Publ. Math. Debrecen
70/3-4 (2007), 319–335

On the inverse problem for sprays

By MIKE CRAMPIN (Gent and London)

Abstract. The inverse problem for sprays, or in other words the Finsler-metrizability problem, is discussed from the point of view of the holonomy of the non-linear connection associated with a spray.

1. Introduction

In Finsler geometry, the geodesics of a Finsler function, parametrized so that the tangent vector has constant Finsler length, define a spray. However, not every spray is obtainable in this way. The inverse problem for sprays is the problem of determining, for a given spray, whether or not there is a Finsler function of which it is the geodesic spray; or more broadly, of giving criteria for distinguishing those sprays which are geodesic.

The inverse problem for sprays, or the Finsler-metrizability problem as it is sometimes called (for example in [16], [23]), is thus apparently a special case of the inverse problem of the calculus of variations for arbitrary systems of second-order ordinary differential equations, or in other words for semi-sprays, which has been studied in numerous papers, starting with the celebrated paper of Douglas in 1941 [8], and including for example [1], [2], [10], [14], [17], [18]. On the other hand, the inverse problem for sprays includes as a special case its affine version, namely the problem of determining whether a symmetric affine connection is the Levi–Civita connection of some metric.

Mathematics Subject Classification: 53B40, 53C22, 53C29, 53C60.
Key words and phrases: Finsler function, holonomy, inverse problem of the calculus of variations, non-linear connection, spray.

For the most part, the inverse problem of the calculus of variations, and ipso facto the inverse problem for sprays, has been tackled by formulating partial differential equations which the unknown Lagrangian or Finsler function, or some quantities derived from it, must satisfy, and subjecting them to some form of integrability analysis: for example, DOUGLAS [8] worked with the so-called Helmholtz conditions for a multiplier matrix, and analysed them using the Riquier theory of systems of partial differential equations. However, it was pointed out as long ago as 1973 by SCHMIDT [19] that an alternative approach is possible, at least in the case of the problem of affine connections and metrics. Here the relevant partial differential equations, for the unknown functions g_{ij} which are the components of the metric, are those that state that the metric has zero covariant derivative:

$$\frac{\partial g_{ij}}{\partial x^k} = g_{lj}\Gamma^l_{ik} + g_{il}\Gamma^l_{jk}.$$

Repeated cross-differentiation gives a succession of conditions on the components of the unknown metric and the connection coefficients Γ^k_{ij} and their derivatives, or better the curvature and its covariant derivatives, which must be satisfied if the connection is to be the Levi–Civita connection of the metric. Schmidt says, in criticism of this method: 'Very little insight however is gained ... into the geometrical meaning of the integrability conditions and the restrictions imposed by them on the connection.' He then proposes an alternative approach which is not susceptible to this criticism, as follows. Since we have an affine connection at our disposal, we can parallelly translate vectors along any path in our manifold M. So given any pair of points x, y, and any curve joining them, we have a diffeomorphism – indeed linear isomorphism – of T_xM with T_yM. Now the defining characteristic of the Levi–Civita connection of a metric is that parallel transport preserves lengths and angles. Let us specify a metric g_x at some point $x \in M$ (that is, a positive-definite symmetric bilinear form on T_xM): we may then seek to extend the definition of g to the whole of M by setting $g_y(u, v) = g_x(u_0, v_0)$ for any $y \in M$, where $u, v \in T_yM$ are the parallel translates of $u_0, v_0 \in T_xM$ along a path joining x to y. The drawback is that u and v, and therefore g_y, will in general depend on the chosen path. This will not be the case, and g will be well-defined, if and only if the isomorphism of T_xM induced by parallel transport around any piecewise smooth closed curve beginning and ending at x leaves g_x invariant. Thus the necessary and sufficient condition for a symmetric affine connection to be derivable from a metric is that at a chosen point $x \in M$ there is a positive-definite symmetric bilinear form on T_xM such that the holonomy group of the connection is contained in the orthogonal group of the form. In short, a symmetric affine connection is Riemann-metrizable if and only if its holonomy group at any one point is a subgroup of an orthogonal group.

As Schmidt himself concedes, the advantage of this approach is not a practical one; indeed, the integrability conditions of the partial differential equations are essentially an infinitesimal version of the holonomy criterion, as is apparent from the fact that the infinitesimal generators of the holonomy group are expressible as repeated covariant derivatives of the curvature. The value of Schmidt's method is the not inconsiderable geometrical light it throws on the problem.

The inverse problem for sprays may be tackled by a method similar to Schmidt's, with equal benefit in terms of geometrical insight. The 2-dimensional case was discussed by Matsumoto and Tamássy in 1980 [15]; aspects of the general case have been considered more recently by Kozma (see Section 2.5 of his review article on holonomy in Finsler geometry [13], and references therein). The method is similar to that of Schmidt described above in that it makes no use of partial differential equations and their integrability conditions, and that it is based on the idea of starting with a Finsler function at one point x of the base manifold M (considered as a function on $T^\circ_x M$, the tangent space at x with its origin deleted) and then spreading it out along curves, thereby arriving at a consistency condition on the initial choice. This consistency condition can be expressed in terms of the holonomy of the non-linear connection associated with the spray. Whether or not a given spray is the geodesic spray of a Finsler function will be determined by whether or not it is possible to choose for that spray an initial function consistent with the condition (which depends of course on the spray).

