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Product closed modules need not be artinian

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Abstract. A module M is called *product closed* if every hereditary pretorsion class in $\sigma[M]$ is closed under products in $\sigma[M]$. It is known that if M is product closed, finitely generated, projective in $\sigma[M]$ and is such that every hereditary pretorsion class in $\sigma[M]$ is M-dominated, then M has finite length. In this paper a module M is constructed which is product closed, finitely generated and projective but which is not artinian.

It was shown by Beachy and Blair [1, Proposition 1.4 and Corollary 3.3] that the following three conditions on a ring R with identity are equivalent:

(1) every hereditary pretorsion class in R-Mod is closed under arbitrary (and not just finite) direct products, or equivalently, every left topologizing filter on R is closed under arbitrary (and not just finite) intersections;

(2) every left R-module M is finitely annihilated, meaning $(0 : M) = (0 : X)$ for some finite subset X of M;

(3) R is left artinian.

This result provided the motivation for [7] which is devoted to the more general investigation of left R-modules M with the property that every hereditary pretorsion class in $\sigma[M]$ is closed under arbitrary products in $\sigma[M]$. A module M with this property is called *product closed*. Beachy and Blair's theorem thus states that the ring R, when considered as left module over itself, is product closed precisely when R is left artinian. Results in [7] show that this characterization fails for a general module M. Whilst a module M with finite length is necessarily

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product closed, the converse need not be true. Indeed, [7, Example 11] exhibits a nonzero product closed module with zero socle and such a module is manifestly nonartinian. This failure in the case of a general M is not surprising, for the ring R, when considered as a module over itself, enjoys a number of special properties: it is finitely generated, projective and is a generator for the category R-Mod. It is thus natural to ask under what conditions, similar to those satisfied by ${}_RR$, is a product closed module M necessarily artinian. A main theorem in [7, Theorem 16] shows that if M is a product closed left R-module which is (1) finitely generated, (2) projective in $\sigma[M]$, and (3) such that all hereditary pretorsion classes $\mathcal{T}$ in $\sigma[M]$ are M-dominated (this means that $\mathcal{T}$ has an M-generated subgenerator), then M has finite length and is thus artinian.

Few examples of nonartinian product closed modules have thus far been identified. The main purpose of this paper is to construct a left R-module M with these properties but which is also '${}_RR$-like' in the sense that M is both finitely generated and projective. Such an example would also serve to show that condition (3) of the aforementioned theorem cannot be dispensed with.

The construction is lengthy and in the interests of manageability has been divided into two parts. The first part (§2) has as its objective the construction of a Loewy bimodule N meeting a number of very particular specifications. In the second part (§3), the ring R is defined as a set of upper triangular matrices all of whose members have entries from N in the off-diagonal position. The left R-module M is then chosen to be a certain direct summand of ${}_RR$.

In the course of proving that the aforementioned M is product closed, we shall have need to describe completely all left topologizing filters on the ring R and to depict these in a lattice diagram (see Theorem 12 and Figure 1). The paper should therefore also be of interest to researchers with an interest in the topologizing filter lattices of rings.

1. Preliminaries

We have included in this section only those definitions, rudimentary results and explanations of notational conventions that are not provided in [7].

If L is a complete lattice we denote by 0_L [resp. 1_L] the smallest [resp. largest] element of L. We omit the subscript L in cases where there is no ambiguity.

If R and S are rings we shall write ${}_RM$ [resp. M_S] (resp. ${}_RM_S$) to indicate that M is a (unital) left R-module [resp. (unital) right S-module] (resp. (unital) R-S-bimodule).

Recall that if $M \in R$-Mod, then the ascending chain $\{\mathrm{soc}^{\alpha} M\}$ of submodules of M is called the *ascending Loewy series of* M. We call $\mathrm{soc}^{\alpha} M$ the *αth Loewy submodule of* M. We say that M is *semiartinian* (or a *Loewy module*) if $M = \mathrm{soc}^{\alpha} M$ for some ordinal α. In this situation if α is the smallest ordinal for which $M = \mathrm{soc}^{\alpha} M$, we call α the *Loewy length of* M. For each ordinal α, the cardinality of a maximal independent family of nonzero submodules of the factor module $\mathrm{soc}^{\alpha+1} M/\mathrm{soc}^{\alpha} M$, is called the *$\alpha$th Loewy invariant of* M and is denoted $d_{\alpha}(M)$.

The reader is referred to [2], [4], [5], [9] and [10] for background information on hereditary pretorsion classes and topologizing filters.

We call $M \in R$-Mod *product closed* if every $\mathcal{T} \in M$-torsp (set of all hereditary pretorsion classes in $\sigma[M]$) is closed under products in $\sigma[M]$, that is, $\prod_{i\in\Gamma}^{\sigma[M]} N_i := \mathrm{Tr}\,(\sigma[M], \prod_{i\in\Gamma} N_i) \in \mathcal{T}$ whenever $\{N_i \mid i \in \Gamma\} \subseteq \mathcal{T}$.

If $M \in R$-Mod we call $\mathcal{T} \in M$-torsp *M-dominated* if $\mathcal{T}$ has an M-generated subgenerator.

We shall denote by R-fil the set of all left topologizing filters on R. If $\mathcal{A}$ is any nonempty family of left ideals of R, we denote by $\eta(\mathcal{A})$ the unique smallest element of R-fil for which $\eta(\mathcal{A}) \supseteq \mathcal{A}$. If $K \leq {}_R R$ then $K \in \eta(\mathcal{A})$ if and only if there exist $A_1, A_2, \ldots, A_n \in \mathcal{A}$ and $r_1, r_2, \ldots, r_n \in R$ such that $K \supseteq \bigcap_{i=1}^{n} (A_i : r_i)$ [4, Example 2.1, p. 15].

Following Cohn [3, p. 130] we call a domain R a *left principal valuation ring* if R contains a nonunit p such that every nonzero $r \in R$ is expressible in the form $r = up^n$ for some unit $u \in R$ and $n \geq 0$. It is easily shown that every nonzero left ideal of R is two-sided and of the form $Rp^n = (Rp)^n$ for some $n \geq 0$. Note that the left ideals of R are linearly ordered by inclusion (i.e., R is a left chain ring) with Rp the unique maximal proper left ideal of R.

2. The bimodule construction

Our objective in this section is to construct rings S and F and a bimodule ${}_S N_F$ which satisfy the following conditions:

(I) S is a left principal valuation ring.

(II) F is a field.

(III) Every S-F-bisubmodule of N is a member of the strictly ascending chain

$$0 \subset \mathrm{soc}_S N \subset \mathrm{soc}_S^2 N \subset \cdots \subset \mathrm{soc}_S^{\omega} N = \bigcup_{n\geq 1} \mathrm{soc}_S^n N = N.$$

Thus ${}_SN$ is a Loewy module with Loewy length ω. Moreover, each $\operatorname{soc}^n_S N$ has finite length as a left S-module and ${}_SN$ is non-artinian.

If F is a field with $\sigma : F \to F$ a field endomorphism and X an indeterminate, we shall denote by $F\big[[X,\sigma]\big]$ the *left skew power series ring over* F. Recall that the elements of $F\big[[X,\sigma]\big]$ are formal power series of the form

$$\sum_{n\geq 0} a_n X^n = \sum_{n=0}^{\infty} a_n X^n$$

where $a_n \in F$ for all $n \geq 0$. Addition of power series is natural and multiplication is induced by the rule

$$Xa = \sigma(a)X \quad \text{for all } a \in F. \tag{1}$$

The following result identifies the prototype of one-sided principal valuation ring.

