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Differentiable loops on the real line

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Dedicated to Professor Adalbert Bovdi on his 70th birthday

Abstract. The paper is devoted to the study of differentiable loops L on the real line such that the group G topologically generated by the left translations is locally compact and hence it is isomorphic to the universal covering group of $PSL_2(\mathbb{R})$. Using the methods developed in [3] we introduce a class of natural parametrizations of the loop manifold L corresponding to the Iwasava decompositions of G and find explicit expressions for the loop multiplication with respect to the given parametrizations. We characterize the differentiable curves $\mathbb{R} \to G$ consisting of the left translations of a loop L in the biinvariant Lorentzian geometry of G.

1. Introduction

An algebra $(L, \circ, e)$ with a binary operation $(x, y) \mapsto x \circ y : L \times L \to L$ and unit element $e \in L$ is called a *left loop* if for any given $a, b \in L$ the equation $a \circ x = b$ is uniquely solvable for x, and if $e \in L$ satisfies $e \circ x = x \circ e = x$ for any $x \in L$. The left loop $(L, \circ, e)$ is called a *loop* if for any given $a, b \in L$ the equation $x \circ a = b$ is uniquely solvable, too. The operation $\circ : L \times L \to L$ is called *multiplication*, the solution of the equation $a \circ x = b$ is denoted by $a \backslash b$, the solution of the equation $x \circ a = b$ is denoted by a/b. The binary *left division* operation $(a, b) \mapsto a \backslash b$ can be defined in a left loop, the binary *right division* operation $(a, b) \mapsto a/b$ can be defined in a loop. The bijective maps $\lambda_x : L \to L : y \mapsto x \circ y$, $x \in L$, are called *left translations* of $(L, \circ, e)$. The group G generated by the

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set $\Lambda = \{\lambda_x, x \in L\}$ of left translations acts transitively on L. Hence there is a bijective correspondence $L \to G/H : x \mapsto \lambda_x H$, where H is the stabilizer of $e \in L$ in the group G.

A loop $(L, \circ, e)$ is called *differentiable* of class C^r if L is a differentiable manifold of class C^r and the multiplication $\circ$, the left and the right divisions of $(L, \circ)$ are C^r-differentiable maps $L \times L \to L$. In the study of the Lie theory of differentiable loops is very fruitful the investigation of the differentiable submanifold $\Lambda = \{\lambda_x, x \in L\} \subset G$ of the group topologically generated by the left translations (cf. [3]), the mapping $x \mapsto \lambda_x : L \to G$ corresponds to a section with respect to the projection map $G \to G/H$.

Differentiable loops defined on the unit circle having a locally compact group G as the group topologically generated by the left translations, were studied in Section 18 of the monograph [3] by P. T. Nagy and Karl Strambach. Particularly they proved the following assertion (Proposition 18.2, p. 235.) by an application of a theorem of Brouwer [1]:

Let L be a topological proper loop on a connected 1-manifold, such that the group G topologically generated by the left translations is locally compact. Then G is a finite covering of the group $PSL_2(\mathbb{R})$ if L is a circle, and G is the universal covering of $PSL_2(\mathbb{R})$ if L is homeomorphic to the real line $\mathbb{R}$.

Section 18 of [3] contains a detailed investigation of differentiable loops on the circle for which the group G topologically generated by the left translations is isomorphic to the group $PSL_2(\mathbb{R})$. The corresponding loop multiplication were determined by a pair of real periodic differentiable functions satisfying the conditions some differential inequality. The investigation of differentiable loops on the circle is extended to the case where the group G is isomorphic to $SL_2(\mathbb{R})$ in [4]. This paper is devoted to the study of differentiable proper loops on the real line $\mathbb{R}$ such that the group G topologically generated by the left translations is locally compact. In this case the group G is isomorphic to the universal covering group of $PSL_2(\mathbb{R})$. Modifying the description of [3] we give explicit expressions of the loop multiplications with respect to a class of natural parametrizations of the loop manifold L corresponding to the Iwasava decompositions of G. We characterize within the biinvariant Lorentzian geometry of the group G the one-dimensional differentiable submanifolds $\Lambda = \{\lambda_x, x \in \mathbb{R}\}$ of $PSL_2(\mathbb{R})$, the point of which are the left translations of a differentiable loop on the real line $\mathbb{R}$.