I have three aims in the present paper. In the first place I shall discuss the relationship between the conditions for metrizability obtained by the holonomy method just described and those obtained in previous work. In particular a comparison should be made with the results obtained by analysing the integrability of partial differential equations satisfied by the unknown Finsler function; I use as my example of this approach the recent paper by Muzsnay [16], which appears to contain the most complete and definite results. In order to carry out the comparison it is necessary to formulate an infinitesimal version of the conditions obtained by the holonomy method.

Secondly, I shall discuss certain aspects of the holonomy construction in the case where the spray is affine; I shall point out in particular that the holonomy approach offers some illuminating insights into Szabó's theorem [21] that any Berwald space admits a Riemannian structure with the same geodesics, and especially into the new proof of this theorem given by Vincze [24].

I said above that the inverse problem for sprays is *apparently* a special case of the inverse problem of the calculus of variations for systems of second-order ordinary differential equations. In fact there is a method of passing between such systems on the one hand, and sprays on the other, which makes the two problems essentially equivalent at the local level. For my third topic I will use this fact to illustrate the results for sprays by giving a new proof of the variationality of a class of systems of second-order differential equations discussed (in the case of a pair of equations) by Douglas, namely case I of his case-by-case analysis; the specification of case I and the proof of its variationality were extended to an arbitrary number of equations in [18]. The very fact that this new proof is possible reveals an interesting and unexpected fact about the nature of the case I criterion.

I briefly review the relevant results from Finsler geometry and the geometry of sprays in the next section; I then describe the holonomy construction, in rather more detail than is to be found in [13], give the infinitesimal version, and discuss the relationship of the results obtained by this approach with those of other authors. Section 3 is devoted to a discussion of the affine case. Section 4 prepares the ground for the consideration of the Douglas case I problem, which is carried out in Section 5.

2. Sprays and Finsler functions

The geodesic spray Γ of a Finsler function F satisfies

$$\Gamma\left(\frac{\partial F}{\partial u^i}\right) - \frac{\partial F}{\partial x^i} = 0$$

(the Euler–Lagrange equations) and $\Gamma(F) = 0$ (constant speed parametrization). Indeed, Γ is uniquely determined by these conditions (and the fact that it is a spray), assuming that F is strongly convex, as is required for it to be a Finsler function. For then the fundamental tensor

$$g_{ij} = F\frac{\partial^2 F}{\partial u^i \partial u^j} + \frac{\partial F}{\partial u^i}\frac{\partial F}{\partial u^j}$$

is positive-definite, and in particular is non-singular, from which it follows that if

$$V^j \frac{\partial^2 F}{\partial u^i \partial u^j} = 0$$

then (V^i) is a scalar multiple of (u^i). Therefore since two sprays differ by a vertical vector field, two sprays which both satisfy the Euler–Lagrange equations for F differ by a multiple of the Liouville field $\Delta = u^i\partial/\partial u^i$. But $\Delta(F) = F$ by homogeneity, so the spray which satisfies the Euler–Lagrange equations for F and has F as a first integral is unique.

One might describe this spray as the canonical geodesic spray. If on the other hand one doesn't insist on constant speed parametrization then the Euler–Lagrange equations determine a projective equivalence class of sprays (in fact the projective equivalence class of the canonical geodesic spray).

The inverse problem for a spray in the natural sense (in the terminology of [23]) is the problem of determining whether or not there is a Finsler function of which it is the canonical geodesic spray. When there is, the spray is said to be Finsler-metrizable in the natural sense [23], or Finslerian [20]. The inverse problem for a spray in the broad sense is the problem of determining whether or not there is a Finsler function of which the base integral curves of the spray are the geodesic paths (with parametrization unspecified). When there is, the spray is said to be Finsler-metrizable in the broad sense, or projectively Finslerian. In fact the inverse problem in the broad sense more properly refers to a whole projective class of sprays rather than an individual one. To begin with I deal with the inverse problem in the natural sense.

Recall that the horizontal distribution of an arbitrary spray

$$\Gamma = u^i\frac{\partial}{\partial x^i} - 2\Gamma^i\frac{\partial}{\partial u^i}$$

is spanned by the vector fields

$$H_i = \frac{\partial}{\partial x^i} - \Gamma^j_i\frac{\partial}{\partial u^j}, \qquad \Gamma^j_i = \frac{\partial\Gamma^j}{\partial u^i};$$

furthermore

$$\left[\Gamma, \frac{\partial}{\partial u^i}\right] = -\frac{\partial}{\partial x^i} + 2\Gamma^j_i\frac{\partial}{\partial u^j} = \frac{\partial}{\partial x^i} - 2H_i.$$

I now apply this observation when the spray is the canonical geodesic spray of a Finsler function F, to obtain

$$0 = \Gamma\left(\frac{\partial F}{\partial u^i}\right) - \frac{\partial F}{\partial x^i} = \left[\Gamma, \frac{\partial}{\partial u^i}\right](F) + \frac{\partial}{\partial u^i}(\Gamma(F)) - \frac{\partial F}{\partial x^i} = -2H_i(F).$$

Thus a Finsler function F has the property that for any vector H which is horizontal (with respect to the horizontal distribution of its geodesic spray), $H(F) = 0$.

(This is a straightforward proof of a result which is quite well known, though it is usually expressed in terms of the energy of the Finsler function rather than the Finsler function itself; the result appears in this form in for example [12], [20], [22], and it is the basis also of MUZSNAY's account of the Finsler-metrizability problem in [16].)