Proposition 1. *Let F be a field and $\sigma : F \to F$ a field endomorphism. The left skew power series ring $F\big[[X,\sigma]\big]$ is a left principal valuation ring with every nonzero $r \in F\big[[X,\sigma]\big]$ expressible in the form $r = uX^m$ for some $m \geq 0$ and unit $u \in F\big[[X,\sigma]\big]$.*

Proof. If $0 \neq r \in F\big[[X,\sigma]\big]$ then r can be written in the form $r = \big(\sum_{n\geq 0} a_n X^n\big)X^m$ for some $m \geq 0$ with $a_0 \neq 0$. But $\sum_{n\geq 0} a_n X^n$ is a unit of $F\big[[X,\sigma]\big]$ since $a_0 \neq 0$. $\square$

If $\sigma : F \to F$ is a field automorphism we denote by $F\big[[X^{-1},X,\sigma]\big]$ the *left skew Laurent series ring over* F. Recall that the elements of $F\big[[X^{-1},X,\sigma]\big]$ are formal series of the form

$$\sum_{n\in\mathbb{Z}} a_n X^n$$

where $a_n \in F$ for all $n \in \mathbb{Z}$ and with well-ordered support (meaning, there exists $m \in \mathbb{Z}$ such that $a_i = 0$ for all $i < m$). Addition is natural and multiplication induced by (1). Note that in the Laurent series ring, (1) is equivalent to

$$X^{-1}a = \sigma^{-1}(a)X^{-1} \quad \text{for all } a \in F. \tag{2}$$

The ring $F\big[[X^{-1},X,\sigma]\big]$ is easily seen to be a skew field. Indeed, it is the (left and right) skew field of quotients for the left skew power series ring $F\big[[X,\sigma]\big]$.

We refer the reader to [3] for a source on skew power series and skew Laurent series rings.

Let $\{x_n \mid n \in \mathbb{Z}\}$ be a set of indeterminates indexed by $\mathbb{Z}$. Define

$$E = \mathbb{Q}(\{x_n \mid n \in \mathbb{Z}\}) \tag{3}$$

to be the field of rational functions in $\{x_n \mid n \in \mathbb{Z}\}$ over $\mathbb{Q}$. Define σ to be the field automorphism on E induced by the mapping

$$x_n \stackrel{\sigma}{\longmapsto} x_{n-1} \quad \text{for all } n \in \mathbb{Z}. \tag{4}$$

For each $n \in \mathbb{Z}$, define

$$F_n = \mathbb{Q}(\dots, x_{n-2}, x_{n-1}, x_n^2, x_{n+1}^2, \dots) = \mathbb{Q}(\{x_i \mid i < n\} \cup \{x_i^2 \mid i \geq n\}). \tag{5}$$

Observe that $\{F_n\}_{n\in\mathbb{Z}}$ constitutes a strictly ascending chain of subfields of E and $E = \bigcup_{n\in\mathbb{Z}} F_n$. Note also that

$$\sigma[F_n] = F_{n-1} \quad \text{for all } n \in \mathbb{Z}. \tag{6}$$

Lemma 2. *Let F_n ($n \in \mathbb{Z}$) be defined as in* (5). *Then the index of F_{n-1} in F_n is* 2, *i.e.*, $\dim_{F_{n-1}} F_n = 2$ *for all $n \in \mathbb{Z}$.*

Proof. Follows from the fact that for each $n \in \mathbb{Z}$, $F_n = F_{n-1}(x_{n-1})$ with $x_{n-1} \notin F_{n-1}$ and $x_{n-1}^2 \in F_{n-1}$. □

Since $\sigma[F_n] = F_{n-1} \subseteq F_n$ for all $n \in \mathbb{Z}$, the automorphism σ restricts to a field endomorphism (but not an automorphism) on each subfield F_n of E. We shall identify σ with its restriction to F_n.

Define

$$F = F_0 \tag{7}$$

and

$$S = F\big[[X, \sigma]\big]. \tag{8}$$

We have $F \subseteq S \subseteq E\big[[X, \sigma]\big] \subseteq E\big[[X^{-1}, X, \sigma]\big]$ as subrings. Define

$$D = E\big[[X^{-1}, X, \sigma]\big] / E\big[[X, \sigma]\big]. \tag{9}$$

We shall view D as an S-F-bimodule. For notational convenience we shall write the elements of D in the form

$$\sum_{n<0} b_n X^n$$

where $b_n \in E$ for all $n < 0$ and the sum has well-ordered (or equivalently, finite) support. When multiplying an element of this form with an element from the ring S, we shall identify non-negative powers of the indeterminate X with zero.

Proposition 3 below sheds light on the submodule structure of D.

Proposition 3. *Let E, σ, F_n ($n \in \mathbb{Z}$), F, S and D be as in* (3), (4), (5), (7), (8) *and* (9)*, respectively. Let $\{W_n\}_{n<0}$ be a family of additive subgroups of E and put*

$$C = \sum_{n<0} W_n X^n = \left\{ \sum_{n<0} b_n X^n \mid b_n \in W_n \text{ for all } n < 0 \right\} \subseteq D.$$

Then:

(i) *C is a left S-submodule of D if and only if each W_n is an F-subspace of E and $\sigma[W_{n-1}] \subseteq W_n$ for all $n < 0$.*

(ii) *C is a right F-submodule of D if and only if each W_n is an F_{-n}-subspace of E.*

Proof. (i) Suppose $C \leq {}_S D$. Since $F \subseteq S$ and $FC \subseteq SC \subseteq C$, it is clear that each W_n is an F-subspace of E. Moreover,

$$\begin{aligned} XC &= \sum_{n<0} X W_n X^n = \sum_{n<0} \sigma[W_n] X^{n+1} \\ &= \sum_{n<0} \sigma[W_{n-1}] X^n \subseteq C = \sum_{n<0} W_n X^n, \end{aligned}$$

so $\sigma[W_{n-1}] \subseteq W_n$ for all $n < 0$.

We now establish the converse. Certainly C is an additive subgroup of D. It remains to show that $SC \subseteq C$. Take $s = \sum_{n\geq 0} a_n X^n \in S$ where $a_n \in F$ for all $n \geq 0$ and $x = \sum_{n<0} b_n X^n \in C$ where $b_n \in W_n$ for all $n < 0$. Then $sx = \sum_{n<0} c_n X^n$ where $c_n = \sum_{i\geq 0} a_i \sigma^i(b_{n-i})$ for all $n < 0$. (Note that this sum is defined because the family $\{b_n\}_{n<0}$ has finite support.) For each $i \geq 0$, $b_{n-i} \in W_{n-i}$ and therefore, by hypothesis, $\sigma^i(b_{n-i}) \in \sigma^i[W_{n-i}] \subseteq W_{n-i+i} = W_n$. Since $a_i \in F$ and W_n is an F-subspace of E, $a_i \sigma^i(b_{n-i}) \in W_n$. It follows that $c_n \in W_n$ for all $n < 0$, so $sx \in C$. We conclude that $SC \subseteq C$, so C is a left S-submodule of D.

(ii) Note first that

$$CF = \left(\sum_{n<0} W_n X^n \right) F = \sum_{n<0} W_n \sigma^n[F] X^n = \sum_{n<0} W_n F_{-n} X^n.$$

Therefore

$$\begin{aligned} C \leq D_F &\Leftrightarrow CF \subseteq C \\ &\Leftrightarrow \sum_{n<0} W_n F_{-n} X^n \subseteq \sum_{n<0} W_n X^n \end{aligned}$$

$$\Leftrightarrow W_n F_{-n} \subseteq W_n \text{ for all } n < 0$$

$$\Leftrightarrow W_n \text{ is an } F_{-n}\text{-subspace of } E \text{ for all } n < 0. \qquad \square$$

Lemma 4. *Let F, S and D be as in* (7), (8) *and* (9), *respectively. If C is an S-F-bisubmodule of D and $x = \sum_{n<0} b_n X^n \in C$, then $b_n X^n \in C$ for all $n < 0$.*

Proof. We shall use the abbreviation $\operatorname{supp} x$ to denote the support of x which is $\{n \in \mathbb{Z} \mid b_n \neq 0\}$. If $x = 0$ there is clearly nothing to prove. Suppose $x \neq 0$, so that $\operatorname{supp} x \neq \emptyset$. Choose m to be the maximal element in $\operatorname{supp} x$. We claim that $b_m X^m \in C$. Define $\mathcal{S} = \{\sum_{n<0} c_n X^n \in C \mid c_m = b_m \text{ and } c_n = 0$ for $m < n < 0\}$. Clearly $x \in \mathcal{S}$, so $\mathcal{S} \neq \emptyset$. Let $y = \sum_{n<0} c_n X^n$ be an element in $\mathcal{S}$ for which $|\operatorname{supp} y|$ is minimal. To establish the above claim it suffices to show that $|\operatorname{supp} y| = 1$, for then $y = c_m X^m = b_m X^m \in C$. Suppose $|\operatorname{supp} y| > 1$ so that $c_l \neq 0$ for some $l < m$. Pick $a \in F \backslash \mathbb{Q}$ so that $\sigma^n(a) \neq a$ whenever $n \neq 0$. In particular, $\sigma^{m-l}(a) \neq a$. Define

$$y' = (\sigma^{m-l}(a) - a)^{-1}(y\sigma^{-l}(a) - ay).$$

Observe that $a, \sigma^{-l}(a), \sigma^{m-l}(a) \in F$ whence $(\sigma^{m-l}(a) - a)^{-1} \in F$ and so $y' \in C$. Write $y' = \sum_{n<0} d_n X^n$ so that $d_n = (\sigma^{m-l}(a) - a)^{-1} c_n (\sigma^{n-l}(a) - a)$ for all $n < 0$. Note that $\operatorname{supp} y' \subseteq \operatorname{supp} y$ and $d_m = c_m = b_m$, whence $y' \in \mathcal{S}$. Note, however, that $d_l = 0$, so $l \in \operatorname{supp} y \backslash \operatorname{supp} y'$ which contradicts the minimality of $|\operatorname{supp} y|$. We conclude that $|\operatorname{supp} y| = 1$ and $b_m X^m \in C$.