2. Action of $\widetilde{PSL_2}(\mathbb{R})$ on $\mathbb{R}$

Let $\mathcal{A}$ be the algebra of real 2×2 matrices. We denote by $\widetilde{PSL_2}(\mathbb{R})$ the universal covering group of the groups

$$SL_2(\mathbb{R}) = \{A \in \mathcal{A};\ \det A = 1\}, \quad PSL_2(\mathbb{R}) = \{\pm A \in \mathcal{A};\ \det A = 1\}.$$

The matrices

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

form a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $SL_2(\mathbb{R})$. Let $\kappa : X \mapsto \frac{1}{2}\mathrm{Trace}(X^2)$ be the normalized Cartan–Killing form on $\mathfrak{sl}_2(\mathbb{R})$ given by $\kappa(X) = -\det(X) = h^2 + t^2 - u^2$ for $X = h\mathbf{H} + t\mathbf{T} + u\mathbf{U}$, which determines a pseudo-euclidean vector space structure on $\mathfrak{sl}_2(\mathbb{R})$ such that the unit vectors $\mathbf{H}$, $\mathbf{T}$, $\mathbf{U}$ form a pseudo-orthonormal basis. We denote by $\exp : \mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$ the exponential map of $SL_2(\mathbb{R})$ and by $\widetilde{\exp} : \mathfrak{sl}_2(\mathbb{R}) \to \widetilde{PSL_2}(\mathbb{R})$ the exponential map of the universal covering group $\widetilde{PSL_2}(\mathbb{R})$. Let $p : \widetilde{PSL_2}(\mathbb{R}) \to SL_2(\mathbb{R})$ be the covering map. The connected 1-parameter subgroups $\{\widetilde{\exp}(t\mathbf{X});\ t \in \mathbb{R}\}$ with $\mathbf{X} \neq \mathbf{0}$ are called *elliptic, hyperbolic* or *parabolic*, if $\kappa(\mathbf{X}) < 0$, $\kappa(\mathbf{X}) > 0$ or $\kappa(\mathbf{X}) = 0$, respectively. The covering map p induces isomorphisms $\{\widetilde{\exp}(t\mathbf{X});\ t \in \mathbb{R}\} \to \{\exp(t\mathbf{X});\ t \in \mathbb{R}\}$ between hyperbolic or parabolic 1-parameter subgroups, it yields proper covering homomorphisms for elliptic 1-parameter subgroups. The connected 2-dimensional subgroups of $\widetilde{PSL_2}(\mathbb{R})$ have the form $\{\widetilde{\exp}(s(\mathbf{Y} - \mathbf{Z})) \cdot \widetilde{\exp}(t\mathbf{X});\ s, t \in \mathbb{R}\}$, where $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is a pseudo-orthonormal basis of $(\mathfrak{sl}_2(\mathbb{R}), \kappa)$, which is conjugate to the basis $(\mathbf{H}, \mathbf{T}, \mathbf{U})$ of $\mathfrak{sl}_2(\mathbb{R})$. Clearly, the covering map p induces isomorphisms

$$\{\widetilde{\exp}(s(\mathbf{Y} - \mathbf{Z})) \cdot \widetilde{\exp}(t\mathbf{X});\ s, t \in \mathbb{R}\} \to \{\exp(s(\mathbf{Y} - \mathbf{Z})) \exp(t\mathbf{X});\ s, t \in \mathbb{R}\}$$

between 2-dimensional subgroups.

The pseudo-euclidean scalar product induced by the Cartan–Killing form on $\mathfrak{sl}_2(\mathbb{R})$ is invariant with respect to the adjoint representation of $\widetilde{PSL_2}(\mathbb{R})$. Hence it can be extended by left or right translations of $\widetilde{PSL_2}(\mathbb{R})$ to a bi-invariant pseudo-riemannian metric $g(X, Y)$ on the manifold $\widetilde{PSL_2}(\mathbb{R})$ such that $g(X, X)$ at the identity element of $\widetilde{PSL_2}(\mathbb{R})$ coincides with $\kappa(X)$. A curve $t \mapsto \gamma(t) : \mathbb{R} \to \widetilde{PSL_2}(\mathbb{R})$ will be called *time-like* if its tangent vectors $\dot{\gamma}(t)$ satisfy $g(\dot{\gamma}(t), \dot{\gamma}(t)) < 0$ for any $t \in \mathbb{R}$.

