacing K by $\frac{1}{2}J$ and taking into account that $J^\circ = C$, $(d_J E)^\# = C$, (20) takes the form

$$\begin{aligned}\mathbf{h}_{\frac{1}{2}f^cJ-\frac{1}{2}df^v\otimes C} &= \mathbf{h}_0 + \frac{1}{2}f^cJ - \frac{1}{2}df^v\otimes C + \frac{1}{2}f^c[J,C] + \frac{1}{2}df^v\otimes C\\ &\quad - \frac{1}{2}CE[J,\operatorname{grad} f^v] - \frac{1}{2}d_J(CE)\otimes \operatorname{grad} f^v\\ &= \mathbf{h}_0 + f^cJ - E[J,\operatorname{grad} f^v] - d_JE\otimes \operatorname{grad} f^v = \overline{\mathbf{h}},\end{aligned}$$

proving our claim. □

5. The effect of conformal changes

We continue to assume that (M,E) is a Finsler manifold.

5.1. Let f be a smooth function on M and define a positive function on TM by

$$\varphi := \exp\circ f^v. \tag{21}$$

If $\widetilde{E} := \varphi E$, then $(M,\widetilde{E})$ is also a Finsler manifold (see [11] Lemma 1). We say that $(M,\widetilde{E})$ has been obtained by a *conformal change* of E given by the *scale function* φ. It is known ([8]) that the Barthel endomorphism $\widetilde{\mathbf{h}}_0$ and the canonical spray $\widetilde{S}_0$ of $(M,\widetilde{E})$ are related to the corresponding data of (M,E) by

$$\widetilde{\mathbf{h}}_0 = \mathbf{h}_0 - \frac{1}{2}(f^cJ + df^v\otimes C) + \frac{1}{2}E[J,\operatorname{grad} f^v] + \frac{1}{2}d_JE\otimes \operatorname{grad} f^v, \tag{22}$$

and

$$\widetilde{S}_0 = S_0 - f^cC + E\operatorname{grad} f^v, \tag{23}$$

respectively.

Lemma 4. *Let β be a semibasic 1-form on TM. Under the conformal change with scale function given by* (21) *the vector field $\beta^\#$ changes by the rule $\varphi\beta^{\widetilde{\#}} = \beta^\#$, where $\widetilde{\#}$ is the sharp operator in the Finsler manifold $(M,\widetilde{E})$.*

Proof. Let $\widetilde{\omega}$ be the fundamental 2-form of $(M,\widetilde{E})$. Then

$$\widetilde{\omega} = dd_J\widetilde{E} = d(d_J\varphi E) = d(\varphi d_JE) = d\varphi\wedge d_JE + \varphi\omega,$$

so for any vector field X on M we have

$$i_{\beta^{\widetilde{\#}}}\widetilde{\omega}(X^c) = i_{\beta^{\widetilde{\#}}}(d\varphi\wedge d_JE + \varphi\omega)(X^c) = d\varphi\wedge d_JE(\beta^{\widetilde{\#}}, X^c)$$

$$+\varphi i_{\widetilde{\beta^{\#}}}\omega(X^c) = d\varphi(\widetilde{\beta^{\#}})d_J E(X^c) - d\varphi(X^c)d_J(\widetilde{\beta^{\#}})$$
$$+ i_{\varphi\widetilde{\beta^{\#}}}\omega(X^c) \overset{(14a)}{=} i_{\varphi\widetilde{\beta^{\#}}}\omega(X^c).$$

On the other hand, $i_{\widetilde{\beta^{\#}}}\,\widetilde{\omega} = \beta = i_{\beta^{\#}}\,\omega$; therefore $\varphi\widetilde{\beta^{\#}} = \beta^{\#}$. □

Proposition 3. *If L is a semibasic vector 1-form on TM, then the vector fields $(d_L E)^{\#}$ and $(\mathcal{L}_C d_L E)^{\#}$ are invariant under any conformal change of E.*