We now repeat the above argument with $x' = x - b_m X^m \in C$ in place of x and conclude that $b_{m-1} X^{m-1} \in C$. Continuing in this way we obtain $b_n X^n \in C$ for all $n < 0$. $\square$

We now characterize the S-F-bisubmodules of D.

Theorem 5. *Let E, σ, F_n ($n \in \mathbb{Z}$), F, S and D be as in* (3), (4), (5), (7), (8) *and* (9), *respectively. Then the following statements are equivalent for $C \subseteq D$:*

(i) *C is an S-F-bisubmodule of D;*

(ii) *there exists a family $\{W_n\}_{n<0}$ such that each W_n is an F_{-n}-subspace of E, $\sigma[W_{n-1}] \subseteq W_n$ for all $n < 0$ and $C = \sum_{n<0} W_n X^n$.*

Proof. (ii)⇒(i) is an immediate consequence of Proposition 3.

(i)⇒(ii) For each $n < 0$, let $\pi_n : D \to E$ denote the nth projection map defined by

$$\sum_{i<0} b_i X^i \xmapsto{\pi_n} b_n$$

for $\sum_{i<0} b_i X^i \in D$. Define $W_n = \pi_n[C]$ for all $n < 0$. Each W_n is clearly an additive subgroup of E and $C \subseteq \sum_{n<0} W_n X^n$. Take $x = \sum_{n<0} b_n X^n \in \sum_{n<0} W_n X^n$ with $b_n \in W_n$ for all $n < 0$. Inasmuch as $b_n \in W_n = \pi_n[C]$ we can choose, for each $n < 0$, $y_n \in C$ such that $\pi_n[y_n] = b_n$. Since $y_n = \sum_{i<0} \pi_i[y_n] X^i \in C$, it follows from Lemma 4 that $\pi_i[y_n] X^i \in C$ for all $i < 0$. In particular, $\pi_n[y_n] X^n = b_n X^n \in C$, whence $x = \sum_{n<0} b_n X^n \in C$. We conclude that $C = \sum_{n<0} W_n X^n$. It follows from Proposition 3 that each W_n is an F_{-n}-subspace of E and $\sigma[W_{n-1}] \subseteq W_n$ for all $n < 0$. □

Proposition 6. *Let E, σ, S and D be as in* (3), (4), (8) *and* (9), *respectively. Then:*

$$\operatorname{soc}_S^n D = \sum_{i=-n}^{-1} EX^i \quad \textit{for all } n \geq 1$$

$$\textit{and } D = \operatorname{soc}_S^\omega D = \bigcup_{n \geq 1} \operatorname{soc}_S^n D.$$

Proof. Since S is a local ring with unique maximal proper left ideal SX we have, for each $n \geq 1$, $\operatorname{soc}_S^n D = \{y \in D \mid (SX)^n y = 0\} = \{y \in D \mid SX^n y = 0\} = \{y \in D \mid X^n y = 0\}$. If $y = \sum_{i<0} b_i X^i \in D$ then $X^n y = \sum_{i<0} \sigma^n(b_i) X^{n+i} = \sum_{i<-n} \sigma^n(b_i) X^{n+i}$ because $X^{n+i} = 0$ whenever $n + i \geq 0$, i.e., $i \geq -n$. Hence

$$\begin{aligned} X^n y = 0 &\Leftrightarrow \sigma^n(b_i) = 0 \quad \text{for all } i < -n \\ &\Leftrightarrow b_i = 0 \quad \text{for all } i < -n \\ &\Leftrightarrow y = \sum_{i=-n}^{-1} b_i X^i \in \sum_{i=-n}^{-1} EX^i. \end{aligned}$$

This shows that $\operatorname{soc}_S^n D = \sum_{i=-n}^{-1} EX^i$ for all $n \geq 1$. Since the sum $y = \sum_{i<0} b_i X^i$ has finite support, y belongs to $\sum_{i=-n}^{-1} EX^i$ for a suitably large n, whence $D = \operatorname{soc}_S^\omega D$. □

Now take $W_n = F_{-n}$ for all $n < 0$ in Theorem 5(ii) and define

$$N = \sum_{n<0} F_{-n} X^n = \sum_{n<0} X^n F \leq {}_S D_F. \tag{10}$$

In the next result, which is the main theorem of §2, we assemble together the important properties of N.

Theorem 7. *Let F, S and N be defined as in* (7), (8) *and* (10), *respectively. Then:*

(i) *Every S-F-bisubmodule of N is a member of the strictly ascending chain*

$$0 \subset \operatorname{soc}_S N \subset \operatorname{soc}_S^2 N \subset \cdots \subset \operatorname{soc}_S^\omega N = \bigcup_{n\geq 1} \operatorname{soc}_S^n N = N.$$

Thus ${}_SN$ is a Loewy module with Loewy length ω.

(ii) *The nth Loewy invariant $d_n({}_SN) = 2^n$ for all $n \geq 1$.*

(iiii) *${}_SN$ is not artinian.*

Proof. (i) Note first that for each $n \geq 1$,

$$\begin{aligned}\operatorname{soc}_S^n N &= N \cap \operatorname{soc}_S^n D\\ &= \Big(\sum_{i<0} F_{-i}X^i\Big) \cap \Big(\sum_{i=-n}^{-1} EX^i\Big) \quad \text{[by Proposition 6]}\\ &= \sum_{i=-n}^{-1} F_{-i}X^i.\end{aligned}$$

Clearly then, $N = \sum_{n<0} F_{-n}X^n = \bigcup_{n\geq 1} \operatorname{soc}_S^n N = \operatorname{soc}_S^\omega N$.

Let $0 \neq L < {}_SN_F$. By Theorem 5 there exists a family $\{W_i\}_{i<0}$ such that each W_i is an F_{-i}-subspace of E, $\sigma[W_{i-1}] \subseteq W_i$ for all $i < 0$ and $L = \sum_{i<0} W_iX^i$. Since $L \subseteq \sum_{i<0} F_{-i}X^i$, $W_i \subseteq F_{-i}$ for all $i < 0$, so $W_i = 0$ or F_{-i} for all $i < 0$. Note that if $W_i = 0$ for some $i < 0$, then $\sigma[W_{i-1}] \subseteq W_i = 0$, whence $W_{i-1} = 0$. It follows that for some $n > 0$, $L = \sum_{i=-n}^{-1} F_{-i}X^i = \operatorname{soc}_S^n N$.

(ii) For each $n \geq 1$ put $B_n = \operatorname{soc}_S^n N/\operatorname{soc}_S^{n-1} N$. Since ${}_SB_n$ is semisimple, the left S-submodules of B_n coincide with the left F-subspaces of B_n. It therefore suffices to show that $\dim_F B_n = 2^n$ for all $n \geq 1$. As shown in the proof of (i) above, $\operatorname{soc}_S^n N = F_nX^{-n} + \operatorname{soc}_S^{n-1} N$, so $\dim_F B_n = \dim_F F_n$ for all $n \geq 1$.

A routine inductive argument using Lemma 2 shows that $\dim_F F_n = 2^n$ for all $n \geq 1$, so $\dim_F B_n = 2^n$, as required.