We consider the group $G = \widetilde{PSL_2}(\mathbb{R})$, a 2-dimensional subgroup $H \subset G$ and the homogeneous space G/H diffeomorphic to $\mathbb{R}$. Let $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ be a pseudo-orthonormal basis of $\mathfrak{sl}_2(\mathbb{R})$, for which there exists an automorphism $\beta : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$ mapping the basis $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ onto the canonical basis $(\mathbf{H}, \mathbf{T}, \mathbf{U})$, such that

$$H = \{\widetilde{\exp}(s(\mathbf{Y} - \mathbf{Z})) \cdot \widetilde{\exp}(t\mathbf{X});\ s, t \in \mathbb{R}\} \subset G.$$

Let K, N and D denote the 1-parameter subgroups $K = \{\widetilde{\exp}(t\mathbf{Z});\ t \in \mathbb{R}\}$, $N = \{\widetilde{\exp}(t(\mathbf{Y} - \mathbf{Z});\ t \in \mathbb{R}\}$ and $D = \{\widetilde{\exp}(t\mathbf{X});\ t \in \mathbb{R}\}$. Since the vector space direct sum $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}\mathbf{Z} + \mathbb{R}(\mathbf{Y} - \mathbf{Z}) + \mathbb{R}\mathbf{X}$ is an Iwasawa decomposition of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, (cf. e.g. [2] p. 234), the map

$$\Delta : g \mapsto (k, n, d) : \widetilde{PSL_2}(\mathbb{R}) \to K \times N \times D \tag{1}$$

defined by $g = knd$ is an analytic diffeomorphism of $\widetilde{PSL_2}(\mathbb{R})$ onto the product manifold $K \times N \times D$. The triple $\Delta(g) = (k, n, d)$ will be called the Iwasava decomposition of the element $g \in \widetilde{PSL_2}(\mathbb{R})$. It follows that any element $g \in \widetilde{PSL_2}(\mathbb{R})$ can be decomposed uniquely into the product $g = \widetilde{\exp}(t\mathbf{Z})\widetilde{\exp}(u(\mathbf{Y} - \mathbf{Z})) \cdot \widetilde{\exp}(v\mathbf{X})$ with $(t, u, v) \in \mathbb{R}^3$. Using the decomposition $G = K \cdot N \cdot D$ and $H = N \cdot D$ we obtain that any coset of the subgroup $H = N \cdot D$ can be represented uniquely by an element $\widetilde{\exp}(t\mathbf{Z}) \in K$, $t \in \mathbb{R}$. Hence the manifold G/H can be parametrized by elements of K. Since the mapping $t \mapsto \widetilde{\exp}(t\mathbf{Z}) : \mathbb{R} \to K$ is bijective, we obtain that

$$\phi : t \mapsto \widetilde{\exp}(t\mathbf{Z})H : \mathbb{R} \to G/H \tag{2}$$

is a diffeomorphism and hence this map gives a global real parametrization of the loop manifold which will be called an *Iwasava parametrization*. This parametrization ϕ is not canonical in the sense that it depends on the choice of the basis $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ of $\mathfrak{sl}_2(\mathbb{R})$.

We can define the action $\alpha_g : \mathbb{R} \mapsto \mathbb{R}$ of $g \in G$ by

$$\alpha_g t = \phi^{-1}(g \cdot \widetilde{\exp}(t\mathbf{Z})H) = \phi^{-1}(g \cdot \phi t),$$

where $kH \mapsto g \cdot kH : G/H \to G/H$ denotes the natural action of G on the factor space G/H. Hence $\alpha : G \times \mathbb{R} \to \mathbb{R}$ is a natural differentiable transformation group action of $G = \widetilde{PSL_2}(\mathbb{R})$ on $\mathbb{R}$.