Proof. Consider the conformal change given by the scale function (21). Then

$$d_L\widetilde{E} = d_L(\varphi E) = \varphi d_L E + E d_L\varphi = \varphi d_L E,$$

since L is semibasic and φ is a vertical lift. Hence using Lemma 4,

$$\varphi(d_L\widetilde{E})^{\widetilde{\#}} = (d_L\widetilde{E})^{\#} = (\varphi d_L E)^{\#} = \varphi(d_L E)^{\#};$$

therefore $(d_L\widetilde{E})^{\widetilde{\#}} = (d_L E)^{\#}$. Similarly, the 1-form $(\mathcal{L}_C d_L E)$ is also semibasic and

$$\mathcal{L}_C d_L\widetilde{E} = \mathcal{L}_C(\varphi d_L E) = (\mathcal{L}_C\varphi)d_L E + \varphi\mathcal{L}_C d_L E = \varphi\mathcal{L}_C d_L E,$$

so, applying Lemmas 4 again, we get

$$\varphi(\mathcal{L}_C d_L\widetilde{E})^{\widetilde{\#}} = (\mathcal{L}_C d_L\widetilde{E})^{\#} = (\varphi\mathcal{L}_C d_L E)^{\#} = \varphi(\mathcal{L}_C d_L E)^{\#}.$$

This yields the desired second equality. □

Remark. If $L := \frac{1}{2}(f^c J - df^v \otimes C)$, our proposition leads to Proposition 3 of Vincze's paper [11].

Proposition 4. *Let L be a semibasic vector 1-form on TM. Under the conformal change given by the scale function (21) the horizontal endomorphism $\mathbf{h}_L$ and its associated semispray S_L change as follows:*

$$\widetilde{\mathbf{h}}_L = \mathbf{h}_L - \frac{1}{2}(f^c J + df^v \otimes C) + \frac{1}{2}E[J, \operatorname{grad} f^v] + \frac{1}{2}d_J E \otimes \operatorname{grad} f^v, \tag{24}$$
$$\widetilde{S}_L = S_L - f^c C + E \operatorname{grad} f^v. \tag{25}$$

Proof. By the conformal invariance of $(d_L E)^{\#}$, (16) and (22) yield immediately (24). Similarly, applying the conformal invariance of $(\mathcal{L}_C d_L E)^{\#}$, Proposition 1(ii) and (23), we obtain (25). □

5.2. *Now we are in a position to state and prove our main observation.*

Theorem. *The set of all conservative L-horizontal endomorphisms on a Finsler manifold is conformally closed.*

Proof. Consider the conformal change $\widetilde{E} := \varphi E$, $\varphi := \exp \circ f^v$ $(f \in C^\infty(M))$. Let L be a semibasic vector 1-form on TM, and define $K := \widetilde{\mathbf{h}}_L - \mathbf{h}_L$. Then K is semibasic, and we have $K := \widetilde{\mathbf{h}}_0 - \mathbf{h}_0$ by (16) and the confomal invariance of $(d_L E)^\#$. Since $\widetilde{\mathbf{h}}_0$ and $\mathbf{h}_0$ are conservative, we get

$$\begin{aligned} 0 &= d_{\widetilde{\mathbf{h}}_0} \widetilde{E} = d_{\mathbf{h}_0 + K} \widetilde{E} = d_{\mathbf{h}_0}(\varphi E) + d_K \widetilde{E} \\ &= E d_{\mathbf{h}_0} \varphi + \varphi d_{\mathbf{h}_0} E + d_K \widetilde{E} = E d_{\mathbf{h}_0} \varphi + d_K \widetilde{E}, \end{aligned}$$

hence

$$d_K \widetilde{E} = -E d_{\mathbf{h}_0} \varphi. \tag{26}$$

Now suppose that the horizontal endomorphism $\mathbf{h}_L$ is conservative. Then