(iii) We use Proposition 3(i) to construct an infinite strictly descending chain of submodules of ${}_SN$. For each $n \geq 1$ define

$$L_n = \sum_{i<0} W(n, i)X^i$$

where for each $i < 0$ and $n \geq 1$,

$$W(n,i) = \begin{cases} F_1, & \text{if } n \geq -i \\ F_{1-n-i}, & \text{if } n < -i. \end{cases}$$

Observe that $W(n,i)$ may be viewed as the (n,i)th entry in the array

$$\begin{array}{ccccccc} . & . & . & F_4 & F_3 & F_2 & F_1 \\ . & . & . & F_3 & F_2 & F_1 & F_1 \\ . & . & . & F_2 & F_1 & F_1 & F_1 \\ & & & . & . & . & . \\ & & & . & . & . & . \\ & & & . & . & . & . \end{array}$$

The nth row of the array corresponds with L_n. Since $F = F_0 \subset F_1 \subset F_2 \subset \dots$, each entry in the array, that is each $W(n,i)$, is an F-subspace of E. Moreover, since $\sigma[F_i] = F_{i-1}$ for all $i < 0$, it follows that for each $n \geq 1$, $\sigma[W(n,i-1)] \subseteq W(n,i)$ for all $i < 0$. We conclude from Proposition 3(i) that $L_n \leq {}_S N$ for all $n \geq 1$.

Observe that each column of the array, when listed from top to bottom, constitutes a descending chain. Hence

$$L_1 \supset L_2 \supset \dots .$$

We conclude that ${}_S N$ is not artinian. $\square$

3. The ring construction

Our objective in this section is to construct an example of a ring R and left R-module M which is product closed, finitely generated and projective but which is not artinian. Define

$$R = \begin{pmatrix} S & {}_S N_F \\ 0 & F \end{pmatrix}$$

where S, F and ${}_S N_F$ satisfy conditions (I), (II) and (III).

Our initial objective shall be to describe completely all left topologizing filters on R. We first describe the left topologizing filters on S.

Proposition 8. *Let S be a left principal valuation ring with unique maximal proper ideal P. Let $\mathcal{S}$ be the left topologizing filter on S comprising all nonzero (or equivalently, essential) left ideals. Then every left topologizing filter on S is a member of the chain*

$$0 = \eta(S) \subset \eta(P) \subset \eta(P^2) \subset \cdots \subset \mathcal{S} \subset 1.$$

Proof. Inasmuch as S is a left chain ring, every $\mathcal{I} \in S$-fil is of the form $\mathcal{I} = \eta(I)$ or $\mathcal{I} = \{K \leq {}_SS \mid K \supset I\}$ for some ideal I of S [8, Lemma 6]. If $I = 0$ then $\mathcal{I} = \{K \leq {}_SS \mid K \supset I\} = \mathcal{S}$. If $0 \neq I \subset S$ then $I = P^n$ for some $n \in \mathbb{N}$ in which case $\mathcal{I} = \eta(I) = \eta(P^n)$ or $\mathcal{I} = \{K \leq {}_SS \mid K \supset P^n\} = \{K \leq {}_SS \mid K \supseteq P^{n-1}\} = \eta(P^{n-1})$. □

Lemma 9. *Let S, F and N be as in* (I), (II) *and* (III). *Let L be a submodule of ${}_SN$ and suppose that for some integer $n \geq 0$, $L \supseteq \mathrm{soc}^n_S N$ but $L \not\supseteq \mathrm{soc}^{n+1}_S N$. Then there exist nonzero $b_1, b_2, \ldots, b_m \in F$ such that $\bigcap_{i=1}^m Lb_i = \mathrm{soc}^n_S N$.*

Proof. Put $L' = L \cap \mathrm{soc}^{n+1}_S N = \mathrm{soc}^{n+1}_S L$. By hypothesis, $\mathrm{soc}^n_S N \subseteq L' \subset \mathrm{soc}^{n+1}_S N$. If $L' = \mathrm{soc}^n_S N$ then $\mathrm{soc}^n_S N = \mathrm{soc}^n_S L = \mathrm{soc}^{n+1}_S L$. Since ${}_SL$ is semiartinian, this implies $L = \mathrm{soc}^n_S L = \mathrm{soc}^n_S N$ and there is nothing further to prove.

Suppose then $\mathrm{soc}^n_S N \subset L' \subset \mathrm{soc}^{n+1}_S N$. By Condition (III), no S-F-bisubmodule of N lies strictly between $\mathrm{soc}^n_S N$ and $\mathrm{soc}^{n+1}_S N$. Thus $L'c_1 \nsubseteq L'$ for some $c_1 \in F$, whence $L' \nsubseteq L'c_1^{-1}$ and so

$$L' \supset L' \cap L'c_1^{-1}.$$

Suppose $L' \cap L'c_1^{-1} \supset \mathrm{soc}^n_S N$. Again, since no S-F-bisubmodule of N lies strictly between $\mathrm{soc}^n_S N$ and $\mathrm{soc}^{n+1}_S N$, we must have $(L' \cap L'c_1^{-1})c_2 \nsubseteq L'$ for some $c_2 \in F$, whence $L' \cap L'c_1^{-1} \nsubseteq L'c_2^{-1}$ and so

$$L' \cap L'c_1^{-1} \supset L' \cap L'c_1^{-1} \cap L'c_2^{-1}.$$

Continuing in this manner we obtain, for suitable $c_1, c_2, \ldots, c_k \in F$, the descending chain

$$\mathrm{soc}^{n+1}_S N \supset L' \supset L' \cap L'c_1^{-1} \supset \cdots \supset L' \cap L'c_1^{-1} \cap \cdots \cap L'c_k^{-1} \supseteq \mathrm{soc}^n_S N.$$

But since $\mathrm{soc}^{n+1}_S N$ has, by hypothesis, finite length the above chain must terminate, so for a suitably large $k \in \mathbb{N}$, we must have

$$L' \cap L'c_1^{-1} \cap \cdots \cap L'c_k^{-1} = \mathrm{soc}^n_S N.$$

Put $m = k+1$, $b_1 = 1$ and $b_i = c_{i-1}^{-1}$ for $2 \leq i \leq k+1$, so that

$$\bigcap_{i=1}^{m} L'b_i = \mathrm{soc}_S^n N.$$

For each $i \in \{1, 2, \ldots, m\}$,

$$\begin{aligned} L'b_i &= (L \cap \mathrm{soc}_S^{n+1} N) b_i = Lb_i \cap (\mathrm{soc}_S^{n+1} N) b_i \\ &= Lb_i \cap \mathrm{soc}_S^{n+1} N \text{ [because } \mathrm{soc}_S^{n+1} N \text{ is an } S\text{-}F\text{-bisubmodule of } N]. \end{aligned}$$

Hence

$$\mathrm{soc}_S^n N = \bigcap_{i=1}^{m} L'b_i = \bigcap_{i=1}^{m} [Lb_i \cap \mathrm{soc}_S^{n+1} N] = \left(\bigcap_{i=1}^{m} Lb_i\right) \cap \mathrm{soc}_S^{n+1} N.$$

Since $\bigcap_{i=1}^{m} Lb_i$ is semiartinian, the above implies that $\bigcap_{i=1}^{m} Lb_i = \mathrm{soc}_S^n N$. $\square$

Lemma 10. *Let* $R = \left(\begin{smallmatrix} S & N \\ 0 & F \end{smallmatrix}\right)$ *with* S, F *and* N *as in* (I), (II) *and* (III).

(i) *Let* $\{x_1, x_2, \ldots, x_m\} \subseteq N$. *Then for each* n *satisfying* $0 \leq n \leq \omega$,

$$\begin{aligned} &\bigcap_{i=1}^{m} \left[\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}\right] \\ &\qquad = \begin{pmatrix} (\mathrm{soc}_S^n N : X) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \end{aligned}$$

where $X = \{x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_m\}$.