A curve on $T^\circ M$ (the slit tangent bundle of M) is called horizontal if its tangent vector is horizontal at each point of it. Clearly the property of being horizontal is unaffected by a reparametrization, even one which reverses the sense in which the curve is traversed. Given any curve σ in M, any point x on σ, and any $u \in T^\circ_x M$ there is a unique horizontal curve σ^{H} passing through (x, u) and projecting onto σ; it is determined by solving the differential equations $\dot{u}^i + \Gamma^i_j(x^k(t), u^k(t))\dot{x}^j = 0$ for $u^i(t)$, where $\sigma(t) = (x^i(t))$, with initial conditions $(u^i(0)) = u$ (assuming that $x = \sigma(0)$). The curve σ^{H} is called the horizontal lift of σ through (x, u). Every horizontal curve is a horizontal lift (of its projection onto M). Furthermore, given any two points x and y in M, a curve σ joining them defines a map $h_\sigma : T^\circ_x M \to T^\circ_y M$ by $h_\sigma(u) = \sigma^{\mathrm{H}}(1)$ where σ^{H} is the horizontal lift of σ through (x, u) and $y = \sigma(1)$. This map is a diffeomorphism, and it is homogeneous in the sense that it commutes with dilations.

Now since $H(F) = 0$ for every horizontal vector H, F is constant along any horizontal curve, therefore along any piecewise horizontal curve, that is, any piecewise smooth curve each of whose smooth segments is horizontal. Thus in particular, if we take any point $x \in M$, and suppose that $u, u' \in T^\circ_x M$ are such that (x, u') can be joined to (x, u) by a piecewise horizontal curve, then $F(x, u') = F(x, u)$.

Every piecewise smooth curve σ in M beginning and ending at x defines a diffeomorphism h_σ of $T^\circ_x M$; the collection of such diffeomorphisms forms a group, with multiplication induced from joining curves in M. This is the holonomy group $\mathfrak{H}_x$ of the non-linear connection (or horizontal distribution) associated with the spray. Thus for any Finsler function F, and for any point $x \in M$, the restriction F_x of F to $T^\circ_x M$ is invariant under $\mathfrak{H}_x$: for any $h \in \mathfrak{H}_x$, $F_x \circ h = F_x$. (See [13] for an encyclopaedic account of holonomy in Finsler geometry, including a statement of this fact, and numerous references.)

This is the consistency condition referred to in the introduction, and will lead to the criterion for determining whether or not a spray is Finsler-metrizable.

I now wish therefore to show that if a given spray Γ is such that for a point $x \in M$ there is a smooth function F_x on $T^\circ_x M$ which is positively homogeneous and strongly convex and is invariant under $\mathfrak{H}_x$, then at least locally there is a

Finsler function F of which Γ is the geodesic spray, such that F_x is the restriction of F to $T_x^\circ M$. I assume that M is pathwise connected.

I propose to define F as follows. For a point $(y,v) \in T^\circ M$, let $\sigma(t)$ be a curve in M with $\sigma(0) = x$, $\sigma(1) = y$; let σ^{H} be the horizontal lift of σ with $\sigma^{\mathrm{H}}(1) = (y,v)$. Then provisionally I set

$$F(y,v) = F_x(\sigma^{\mathrm{H}}(0)).$$

The key point is to show that F is well-defined by this construction; that is to say, that if $\hat{\sigma}$ is another curve in M with $\hat{\sigma}(0) = x$, $\hat{\sigma}(1) = y$, then $F_x(\hat{\sigma}^{\mathrm{H}}(0)) = F_x(\sigma^{\mathrm{H}}(0))$. But one can construct out of σ and $\hat{\sigma}$ a piecewise smooth curve τ beginning and ending at x, in the usual way:

$$\tau(t) = \begin{cases} \sigma(2t) & 0 \le t \le \dfrac{1}{2} \\ \hat{\sigma}(2-2t) & \dfrac{1}{2} \le t \le 1. \end{cases}$$

By assumption, F_x is invariant under the element h_τ of $\mathfrak{H}_x$ determined by τ. Thus

$$F_x(\sigma^{\mathrm{H}}(0)) = F_x(h_\tau(\sigma^{\mathrm{H}}(0))) = F_x(\hat{\sigma}^{\mathrm{H}}(0))$$

and $F(y,v)$ is well-defined.

When F_x is invariant under $\mathfrak{H}_x$, any choice of curves in M serves to define F. In particular, one can take $t \mapsto (ty^i)$ in a coordinate patch with x as origin as a curve joining x to $y = (y^i)$. To define $F(y,v)$ one needs the solution at $t=0$ of the system of differential equations $\dot{u}^i + \Gamma^i_j(ty^k, u^k)y^j = 0$ with 'initial conditions' $u^i = v^i$ at $t = 1$. It follows from the smooth dependence of solutions of a system of differential equations on initial conditions, v^i, and parameters, y^i, that F is smooth.

The fact that F is positively homogeneous follows from the fact that horizontal lifting commutes with positive dilations, and the assumption that F_x is positively homogeneous. Finally, F will be strongly convex on a neighbourhood of x if F_x is strongly convex, by continuity.