$$\begin{aligned} d_{\widetilde{\mathbf{h}}_L} \widetilde{E} &= d_{\mathbf{h}_L + K} \widetilde{E} = d_{\mathbf{h}_L} \widetilde{E} + d_K \widetilde{E} \overset{(26)}{=} d_{\mathbf{h}_L} \widetilde{E} - E d_{\mathbf{h}_0} \varphi \\ &= d_{\mathbf{h}_L} \varphi E - E d_{\mathbf{h}_0} \varphi = \varphi d_{\mathbf{h}_L} E + E d_{\mathbf{h}_L} \varphi - E d_{\mathbf{h}_0} \varphi \\ &= E d_{\mathbf{h}_L - \mathbf{h}_0} \varphi = 0, \end{aligned}$$

since $\mathbf{h}_L - \mathbf{h}_0$ is semibasic by (16), and φ is a vertical lift. Thus $\widetilde{\mathbf{h}}_L$ is also conservative, and the proof is concluded. □

Acknowledgement. We are indebted to the anonymous referee for her/his useful suggestions.

References

[1] M. de León and P. R. Rodrigues, Methods of differential geometry in analytical mechanics, *North-Holland*, *Amsterdam*, 1989.

[2] A. Frölicher and A. Nijenhuis, Theory of vector-valued differential forms, *Proc. Kon. Ned. Akad. A.* **59** (1956), 338–359.

[3] J. Grifone, Structure presque tangente et connexions, I, *Ann. Inst. Fourier, Grenoble* **22**, no. 1 (1972), 287–334.

[4] J. Klein et A. Voutier, Formes extérieures génératrices de sprays, *Ann. Inst. Fourier, Grenoble* **18**, no. 2 (1968), 241–260.

[5] I. Kolář, P. W. Michor and J. Slovák, Natural operations in differential geometry, *Springer-Verlag*, *Berlin*, 1993.

[6] J. Szilasi, A Setting for Spray and Finsler Geometry, in: Handbook of Finsler Geometry Vol. 2, (P. L. Antonelli, ed.), *Kluwer Academic Publishers*, *Dordrecht*, 2003.

[7] Sz. Szakál and J. Szilasi, A new approach to generalized Berwald manifolds II, *Publ. Math. Debrecen* **60** (2002), 429–453.

[8] J. Szilasi, Notable Finsler connections on a Finsler manifold, *Lecturas Matemáticas* **19** (1998), 7–34.

[9] J. Szilasi and Cs. Vincze, On conformal equivalence of Riemann–Finsler metrics, *Publ. Math. Debrecen* **52** (1998), 167–185.

[10] J. Szilasi and Á. Győry, A generalization of Weyl's Theorem on projectively related affine connections, *Report on Mathematical Physics* **53**, no. 2 (2004), 261–273.

[11] Cs. Vincze, An intrinsic version of Hashiguchi–Ichijyō's theorems for Wagner manifolds, *SUT Journal of Mathematics* **35**, no. 2 (1999), 263–270.

[12] Cs. Vincze, On Wagner connections and Wagner manifolds, *Acta Math. Hungar.* **89** (2000), 111–133.

[13] K. Yano and S. Ishihara, Tangent and Cotangent Bundles: Differential Geometry, *Marcel Dekker Inc.*, *New York*, 1973.

M. M. REZAII
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
AMIRKABIR UNIVERSITY OF TECHNOLOGY (TEHRAN POLYTECHNIC)
424 HAFEZ AVE. 15914 TEHRAN
IRAN

E-mail: mmreza@aut.ac.ir

MANSOOR BARZEGARI
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
AMIRKABIR UNIVERSITY OF TECHNOLOGY (TEHRAN POLYTECHNIC)
424 HAFEZ AVE. 15914 TEHRAN
IRAN

BEHZAD NAJAFI
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
PAYAM NOUR UNIVERSITY
IRAN

E-mail: behzad_najafi@pnu.ac.ir

(Received May 30, 2005; revised September 13, 2005; accepted January 6, 2006)