(ii) *If* $r = \left(\begin{smallmatrix} 0 & x \\ 0 & c \end{smallmatrix}\right) \in R$, *then*

$$\left(\begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} : r\right) = \begin{cases} \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -xc^{-1} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}, & \text{if } c \neq 0 \\ \begin{pmatrix} (\mathrm{soc}_S^n N : x) & N \\ 0 & F \end{pmatrix}, & \text{if } c = 0. \end{cases}$$

Proof. (i) Suppose $0 \leq n \leq \omega$. Then

$$\begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix} \in \bigcap_{i=1}^{m} \left[\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}\right]$$

$\Leftrightarrow \quad \exists w_1, w_2, \ldots, w_m \in \mathrm{soc}_S^n N$ such that $x = ax_i + w_i \ \forall i \in \{1, 2, \ldots, m\}$

$\Leftrightarrow \quad \exists w_1, w_2, \ldots, w_m \in \mathrm{soc}_S^n N$ such that $x = ax_1 + w_1$ and $ax_1 + w_1 = ax_i + w_i \ \forall i \in \{1, 2, \ldots, m\}$

$\Leftrightarrow \quad \exists w_1, w_2, \ldots, w_m \in \mathrm{soc}_S^n N$ such that $x = ax_1 + w_1$ and $a(x_1 - x_i) = w_i - w_1 \ \forall i \in \{1, 2, \ldots, m\}$

$\Leftrightarrow \quad x \in ax_1 + \mathrm{soc}_S^n N$ and $a(x_1 - x_i) \in \mathrm{soc}_S^n N \ \forall i \in \{1, 2, \ldots, m\}$

$\Leftrightarrow \quad x \in ax_1 + \mathrm{soc}_S^n N$ and $a \in (\mathrm{soc}_S^n N : X)$ where $X = \{x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_m\}$

$$\Leftrightarrow \quad \begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} (\mathrm{soc}_S^n N : X) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}.$$

(ii) Note first that

$$\Leftrightarrow \quad \begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \in \left(\begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} : r \right) = \left(\begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} 0 & x \\ 0 & c \end{pmatrix} \right)$$

$$\Leftrightarrow \quad \begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & ax + yc \\ 0 & bc \end{pmatrix} \in \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \quad \left. \begin{aligned} ax + yc &\in \mathrm{soc}_S^n N \\ \text{and} \quad bc &= 0 \end{aligned} \right\} \tag{11}$$

If $c \neq 0$, then (11) is equivalent to

$$\left. \begin{aligned} b &= 0 \\ \text{and} \quad axc^{-1} + y &\in (\mathrm{soc}_S^n N)c^{-1} \end{aligned} \right\} \tag{12}$$

But $(\mathrm{soc}_S^n N)c^{-1} = \mathrm{soc}_S^n N$ because $\mathrm{soc}_S^n N$ is an S-F-bisubmodule of N, so (12) yields $axc^{-1} + y \in \mathrm{soc}_S^n N$, i.e., $y \in -axc^{-1} + \mathrm{soc}_S^n N$. Thus

$$\left(\begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} : r \right) = \left\{ \begin{pmatrix} a & -axc^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \;\middle|\; a \in S \right\}$$

$$= \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -xc^{-1} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}.$$

If $c = 0$, then (11) reduces to $ax \in \mathrm{soc}_S^n N$, i.e., $a \in (\mathrm{soc}_S^n N : x)$, in which case $\left(\left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) : r \right) = \left(\begin{smallmatrix} (\mathrm{soc}_S^n N : x) & N \\ 0 & F \end{smallmatrix}\right)$. □

For notational convenience we introduce the following: if $\mathcal{F}$ is a family of left ideals of S then

$$\begin{pmatrix} \mathcal{F} & 0 \\ 0 & 0 \end{pmatrix} := \left\{ \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, K \in \mathcal{F} \right\}.$$

If $L \leq {}_S N$ then $\left(\begin{smallmatrix} \mathcal{F} & L \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathcal{F} & L \\ 0 & F \end{smallmatrix}\right)$ will have the obvious meanings.

Proposition 11. *Let* $R = \left(\begin{smallmatrix} S & N \\ 0 & F \end{smallmatrix}\right)$ *with* S, F *and* N *as in* (I), (II) *and* (III). *Let* $\mathcal{S}$ *denote the left topologizing filter on* S *comprising all nonzero left ideals.*

(i) $\eta \left(\begin{smallmatrix} \mathcal{S} & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) = \eta \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$ *for all finite* $n \geq 0$.

(ii) $\eta \left(\begin{smallmatrix} \mathcal{S} & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) = \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \cap \eta \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$ *for all* n *satisfying* $0 \leq n \leq \omega$.

Proof. It is clear that $\eta \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \subseteq \eta \left(\begin{smallmatrix} \mathcal{S} & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \subseteq \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \cap \eta \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$ for all n satisfying $0 \leq n \leq \omega$. It thus remains to establish the reverse containments.

(i) Suppose n is finite. It suffices to demonstrate that $\left(\begin{smallmatrix} \mathcal{S} & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \subseteq \eta \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$. Let $K \in \mathcal{S}$. If $K = S$ then obviously $\left(\begin{smallmatrix} K & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \in \eta \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$. Suppose then $K \neq S$. Since S is a left principal valuation ring with unique maximal proper ideal P, $K = P^m$ for some $m \in \mathbb{N}$. Choose $x \in \mathrm{soc}_S^{n+m} N \backslash \mathrm{soc}_S^{n+m-1} N$. Observe that $(\mathrm{soc}_S^n N : x) = P^m = K$. Put $x_1 = 0$, $x_2 = -x$ and $X = \{x_1 - x_2\} = \{x\}$. By Lemma 10(i),

$$\begin{aligned} \bigcap_{i=1}^{2} \left[\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \right] &= \begin{pmatrix} (\mathrm{soc}_S^n N : X) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (\mathrm{soc}_S^n N : X) & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} K & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Inasmuch as each $\left(\begin{smallmatrix} 1 & x_i \\ 0 & 1 \end{smallmatrix}\right)$ is a unit of R and $\left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$ is an ideal of R, $\left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_i \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$, so $\left(\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_i \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_i \\ 0 & 1 \end{smallmatrix}\right)$ for $i \in \{1, 2\}$. Since each $\left(\begin{smallmatrix} 1 & x_i \\ 0 & 1 \end{smallmatrix}\right)$ is a unit of R, we must have that $\left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_i \\ 0 & 1 \end{smallmatrix}\right) \in \eta \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$ for $i \in \{1, 2\}$. Hence $\left(\begin{smallmatrix} K & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) = \bigcap_{i=1}^{2} \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x_i \\ 0 & 1 \end{smallmatrix}\right) \in \eta \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$, as required.

(ii) It suffices to show that $\eta \left(\begin{smallmatrix} \mathcal{S} & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \supseteq \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \cap \eta \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$ for all n satisfying $0 \leq n \leq \omega$. Let $I \in \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \cap \eta \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$. Taking $n = 0$ in (i), we see that

$\eta\left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0\end{smallmatrix}\right) = \eta\left(\begin{smallmatrix} S & 0 \\ 0 & 0\end{smallmatrix}\right)$, so there exist $r_1, r_2, \ldots, r_m \in R$ such that $I \supseteq \bigcap_{i=1}^m \left(\left(\begin{smallmatrix} S & 0 \\ 0 & 0\end{smallmatrix}\right) : r_i\right)$. Note, however, that for each $i \in \{1, 2, \ldots, m\}$, $\left(\left(\begin{smallmatrix} S & 0 \\ 0 & 0\end{smallmatrix}\right) : r_i\right) = (R(1-e) : r_i) = (0 : r_i e)$, where $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right)$. Write $r_i e = \left(\begin{smallmatrix} 0 & x_i \\ 0 & c_i\end{smallmatrix}\right)$ for each $i \in \{1, 2, \ldots, m\}$. No generality is lost if we assume $c_i \neq 0$ for $1 \leq i \leq l$ and $c_i = 0$ for $l+1 \leq i \leq m$. Taking $n = 0$ in Lemma 10(ii), we obtain

$$(0 : r_i e) = \begin{cases} \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_i c_i^{-1} \\ 0 & 1 \end{pmatrix}, & \text{if } 1 \leq i \leq l \\ \begin{pmatrix} (0 : x_i) & N \\ 0 & F \end{pmatrix}, & \text{if } l+1 \leq i \leq m. \end{cases}$$