Proposition 1. *The mapping* $\alpha_g : \mathbb{R} \mapsto \mathbb{R}$ *with* $g = \widetilde{\exp}(t\mathbf{Z}) \cdot \widetilde{\exp}(u(\mathbf{Y} - \mathbf{Z})) \cdot \widetilde{\exp}(v\mathbf{X})$ *with respect to the Iwasava parametrization (2) can be expressed by*

$$\alpha_g s = \begin{cases} t + \operatorname{arccot}\left(2u + e^{-2v} \cot s\right) + i\pi, & \text{if } \ i\pi < s < (i+1)\pi, \\ t + i\pi, & \text{if } \ s = i\pi, \end{cases}$$

for any $i \in \mathbb{Z}$. *In particular,* $\alpha_{\widetilde{\exp}(t\mathbf{Z})} s = t + s$.

PROOF. The equation $\alpha_{\widetilde{\exp}(t\mathbf{Z})}s = t+s$ follows from the relation

$$\widetilde{\exp}(t\mathbf{Z}) \cdot \widetilde{\exp}(s\mathbf{Z}) \cdot H = \widetilde{\exp}((t+s)\mathbf{Z}) \cdot H.$$

If $t=0$ the element $\alpha_g s$ is determined by

$$\widetilde{\exp}(u(\mathbf{Y}-\mathbf{Z})) \cdot \widetilde{\exp}(v\mathbf{X}) \cdot \widetilde{\exp}(s\mathbf{Z}) \cdot H = \widetilde{\exp}(\alpha_g s\mathbf{Z}) \cdot H.$$

Since there exists an automorphism $\beta : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$ mapping the basis $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ onto the canonical basis $(\mathbf{H}, \mathbf{T}, \mathbf{U})$ of $\mathfrak{sl}_2(\mathbb{R})$ this equation is equivalent to

$$\widetilde{\exp}(u(\mathbf{T}-\mathbf{U})) \cdot \widetilde{\exp}(v\mathbf{H}) \cdot \widetilde{\exp}(s\mathbf{U}) \cdot \bar{H} = \widetilde{\exp}(\alpha_g s\mathbf{U}) \cdot \bar{H},$$

where $\bar{H} = \{\widetilde{\exp}(s(\mathbf{T}-\mathbf{U})) \cdot \widetilde{\exp}(t\mathbf{H});\ s,t \in \mathbb{R}\} \subset G$. Applying the covering homomorphism $p : \widetilde{PSL_2}(\mathbb{R}) \to SL_2(\mathbb{R})$ we obtain

$$\begin{pmatrix} e^v & 0 \\ 2ue^v & e^{-v} \end{pmatrix} \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} p(\bar{H}) = \begin{pmatrix} \cos \alpha_g s & \sin \alpha_g s \\ -\sin \alpha_g s & \cos \alpha_g s \end{pmatrix} p(\bar{H})$$

or equivalently

$$\begin{pmatrix} e^v & 0 \\ 2ue^v & e^{-v} \end{pmatrix} \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} = \begin{pmatrix} \cos \alpha_g s & \sin \alpha_g s \\ -\sin \alpha_g s & \cos \alpha_g s \end{pmatrix} \begin{pmatrix} e^x & 0 \\ 2ye^x & e^{-x} \end{pmatrix}$$

for some $x, y \in \mathbb{R}$. Comparing the second columns of the product matrices we obtain

$$\cot \alpha_g s = \frac{2ue^v \sin s + e^{-v} \cos s}{e^v \sin s} = 2u + e^{-2v} \cot s.$$

Since the map $\alpha_g : \mathbb{R} \to \mathbb{R}$ is a bijection and hence it is monotone. The shape of the last expression shows that α_g is monotone increasing. The assertion follows from this and from $\alpha_g 0 = 0$. □

3. The left multiplication curve

We consider a differentiable left loop defined on a one-manifold and assume that the group topologically generated by the left translations is isomorphic to $G = \widetilde{PSL_2}(\mathbb{R})$. Since G acts transitively, the left loop manifold may be identified with the factor space G/H, where the 2-dimensional subgroup H of G is the identity of the left loop $(G/H, \circ, H)$. Let $\pi : G \to G/H$ be the canonical projection map and let ϕ be the parametrization $\mathbb{R} \to G/H$ introduced in equation (2). We define a differentiable left loop $(\mathbb{R}, \circ, 0)$ on the real line $\mathbb{R}$ assuming that the map $\phi : (G/H, \circ, H) \to (\mathbb{R}, \circ, 0)$ is an isomorphism. The left translations $\lambda_x, x \in \mathbb{R}$, of $(\mathbb{R}, \circ, 0)$ have the shape $\lambda_x = \alpha_{\sigma_x}$ where $\sigma_x \in G$ is the left translation on G/H corresponding to the element $\phi(x) \in G/H$. Hence σ_x is contained in $\phi(x) \in G/H$ for any $x \in \mathbb{R}$ and has the same action on G/H as the mapping given by $\phi\lambda_x\phi^{-1}$.