It remains to show that Γ is the canonical geodesic spray of F. By construction, F is constant along horizontal curves. Thus $H(F) = 0$ for any horizontal vector field H. But Γ is horizontal, so $\Gamma(F) = 0$. It follows from the calculations at the beginning of this section that

$$\Gamma\left(\frac{\partial F}{\partial u^i}\right) - \frac{\partial F}{\partial x^i} = \frac{\partial}{\partial u^i}(\Gamma(F)) - 2H_i(F) = 0,$$

so Γ satisfies the Euler–Lagrange equation, and since $\Gamma(F) = 0$, Γ is the geodesic spray of F.

I summarize the foregoing discussion in the form of a theorem.

Theorem. *A spray Γ is locally Finsler-metrizable in the natural sense if and only if for a point $x \in M$ there is a function F_x on $T_x^\circ M$ which is positively homogeneous and strongly convex and is invariant under $\mathfrak{H}_x$, the holonomy group at x of the non-linear connection associated with Γ.*

I now draw some consequences from this theorem, which extend and illuminate results obtained previously by other authors.

As I pointed out earlier, elements of $\mathfrak{H}_x$ commute with dilations, and therefore map rays in $T_x^\circ M$ to rays; that is, $\mathfrak{H}_x$ may be considered to act on $T_x^\circ M/\mathbf{R}^+$, the space of rays. Note in particular that if some element h of $\mathfrak{H}_x$ maps a ray to itself then if the spray is metrizable h must leave the ray pointwise fixed. Suppose now that $\mathfrak{H}_x$ acts transitively on $T_x^\circ M/\mathbf{R}^+$, in such a way that the elements of the isotropy group of any ray leave that ray pointwise fixed. Then there is a unique positively homogeneous $\mathfrak{H}_x$-invariant function F_x on $T_x^\circ M$ taking the value 1 at a chosen point $u \in T_x^\circ M$, and it is positive-valued. Moreover, if $\hat{F}_x$ is the corresponding function which takes the value 1 at some other point $\hat{u} \in T_x^\circ M$ then $\hat{F}_x = cF_x$ for some constant c (in fact $c = \hat{F}_x(u)$). Whether or not the spray is metrizable in this case depends entirely on whether or not the function F_x is strongly convex; when it is the Finsler function is unique, up to homothety. This result generalizes one given by Matsumoto and Tamássy in [15].

More generally, one can consider the case in which $\mathfrak{H}_x$ does not act transitively, but still has the property that any element of $\mathfrak{H}_x$ which leaves a ray fixed leaves it pointwise fixed, as follows. Let Σ be a section of the action of $\mathfrak{H}_x$ on $T_x^\circ M/\mathbf{R}^+$, that is, a subset of $T_x^\circ M/\mathbf{R}^+$ containing one point of each orbit of $\mathfrak{H}_x$ on $T_x^\circ M/\mathbf{R}^+$. For each point, in other words ray, in Σ choose a point u on that ray: then there is a unique positively homogeneous $\mathfrak{H}_x$-invariant function F_x on $T_x^\circ M$ taking the value 1 at each chosen point u, and it is positive-valued. If this function is smooth and strongly convex it is a Finsler function. Moreover, if the spray admits Finsler functions F_x and $\hat{F}_x$, and for each ray in Σ there is a point u on the ray such that $\hat{F}_x(u) = F_x(u)$ (in which case equality will hold throughout the ray by homogeneity) then $\hat{F}_x = F_x$.

Kozma, in his theorem on metrizability in [13], takes a somewhat different approach. He shows that when the holonomy group $\mathfrak{H}_x$ is a compact Lie group the spray is metrizable by constructing an invariant function F_x by averaging over the group with respect to its Haar measure. This method is based on that used by Szabó in the Berwald case, which I shall come back to later. However, there is no reason to suppose in general that the holonomy group is compact.

I next describe an infinitesimal version of the holonomy criterion for Finsler-metrizability.

For any spray, let $\mathcal{D}$ be the smallest integrable distribution on $T^\circ M$ containing its horizontal distribution $\mathcal{H}$; assume that $\mathcal{D}$ is regular (that is, that the dimension of $\mathcal{D}$ is constant). Then the set of points of $T^\circ M$ that can be connected to a given point (x, u) by piecewise horizontal curves is the leaf of the distribution $\mathcal{D}$ through (x, u). In particular, the set of points in $T_x^\circ M$ that can be connected to (x, u) by piecewise horizontal curves is the intersection of the leaf with $T_x^\circ M$.

Let $\mathcal{V}$ be the vertical distribution on $T^\circ M$; it is of course integrable, and its leaves are the fibres of $T^\circ M \to M$. Let $\mathcal{W} = \mathcal{D} \cap \mathcal{V}$. Since $\mathcal{H} \subseteq \mathcal{D}$ and $\mathcal{H}$ and $\mathcal{V}$ are complementary, $\dim \mathcal{W} = \dim \mathcal{D} - \dim \mathcal{H}$; in particular, $\mathcal{W}$ is regular since $\mathcal{D}$ is, by assumption. Moreover, $\mathcal{W}$ is integrable. The leaf of $\mathcal{W}$ through a point (x, u) is the intersection of the leaf of $\mathcal{D}$ through (x, u) with $T_x^\circ M$, the leaf of $\mathcal{V}$ through (x, u).