Taking $X = \{-x_1 c_1^{-1} + x_i c_i^{-1} \mid 2 \leq i \leq l\}$ and $Y = \{x_i \mid l+1 \leq i \leq m\}$, we obtain

$$\begin{aligned} \bigcap_{i=1}^m \left(\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} : r_i\right) &= \bigcap_{i=1}^m (0 : r_i e) \\ &= \left[\bigcap_{i=1}^l \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_i c_i^{-1} \\ 0 & 1 \end{pmatrix}\right] \cap \left[\bigcap_{i=l+1}^m \begin{pmatrix} (0 : x_i) & N \\ 0 & F \end{pmatrix}\right] \\ &= \begin{pmatrix} (0 : X) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_1 c_1^{-1} \\ 0 & 1 \end{pmatrix} \cap \begin{pmatrix} (0 : Y) & N \\ 0 & F \end{pmatrix} \text{ [by Lemma 10(i)]} \\ &= \begin{pmatrix} (0 : X \cup Y) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_1 c_1^{-1} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Since $X \cup Y$ is a finite subset of N, $(0 : X \cup Y)$ is a nonzero left ideal of S, i.e., $(0 : X \cup Y) \in \mathcal{S}$. Moreover, $I \supseteq \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0\end{smallmatrix}\right)$ because $I \in \eta\left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0\end{smallmatrix}\right)$, so

$$\begin{aligned} I &\supseteq \begin{pmatrix} (0 : X \cup Y) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_1 c_1^{-1} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (0 : X \cup Y) & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_1 c_1^{-1} \\ 0 & 1 \end{pmatrix} \\ &\qquad \left[\text{because } \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_1 c_1^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}\right] \end{aligned}$$

$$\in \eta \begin{pmatrix} \mathcal{S} & \operatorname{soc}_S^n N \\ 0 & 0 \end{pmatrix} \left[\text{because } \begin{pmatrix} 1 & -x_1 c_1^{-1} \\ 0 & 1 \end{pmatrix} \text{ is a unit of } R\right].$$

We conclude that $\eta \left(\begin{smallmatrix} \mathcal{S} & \operatorname{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \supseteq \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \cap \eta \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$, as required. □

We are now in a position to prove the main theorem of §3.

Theorem 12. *Let* $R = \left(\begin{smallmatrix} S & N \\ 0 & F \end{smallmatrix}\right)$ *with* S, F *and* N *as in* (I), (II) *and* (III). *Let* $\mathcal{S}$ *denote the left topologizing filter on* S *comprising all nonzero left ideals. The lattice diagram shown in Figure 1 is a complete description of all left topologizing filters on* R.

PROOF. Let $\mathcal{G} \in R$-fil. We divide our argument into two cases.

Case 1 $\left[\boldsymbol{\mathcal{G} \not\subseteq \eta \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)}\right]$.

If $\mathcal{G} \subseteq \eta \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$ for all finite $n \geq 0$, then $\mathcal{G} \subseteq \bigcap_{n<\omega} \eta \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) = \eta \left(\begin{smallmatrix} 0 & \bigcup_{n<\omega} \operatorname{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) = \eta \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)$ which contradicts our hypothesis. Thus $\mathcal{G} \not\subseteq \eta \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^{n+1} N \\ 0 & 0 \end{smallmatrix}\right)$ for some finite $n \geq 0$. Take n minimal with this property and choose $K \in \mathcal{G}$ such that $K \not\supseteq \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^{n+1} N \\ 0 & 0 \end{smallmatrix}\right)$. It follows from the minimality of n that $K \supseteq \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$. Note that K cannot contain an element of the form $\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right)$ with $b \neq 0$, for otherwise $K \supseteq R \left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \supseteq \left(\begin{smallmatrix} 0 & Nb \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right) \supseteq \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^{n+1} N \\ 0 & 0 \end{smallmatrix}\right)$, a contradiction. Thus $K \subseteq \left(\begin{smallmatrix} S & N \\ 0 & 0 \end{smallmatrix}\right)$. Put $K' = K \cap \left(\begin{smallmatrix} 0 & N \\ 0 & F \end{smallmatrix}\right)$. Then $K' = \left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix}\right)$ for some $L \leq {}_S N$. Note that $L \supseteq \operatorname{soc}_S^n N$ and $L \not\supseteq \operatorname{soc}_S^{n+1} N$. By Lemma 9 there exist nonzero $b_1, b_2, \ldots, b_m \in F$ such that $\bigcap_{i=1}^m Lb_i = \operatorname{soc}_S^n N$. Putting $u_i = \left(\begin{smallmatrix} 1 & 0 \\ 0 & b_i \end{smallmatrix}\right)$ for each $i \in \{1, 2, \ldots, m\}$, we obtain $\bigcap_{i=1}^m K' u_i = \bigcap_{i=1}^m \left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & b_i \end{smallmatrix}\right) = \bigcap_{i=1}^m \left(\begin{smallmatrix} 0 & Lb_i \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$. We also have that

$$\begin{aligned} \bigcap_{i=1}^m K' u_i &= \bigcap_{i=1}^m \left[K \cap \begin{pmatrix} 0 & N \\ 0 & F \end{pmatrix}\right] u_i \\ &= \bigcap_{i=1}^m \left[K u_i \cap \begin{pmatrix} 0 & N \\ 0 & F \end{pmatrix} u_i\right] \\ &= \left(\bigcap_{i=1}^m K u_i\right) \cap \begin{pmatrix} 0 & N \\ 0 & F \end{pmatrix} \quad [\text{because each } u_i \text{ is a unit of } R]. \end{aligned}$$

Hence $(\bigcap_{i=1}^m K u_i) \cap \left(\begin{smallmatrix} 0 & N \\ 0 & F \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$. Put $Q = \bigcap_{i=1}^m K u_i \in \mathcal{G}$ and $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ so that $\mathrm{Re} = \left(\begin{smallmatrix} 0 & N \\ 0 & F \end{smallmatrix}\right)$. It follows from the above that $\mathrm{Re} \big/ \left(\begin{smallmatrix} 0 & \operatorname{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \lesssim R/Q$. But $R =$

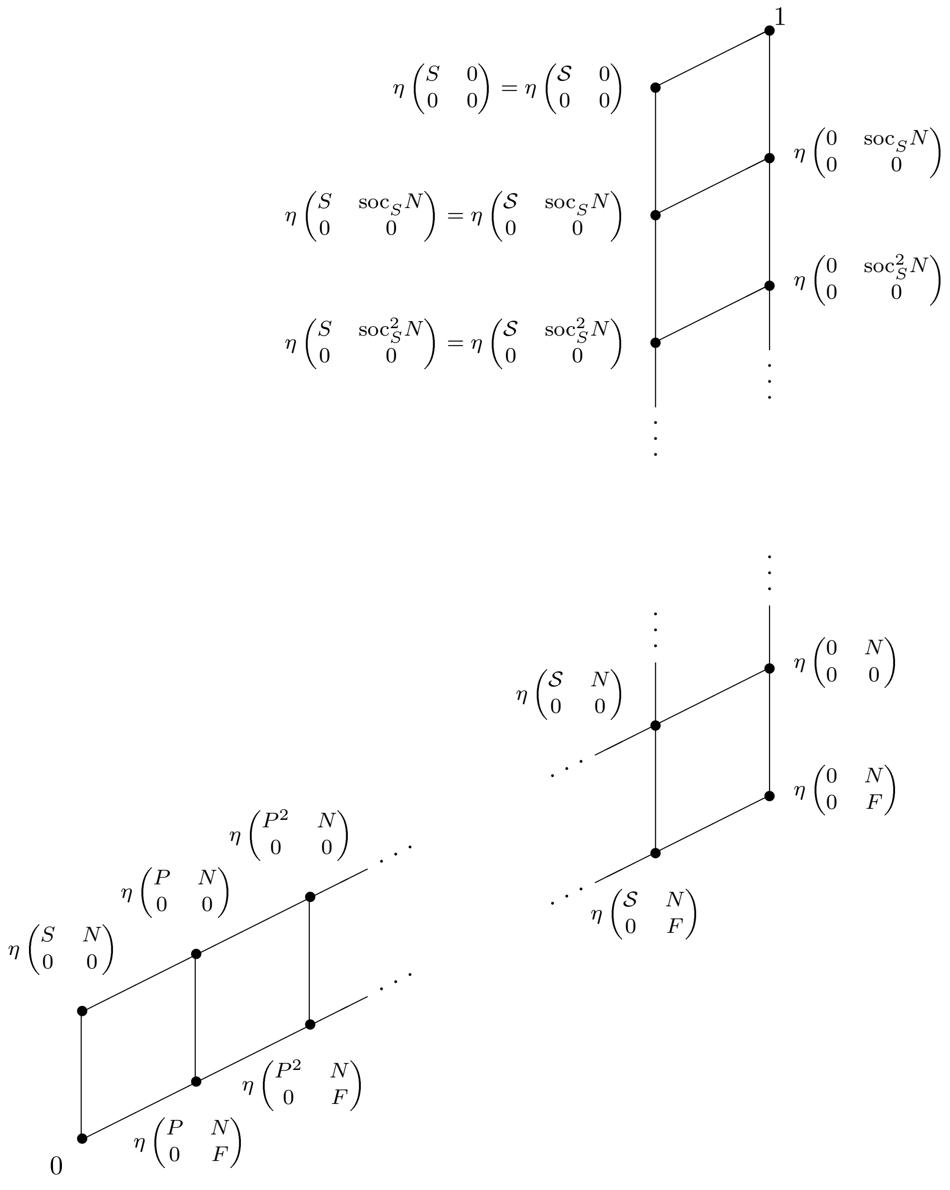

Figure 1. Lattice diagram of all left topologizing filters on R

$\mathrm{Re} \oplus R(1-e)$, so $\mathrm{Re} / \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \cong R/\left[R(1-e) \oplus \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)\right] = R/\left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$. Hence $\eta \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \subseteq \eta(Q) \subseteq \mathcal{G}$. We thus have $\eta \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) \subseteq \mathcal{G} \subseteq \eta \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$. By Proposition 11((i) and (ii)), $\eta \left(\begin{smallmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) = \eta \left(\begin{smallmatrix} \mathcal{S} & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right) = \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \cap \eta \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)$. Thus $\mathcal{G}$ belongs to the interval $\left[\eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \cap \eta \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right), \eta \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)\right]$ in R-fil. Since R-fil is a modular lattice (a proof of this fact may be found in [6, Proposition II.1.6, p. 68]), the above interval is order isomorphic to the interval $\left[\eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right), \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \vee \eta \left(\begin{smallmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{smallmatrix}\right)\right]$ in R-fil. But since $\eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a coatom of R-fil (this is easily seen to be the case for $\mathcal{S}$ comprises all nonzero left ideals of S), the above intervals must each contain precisely two members. We conclude that

$$\mathcal{G} = \eta \begin{pmatrix} \mathcal{S} & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \eta \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}.$$

We have thus shown that if $\mathcal{G} \not\subseteq \eta \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)$ then $\mathcal{G}$ is a member of the lattice diagram shown in Figure 2.

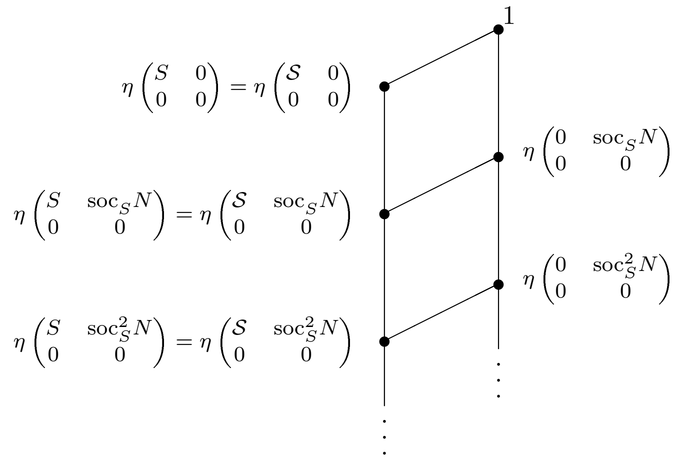

Figure 2. Lattice diagram of all left topologizing filters on R not contained in $\eta \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)$

Case 2 $\left[\boldsymbol{\mathcal{G} \subseteq \eta \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)}\right]$.

Consider the canonical ring epimorphism $\pi : R \to S \times F$ defined by $\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \overset{\pi}{\mapsto} (a, b)$. Note that $\mathrm{Ke}\,\pi = \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)$. The epimorphism π induces a bijection (in fact, a lattice isomorphism) from $(S \times F)$-fil onto the interval $\left[0, \eta \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)\right]$ of R-fil defined by

$$\mathcal{F} \mapsto \{\pi^{-1}[K] \mid K \in \mathcal{F}\} \quad \text{for } \mathcal{F} \in (S \times F)\text{-fil}.$$

Furthermore, every $\mathcal{F} \in (S \times F)$-fil is of the form $\{K_1 \times K_2 \mid K_1 \in \mathcal{I}, K_2 \in \mathcal{H}\}$ for some $\mathcal{I} \in S$-fil and $\mathcal{H} \in F$-fil. Observe that if $K_1 \in \mathcal{I}$ and $K_2 \in \mathcal{H}$, then $\pi^{-1}[K_1 \times K_2] = \begin{pmatrix} K_1 & N \\ 0 & K_2 \end{pmatrix}$. It follows that $\mathcal{G} = \left(\begin{smallmatrix} \mathcal{I} & N \\ 0 & \mathcal{H} \end{smallmatrix}\right)$ for some $\mathcal{I} \in S$-fil and $\mathcal{H} \in F$-fil. By Proposition 8, $\mathcal{I}$ is a member of the chain

$$0 = \eta(S) \subset \eta(P) \subset \eta(P^2) \subset \cdots \subset \mathcal{S} \subset 1.$$

Since F is a field, $\mathcal{H} = 0$ or 1. We have thus shown that if $\mathcal{G} \subseteq \eta \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)$ then $\mathcal{G}$ is a member of the lattice diagram shown in Figure 3.

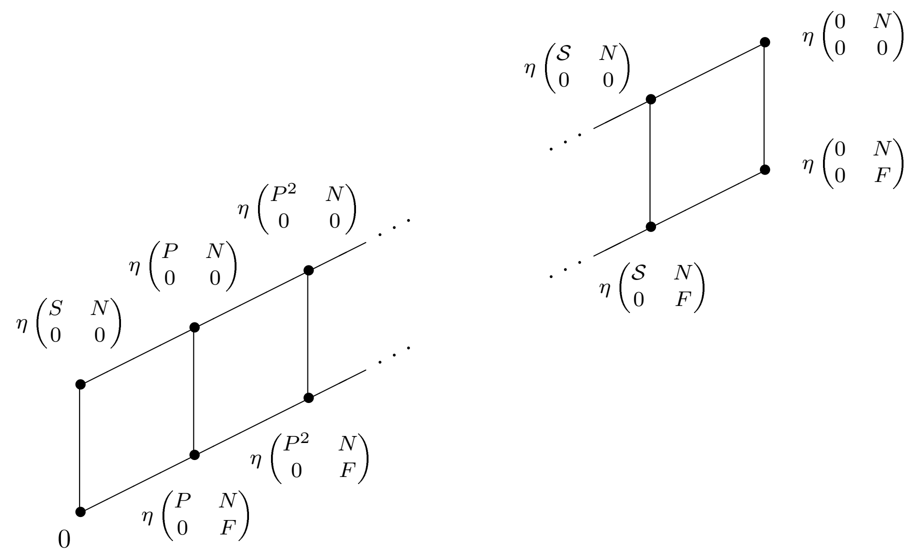

Figure 3. Lattice diagram of all left topologizing filters on R contained in $\eta \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)$

$\square$

We are finally in a position to exhibit a left R-module M that has the properties referred to in the first paragraph of this section.

Theorem 13. *Let* $R = \left(\begin{smallmatrix} S & N \\ 0 & F \end{smallmatrix}\right)$ *with* S, F *and* N *as in* (I), (II) *and* (III). *Let* $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in R$ *and put* $M = \mathrm{Re} = \left(\begin{smallmatrix} 0 & N \\ 0 & F \end{smallmatrix}\right)$. *Then* M *is a left* R*-module with the following properties:*

(i) *M is finitely generated and projective;*

(ii) *M is not artinian;*

(iii) *M is product closed.*

Proof. (i) Since M is a direct summand of ${}_RR$ it is finitely generated and projective.

(ii) By hypothesis, ${}_SN$ is not artinian. If $L_1 \supset L_2 \supset \dots$ is a strictly descending chain of submodules of ${}_SN$ then $\left(\begin{smallmatrix} 0 & L_1 \\ 0 & 0 \end{smallmatrix}\right) \supset \left(\begin{smallmatrix} 0 & L_2 \\ 0 & 0 \end{smallmatrix}\right) \supset \dots$ is a strictly descending chain of submodules of M.

(iii) Note that $\sigma[M] = \sigma[\mathrm{Re}] = \sigma[R/R(1-e)]$. If $\mathcal{F}$ denotes the left topologizing filter on R associated with the hereditary pretorsion class $\sigma[M]$, then $\mathcal{F} = \eta(R(1-e)) = \eta \left(\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right) = \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. It is easily seen from Figure 1 that the interval $\left[0, \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]$ of R-fil has the lattice diagram depicted in Figure 4.