So much is true for any spray for which $\mathcal{D}$ is regular. But for the geodesic spray of a Finsler function F we have in addition that for any $x \in M$, the restriction of F to $T_x^\circ M$ is constant on the leaves of $\mathcal{W}_x$, the restriction of $\mathcal{W}$ to $T_x^\circ M$; in other words, it is an integral function of $\mathcal{W}_x$. By a slight modification of the argument given earlier, it can be seen that this is a sufficient as well as a necessary condition for the existence of a Finsler function of which the given spray is the geodesic spray. We therefore have the following infinitesimal version of the Theorem.

Theorem. *A spray is locally Finsler-metrizable in the natural sense if and only if for any $x \in M$ the distribution $\mathcal{W}_x$ on $T_x^\circ M$ determined by it admits an integral function which is positively homogeneous and strictly convex.*

Note in particular that a spray for which $\mathcal{W}_x$ contains the dilation field of $T_x^\circ M$ cannot be metrizable.

These results appear to be essentially equivalent to those obtained by Muzsnay in [16]. However, Muzsnay works with the energy E rather than the Finsler function, and uses Spencer's technique of formal integrability applied to the partial differential system $\mathcal{D}(E) = 0$, $\Delta(E) = 2E$. His results are expressed in terms of the distribution $\mathcal{D}$ rather than $\mathcal{W}_x$.

Finally, since a Finsler function is positively homogeneous in the velocity variables this criterion, in either form, can be stated in terms of the indicatrix: for example, the necessary and sufficient condition for local Finsler-metrizability is that for $x \in M$ there is smooth strongly convex codimension 1 submanifold of $T_x^\circ M$ which is invariant under $\mathfrak{H}_x$.

3. The affine case

When the spray is an affine spray (the geodesic spray of a symmetric affine connection), the condition for a curve $t \mapsto (x^i(t), u^i(t))$ to be horizontal is that $\dot{u}^i + \Gamma^i_{jk}\dot{x}^j u^k = 0$; thus the horizontal lift of a curve in M can be identified with that curve with a parallel vector field along it. Then if

$$F(x^i, u^i) = \sqrt{g_{jk}u^j u^k},$$

F is constant along horizontal lifts if and only if any parallelly translated vector field has constant length. But then by polarization the angle between any pair of parallelly translated vector fields is constant too.

The criterion for the existence of a Finsler function which takes the form above at the initial point $x \in M$, say

$$F_x(u^i) = \sqrt{a_{jk}u^j u^k},$$

is that for any piecewise smooth closed curve σ in M beginning and ending at x, if $v \in T_xM$ is the parallel translate of $u \in T_xM$ along σ then (at x)

$$\sqrt{a_{jk}u^j u^k} = \sqrt{a_{jk}v^j v^k}.$$

This means that the holonomy group must preserve lengths (as measured by a_{ij}); but then, again by polarization, it preserves angles too (it consists after all of linear isomorphisms of T_xM). So the criterion is the same as the one given by Schmidt. Furthermore, since the diffeomorphism $T_xM \to T_yM$ generated by the horizontal lift of any curve is in this case a linear isomorphism, the Finsler function obtained will have the form

$$F(x^i, u^i) = \sqrt{g_{jk}u^j u^k},$$

where g_{jk} are the components of a metric on M.

Of course one need not begin with the square root of a quadratic function on T_xM; indeed, if the zero vector is excised from T_xM then any positively homogeneous strictly convex function which satisfies the criterion will do. If there is such a function then the Finsler space obtained from it will be a Berwald space.

Suppose given a Berwald space; then $\mathfrak{H}_x$, which is the holonomy group of a linear connection, leaves the indicatrix invariant. It is easy to see that as a consequence, as linear isomorphisms of T_xM the elements of $\mathfrak{H}_x$ are uniformly bounded, so $\mathfrak{H}_x$ is a compact Lie subgroup of the linear isomorphism group of T_xM. One can then construct a positive-definite bilinear form on T_xM which is invariant under $\mathfrak{H}_x$ by averaging an arbitrary one over the group with respect to an invariant measure. This shows how SZABÓ's theorem [21] that any Berwald space admits a Riemannian structure with the same geodesics fits into the picture.

There is a little bit more to this part of the story than meets the eye. For a general spray, and for a fixed curve σ joining x to y in M, I have denoted by $h_\sigma : T_x^\circ M \to T_y^\circ M$ the diffeomorphism defined by the horizontal lift. The differential $h_{\sigma *} : T_u(T_x^\circ M) \to T_{h_\sigma(u)}(T_y^\circ M)$ of h_σ represents the Berwald connection associated with the spray, in the following sense: if $v \in T_u(T_x^\circ M)$ then $h_{\sigma *}(v)$ is the element of $T_{h_\sigma(u)}(T_y^\circ M)$ obtained by parallelly translating v, relative to the Berwald connection, along the horizontal curve σ^{H} from (x,u) to $(y, h_\sigma(u))$ (see [3] for a construction of the Berwald connection along these lines). Then $h_{\sigma *}$ is a linear isomorphism of $T_u(T_x^\circ M)$ with $T_{h_\sigma(u)}(T_y^\circ M)$. In the affine case, when h_σ is already linear, it coincides with its differential (when $T_u(T_xM)$ is identified with T_xM and $T_{h_\sigma(u)}(T_yM)$ with T_yM); but in general this will not be so.

Let us consider the Levi–Civita connection of a metric g from this point of view. We may regard g_x as a bilinear form on T_xM for each $x \in M$: then $h_\sigma : T_xM \to T_yM$ preserves bilinear forms. But we may equally regard g_x as defining a metric on T_xM, that is, as a bilinear form on $T_u(T_xM)$ for each $u \in T_xM$: then h_σ is an isometry of the Riemannian spaces T_xM and T_yM. In this case the metric on T_xM is of course constant, and the two points of view are essentially identical.