Let $\mathcal{G} \in \left[0, \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]$. We use [7, Theorem 1(4)] to show that M is product closed. To this end let $\mathcal{G}' \subseteq \mathcal{G}$ and suppose $\bigcap \mathcal{G}' \in \mathcal{F}$. We need to show that $\bigcap \mathcal{G}' \in \mathcal{G}$. If $\mathcal{G} = \eta(I)$ for some ideal I of R, then $\mathcal{G}$ is Jansian and thus closed under arbitrary intersections of left ideals. In this instance there is nothing to prove. Suppose then that $\mathcal{G}$ is not Jansian. Inspection of Figure 4 reveals that the non-Jansian members of $\left[0, \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]$ are precisely those belonging to the interval $\left[\eta \left(\begin{smallmatrix} \mathcal{S} & N \\ 0 & F \end{smallmatrix}\right), \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]$. Suppose $\mathcal{G} = \eta \begin{pmatrix} \mathcal{S} & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$ for some n satisfying $0 \le n \le \omega$. By Proposition 11(ii), $\mathcal{G} = \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \cap \eta \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$. By hypothesis, $\bigcap \mathcal{G}' \in \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since $\eta \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$ is Jansian, $\bigcap \mathcal{G}' \in \eta \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$. We conclude that $\bigcap \mathcal{G}' \in \mathcal{G}$.

Noting that $\eta \left(\begin{smallmatrix} \mathcal{S} & N \\ 0 & F \end{smallmatrix}\right) = \eta \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right) \cap \eta \left(\begin{smallmatrix} 0 & N \\ 0 & F \end{smallmatrix}\right)$, a similar argument shows that $\bigcap \mathcal{G}' \in \mathcal{G}$ in the case where $\mathcal{G} = \eta \left(\begin{smallmatrix} \mathcal{S} & N \\ 0 & F \end{smallmatrix}\right)$. $\square$

Remark 14. (i) The module M of Theorem 13 is local and semiartinian and is such that $\mathrm{End}_R M \cong F$. The first of these assertions follows because every element of M not contained in $\left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right)$, generates M. The second is a consequence of the fact that

$$0 \subset \begin{pmatrix} 0 & \mathrm{soc}_S N \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & \mathrm{soc}_S^2 N \\ 0 & 0 \end{pmatrix} \subset \cdots \subset \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \subset M$$

is an ascending Loewy series for M (of length $\omega + 1$), whilst the third follows because $\mathrm{End}_R M = \mathrm{End}_R \mathrm{Re} \cong eRe \cong F$.

(ii) The module M of Theorem 13 is finitely generated, product closed and certainly projective in $\sigma[M]$, but does not have finite length. This shows that the requirement in [7, Theorem 16] that every $\mathcal{T} \in M$-torsp is M-dominated, cannot be dispensed with.

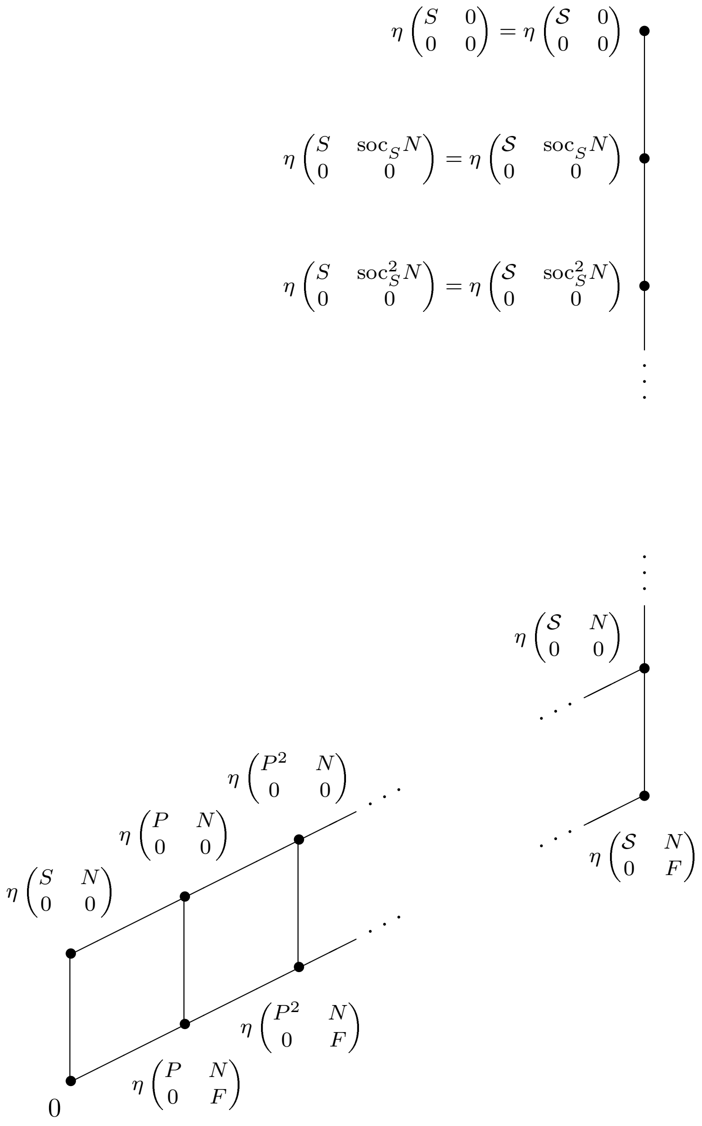

Figure 4. Lattice diagram of all left topologizing filters on R contained in $\eta\left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{smallmatrix}\right)$

In the next result we identify the M-dominated members of M-torsp.

Proposition 15. *Let* $R = \begin{pmatrix} S & N \\ 0 & F \end{pmatrix}$ *with* S, F *and* N *as in* (I), (II) *and* (III). *Let* $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R$ *and put* $M = \mathrm{Re} = \begin{pmatrix} 0 & N \\ 0 & F \end{pmatrix}$. *The following assertions are equivalent for a nontrivial hereditary pretorsion class* $\mathcal{T}$ *in* $\sigma[M]$*:*

(i) $\mathcal{T}$ *is* M*-dominated;*

(ii) *if* $\mathcal{F}$ *is the left topologizing filter on* R *associated with* $\mathcal{T}$, *then* $\mathcal{F} = \eta \begin{pmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$ *for some* n *satisfying* $0 \leq n \leq \omega$.

Proof. (ii)$\Rightarrow$(i) Suppose $\mathcal{F} = \eta \begin{pmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$ with n satisfying $0 \leq n \leq \omega$. Then $\mathcal{T} = \sigma \left[R \Big/ \begin{pmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \right]$. In the proof of Theorem 12 (Case 1), it was noted that $R \Big/ \begin{pmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} \cong \mathrm{Re} \Big/ \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix} = M \Big/ \begin{pmatrix} 0 & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$. It follows that $R \Big/ \begin{pmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$ is M-generated, whence $\mathcal{T}$ is M-dominated.

(i)$\Rightarrow$(ii) By [7, Corollary 13], $\mathcal{T} = \sigma[M/M^{\mathcal{T}}]$. It is clear from Figure 2 that if $\mathcal{F} \nsubseteq \eta \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ then $\mathcal{F} = \eta \begin{pmatrix} S & \mathrm{soc}_S^n N \\ 0 & 0 \end{pmatrix}$ for some finite $n \geq 0$. In this case there is nothing further to prove. Suppose then, $\mathcal{F} \subseteq \eta \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$. It follows that $\mathcal{T} \subseteq \left\{ L \in R\text{-Mod} \mid \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} L = 0 \right\}$. In particular, $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & N \\ 0 & F \end{pmatrix} = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \subseteq M^{\mathcal{T}}$. Since $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ is a maximal proper submodule of M and $\mathcal{T}$ is nontrivial, we must have $M^{\mathcal{T}} = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ in which case $\mathcal{T} = \sigma \left[M / \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \right]$. But $M / \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \cong R / \begin{pmatrix} S & N \\ 0 & 0 \end{pmatrix}$. Thus $\mathcal{T} = \sigma \left[R / \begin{pmatrix} S & N \\ 0 & 0 \end{pmatrix} \right]$, whence $\mathcal{F} = \eta \begin{pmatrix} S & N \\ 0 & 0 \end{pmatrix}$. $\square$

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