It is instructive to take a similar view of the general Finsler case. We may regard the fundamental tensor

$$g_{ij} = F\frac{\partial^2 F}{\partial u^i \partial u^j} + \frac{\partial F}{\partial u^i}\frac{\partial F}{\partial u^j}$$

as defining a Riemannian metric on $T_x^\circ M$ for each $x \in M$ (no longer constant in general). In general, $h_\sigma : T_x^\circ M \to T_y^\circ M$ will not be an isometry. In view of the remarks made earlier about the relation between the differential of h_σ and the Berwald connection, the condition for h_σ to be an isometry is that the fundamental tensor be Berwald parallel along all horizontal lifts of σ. That is, a

Finsler space in which the horizontal lift mapping is an isometry for all curves is a Landsberg space.

For a Landsberg space, therefore, the holonomy group $\mathfrak{H}_x$ is a subgroup of the isometry group of the Riemannian space $T_x^\circ M$ equipped with the fundamental tensor. (For further information on these issues see [13].)

Since a Berwald space is a Landsberg space, the holonomy group at any point is contained in the isometry group of the fundamental tensor; this also follows directly from the fact that $\mathfrak{H}_x$ consists of linear isomorphisms of T_xM and leaves F_x invariant. Thus the volume form on T_xM defined by the Riemannian structure is invariant under $\mathfrak{H}_x$, and so is the volume form it induces on the indicatrix in $T_x^\circ M$, which is itself invariant. These facts lead to an alternative way of proving Szabó's theorem, as has been pointed out by Vincze [24]: one can construct an $\mathfrak{H}_x$-invariant positive-definite bilinear form on T_xM by averaging the fundamental tensor field over the indicatrix with respect to the induced volume, rather than averaging an arbitrary bilinear form over the group with respect to an invariant measure.

4. R-flat and isotropic sprays

One situation in which the criterion for a spray to be Finsler-metrizable is manifestly satisfied is when the spray's horizontal distribution is integrable (so that $\mathcal{W}_x = \{0\}$), or equivalently when the spray has vanishing Riemann curvature, or in the terminology of Shen [20] is R-flat. Clearly, any R-flat spray is Finsler-metrizable, and we can choose the value F_x of a Finsler function F for it on the fibre over $x \in M$ at will, subject only to the restrictions of homogeneity and strong convexity.

So far I have considered only the natural Finsler inverse problem; but once having shown that any particular spray is naturally Finsler-metrizable we can conclude immediately that all sprays projectively equivalent to it are metrizable in the broad sense, or are projectively Finslerian. The question now arises, which sprays are projectively equivalent to an R-flat spray?

To answer this question I must remind the reader about isotropic sprays. The Riemann curvature $R^i{}_{jkl}$ of a spray determines and is determined by the type $(1,1)$ tensor $R^i_j = R^i{}_{kjl}u^ku^l$; the spray is isotropic if there is a function λ and a covector field μ_i such that

$$R^i_j = \lambda\delta^i_j + \mu_j u^i.$$

The property of being isotropic is easily seen to be projectively invariant. It turns out to have another equivalent formulation, which will be highly relevant. One can construct from the (full) Riemann curvature a projectively invariant tensor $P^i{}_{jkl}$, related to it in much the same way as the projective curvature is to the Riemann curvature of an affine connection. (One version of this construction was given already by Douglas in his classic paper [7]; DOUGLAS calls the tensor the Weyl tensor, but Weyl has so many tensors named after him that to avoid confusion it seems best not to use this name.) It turns out that for $n > 2$ a spray is isotropic if and only if $P^i{}_{jkl} = 0$. (In dimension 2, $P^i{}_{jkl}$ is automatically zero, just as in the affine case, so it is advisable to exclude $n = 2$ from the following discussion. See [4], [5], [20] for more details about isotropic sprays.) It is clear that the projective class of an R-flat spray consists of isotropic sprays. But just as in the affine case, the vanishing of $P^i{}_{jkl}$ is sufficient, as well as necessary, for the projective class to include an R-flat spray (see [4] for a proof). (Unlike the situation in the affine case, however, this does not mean that an isotropic spray is projectively flat: there is after all another projectively invariant tensor associated with a spray, namely its Douglas tensor, and in order for the spray to be projectively flat its Douglas tensor must also vanish.)

It follows from the discussion above that all isotropic sprays are projectively Finslerian. (Despite its title, [10] deals with semi-sprays rather than sprays as the terms are used here; and to the extent that isotropic sprays are covered it is in the context of the problem of metrizability in the natural sense.)

5. The inverse problem of the calculus of variations

This observation can be used to give a new and simple proof of the variationality of a system of second-order ordinary differential equations in n independent variables, when the system belongs to Douglas's case I.

Suppose given an $(n + 1)$-dimensional manifold with a projective class of sprays. Take a coordinate patch of the base manifold with coordinates $(x^0, x^i) = (x^a)$, $i = 1, 2, \dots, n$, and over this patch the open subset of the tangent bundle where $u^0 \neq 0$ (the fibre coordinates are $(u^0, u^i) = (u^a)$). In this open subset of the tangent bundle one can select from the projective class of sprays one, say Γ, such that $\Gamma(u^0) = 0$; the restriction of Γ to the submanifold $u^0 = 1$ is tangent to it, and its base integral curves are parametrized by x^0. In fact Γ_0, the restriction of Γ to $u^0 = 1$, is given by

$$\Gamma_0 = \frac{\partial}{\partial x^0} + \dot{x}^i \frac{\partial}{\partial x^i} - 2\Gamma^i(x^0, x^j, 1, \dot{x}^j)\frac{\partial}{\partial \dot{x}^i},$$

where the $\dot{x}^i$ are the restrictions of the u^i to $u^0 = 1$, and so the base integral curves of Γ_0 are the solutions of the system of second-order ordinary differential equations

$$\ddot{x}^i = f^i(t, x^j, \dot{x}^j), \quad \text{where } f^i(t, x^j, \dot{x}^j) = -2\Gamma^i(t, x^j, 1, \dot{x}^j).$$

Conversely, given any such system of differential equations we can at least locally reconstruct a spray by setting

$$\Gamma^0 = 0, \quad \Gamma^i(x^a, u^a) = -\frac{1}{2}(u^0)^2 f^i(x^0, x^j, u^j/u^0),$$

bearing in mind the homogeneity requirement for the Γ^a.

Suppose now that we have a Finsler function F, and we take for the projective class of sprays in the construction just described that determined by the (unparametrized) geodesics of F. Define a function $L(t, x^i, \dot{x}^i)$ by

$$L(t, x^i, \dot{x}^i) = F(t, x^i, 1, \dot{x}^i)$$

(so that L is effectively the restriction of F to $u^0 = 1$). Then Γ_0 given above satisfies

$$\Gamma_0\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0;$$

moreover, the Hessian

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$$

is non-singular (as follows from the assumption of strong convexity of F), which is to say that the second-order differential equations $\ddot{x}^i = f^i(t, x^j, \dot{x}^j)$ are the Euler–Lagrange equations of the Lagrangian L. Incidentally, the equation

$$\Gamma\left(\frac{\partial F}{\partial u^0}\right) - \frac{\partial F}{\partial x^0} = 0,$$

which has so far been left out of consideration here, is

$$\frac{dE}{dt} + \frac{\partial L}{\partial t} = 0, \quad E = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L,$$

the energy equation.

Thus a possible method of solving the inverse problem for a system of second-order ordinary differential equations is to construct the corresponding spray by passing to the homogeneous formalism as described above; if this spray is projectively Finslerian then the original equations are variational. Of course this will in general be only a local construction: that is to say, the spray is only locally defined, and one cannot expect to do more than find a local Finsler function.

Any isotropic spray is projectively Finslerian, so it is natural to ask for the conditions under which a system of second-order differential equations gives rise to an isotropic spray. For this purpose it is necessary to express the Riemann curvature of the spray in terms of quantities derived from the differential equations, that is from the coefficients f^i. It turns out (see [5], [20]) that

$$\begin{aligned} R^0_0 &= R^0_i = 0 \\ R^i_0 &= u^0 u^j \Phi^i_j \\ R^i_j &= -(u^0)^2 \Phi^i_j, \end{aligned}$$

where Φ^i_j is the so-called Jacobi endomorphism of the system of differential equations,

$$\Phi^i_j = \frac{\partial f^i}{\partial x^j} + \frac{d\gamma^i_j}{dt} + \gamma^i_k \gamma^k_j, \quad \gamma^i_j = -\tfrac{1}{2}\frac{\partial f^i}{\partial \dot{x}^j}.$$

Now the differential equations comprising Douglas's case I are those for which the Jacobi endomorphism is a scalar multiple of the identity, that is, for which $\Phi^i_j = \Phi\delta^i_j$ say. For such differential equations the curvature of the corresponding spray is given by

$$\begin{aligned} R^0_0 &= R^0_i = 0 \\ R^i_0 &= u^0 u^i \Phi \\ R^i_j &= -(u^0)^2 \Phi \delta^i_j; \end{aligned}$$

but then $R^a_b = \lambda\delta^a_b + \mu_b u^a$ $(a, b = 0, 1, \ldots, n)$ with $\lambda = -(u^0)^2\Phi$, $\mu_0 = u^0\Phi$, $\mu_i = 0$, so the spray is isotropic. Conversely, if the spray is isotropic, with $R^a_b = \lambda\delta^a_b + \mu_b u^a$ where the R^a_b are given in terms of the Φ^i_j as above, then we have $R^0_i = \mu_i u^0 = 0$, so $\mu_i = 0$; and then $R^i_j = \lambda\delta^i_j = -(u^0)^2\Phi^i_j$, whence Φ^i_j is a multiple of δ^i_j. Thus when expressed in terms of spray geometry, Douglas's case I criterion corresponds exactly to the condition for a spray to be isotropic. It follows that systems of differential equations in case I are variational.

Finally, I want to point out an unexpected and interesting fact about the case I criterion that follows from this analysis. It is clearly necessary that for any such criterion to be useful it must be invariant under coordinate transformations of the form $t \mapsto t$, $x^i \mapsto \chi^i(t, x^j)$. However, as has just been shown this criterion is equivalent to the fact that the corresponding spray is isotropic, and this is a projective property of sprays, that is, it holds for all members of the projective class. To put it another way, the criterion will hold regardless of the choice of parametrization of the base integral curves of the spray; or another way still, it is actually invariant under the larger class of coordinate transformations $t \mapsto \tau(t, x^i)$, $x^i \mapsto \chi^i(t, x^j)$, the so-called point transformations. This observation is in fact implicit in studies of Cartan connections associated with systems of second-order differential equations under point transformations [5], [6], [9], [11], where the trace-free part of the Jacobi endomorphism appears as the torsion of the connection; but the relation of this work to the inverse problem has not been recognized before, to my knowledge.

Acknowledgements. I am a Guest Professor at Ghent University, and a Visiting Senior Research Fellow of King's College, University of London: I should like to express my gratitude to the Department of Mathematical Physics and Astronomy at Ghent and the Department of Mathematics at King's for their hospitality.

References

[1] I. Anderson and G. Thompson, The inverse problem of the calculus of variations for ordinary differential equations, *Mem. Amer. Math. Soc.* **98**, no. 473 (1992).

[2] M. Crampin, On the differential geometry of the Euler–Lagrange equation and the inverse problem of Lagrangian dynamics, *J. Phys. A: Math. Gen.* **14** (1981), 2567–2575.

[3] M. Crampin, Connections of Berwald type, *Publ. Math. Debrecen* **57** (2000), 455–473.

[4] M. Crampin, Isotropic and R-flat sprays, *Houston J. Math. (to appear);* available at `http:/maphyast.ugent.be` Theoretical Mechanics.

[5] M. Crampin and D. J. Saunders, Projective connections, *J. Geom. Phys.* **57** (2007), 691–727.

[6] M. Crampin and D. J. Saunders, Path spaces and almost Grassman structures, *preprint* Ghent University (2005) available at `http:/maphyast.ugent.be` Theoretical Mechanics.

[7] J. Douglas, The general geometry of paths, *Ann. Math.* **29** (1928), 143–168.

[8] J. Douglas, Solution of the inverse problem of the calculus of variations, *Trans. Amer. Math. Soc.* **50** (1941), 71–128.

[9] M. E. Fels, The equivalence problem for systems of second-order ordinary differential equations, *Proc. London Math. Soc.* **71** (1995), 221–240.

[10] J. Grifone and Z. Muzsnay, Sur le problème inverse du calcul des variations: existence de lagrangiens associées à un spray dans le cas isotrope, *Ann. Inst. Fourier Grenoble* **49** (1999), 1387–1421.

[11] D. A. Grossman, Torsion-free path geometries and integrable second order ODE systems, *Selecta Math.* **6** (2000), 399–442.

[12] J. Klein, Geometry of sprays. Lagrangian case. Principle of least curvature, Proceedings of the IUTAM-ISIMM Symposium on Analytical Mechanics, *Torino*, 1982.

[13] L. Kozma, Holonomy structures in Finsler geometry, Handbook of Finsler Geometry, (P. L. Antonelli, ed.), *Kluwer*, 2003.

[14] E. Massa and E. Pagani, Jet bundle geometry, dynamical connections, and the inverse problem of Lagrangian mechanics, *Ann. Inst. H. Poincaré Phys. Theor.* **61** (1994), 17–62.

[15] M. Matsumoto and L. Tamássy, Scalar and gradient vector fields of Finsler spaces and holonomy groups of nonlinear connections, *Demonstratio Math.* **XIII** (1980), 551–563.

[16] Z. Muzsnay, The Euler–Lagrange PDE and Finsler metrizability, *Houston J. Math.* **32** (2006), 79–98.

[17] G. E. Prince, The inverse problem in the calculus of variations and its ramifications, Geometric Approaches to Differential Equations, (P. Vassiliou and I. Lisle, eds.), *Lecture Notes of the Australian Mathematical Society: Cambridge University Press*, 2000.

[18] W. Sarlet, M. Crampin and E. Martínez, The integrability conditions in the inverse problem of the calculus of variations for second-order ordinary differential equations, *Acta Applicandae Math.* **54** (1998), 233–273.

[19] B. G. Schmidt, Conditions on a connection to be a metric connection, *Commun. Math. Phys.* **29** (1973), 55–59.

[20] Zhongmin Shen, Differential Geometry of Spray and Finsler Spaces, *Kluwer*, 2001.

[21] Z. I. Szabó, Positive definite Berwald spaces, *Tensor, N. S.* **35** (1981), 25–39.

[22] J. Szilasi, Notable Finsler connections in a Finsler manifold, *Lect. Math.* **19** (1998), 7–34.

[23] J. Szilasi and Sz. Vattamány, On the Finsler-metrizabilities of spray manifolds, *Period. Math. Hungarica* **44** (2002), 81–100.

[24] Cs. Vincze, A new proof of Szabó's theorem on the Riemann-metrizability of Berwald manifolds, *Acta Math. Acad. Paedagog. Nyházi.* **21** (2005), 199–204.

MIKE CRAMPIN
DEPARTMENT OF MATHEMATICAL PHYSICS AND ASTRONOMY
GHENT UNIVERSITY
KRIJGSLAAN 281
B–9000 GENT
BELGIUM

AND

DEPARTMENT OF MATHEMATICS
KING'S COLLEGE
STRAND
LONDON WC2R 2LS
UK

ADDRESS FOR CORRESPONDENCE:
MIKE CRAMPIN
65 MOUNT PLEASANT
ASPLEY GUISE, BEDS MK17 8JX
UNITED KINGDOM

E-mail: Crampin@btinternet.com

(Received June 7, 2005; revised December 12, 2005)