Definition 2. A differentiable curve $\sigma : x \mapsto \sigma_x : \mathbb{R} \to G$ is called a left multiplication curve on G if σ_x is contained in $\phi(x) \in G/H$ for any $x \in \mathbb{R}$. The map $(x, y) \mapsto x \cdot y = \phi^{-1}\sigma_x\phi(y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called the multiplication associated with the left multiplication curve σ with respect to the Iwasava parametrization (2).

The multiplication $(x, y) \mapsto x \cdot y$ on $\mathbb{R}$, associated with a left multiplication curve σ determines a differentiable left loop $(\mathbb{R}, \circ, 0)$ which has $0 \in \mathbb{R}$ as unit element. One can see immediately, that the left multiplication maps of this left loop topologically generate a group which is isomorpic to G.

Now, let $(G/H, \circ, H)$ be a differentiable left loop having $G = \widetilde{PSL_2}(\mathbb{R})$ as the group topologically generated by the left translations. The mapping $x \mapsto \sigma_x : \mathbb{R} \to G$ satisfying $\lambda_x = \alpha_{\sigma_x}$ determines a left multiplication curve on G, and the multiplication associated with σ is isomorphic to the multiplication of the left loop $(G/H, \circ, H)$. According to $G = K \cdot N \cdot D$ we have the decomposition

$$\sigma_x = \widetilde{\exp}(x\mathbf{Z}) \cdot \widetilde{\exp}(\nu(x)(\mathbf{Y} - \mathbf{Z})) \cdot \widetilde{\exp}(\delta(x)\mathbf{X}),$$

where $\nu(x)$ and $\delta(x)$ are real differentiable functions having the initial value $\nu(0) = \delta(0) = 0$. The left translations of the associated multiplication $(x, y) \mapsto x \cdot y$ have the shape $\lambda_x = \alpha_{\sigma_x}$. If the left multiplication curve is not a one-parameter subgroup of G then the group topologically generated by the left translations is isomorphic to $G = \widetilde{PSL_2}(\mathbb{R})$.

We obtain from Proposition 1 the following

Proposition 3. *Let $\sigma : x \mapsto \sigma_x : \mathbb{R} \to \widetilde{PSL_2}(\mathbb{R})$ be a left multiplication curve of the shape $\sigma_x = \widetilde{\exp}(x\mathbf{Z}) \cdot \widetilde{\exp}(\nu(x)(\mathbf{Y} - \mathbf{Z})) \cdot \widetilde{\exp}(\delta(x)\mathbf{X})$, where $\nu(x)$ and $\delta(x)$ are differentiable functions having the initial value $\nu(0) = \delta(0) = 0$. The multiplication $(x, y) \mapsto x \cdot y$ associated with the left multiplication curve σ with respect to the Iwasava parametrization (2) can be expressed by*

$$x \cdot y = \begin{cases} x + \operatorname{arccot}\left(2\nu(x) + e^{-2\delta(x)} \cot y\right) + i\pi, & \text{if} \quad i\pi < y < (i+1)\pi, \\ x + i\pi, & \text{if} \quad y = i\pi, \end{cases}$$

for any $i \in \mathbb{Z}$.

Now, we want characterize the loop multiplications among the differentiable left loop multiplications described in the previous proposition. Clearly, if $(\mathbb{R}, \cdot, 0)$ is a loop, then for any $y \in \mathbb{R}$ the right multiplication map $\rho_y : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism. It follows that the map $\rho_y : \mathbb{R} \to \mathbb{R}$ and its inverse $\rho_y^{-1} : \mathbb{R} \to \mathbb{R}$ are monotone differentiable functions and hence the function $m(x, y) = x \cdot y$ satisfies $\frac{\partial}{\partial x} m(x, y) \neq 0$ for any $x, y \in \mathbb{R}$. Since $m(x, 0) = x$ one has $\frac{\partial}{\partial x} m(x, 0) = 1$ from which follows $\frac{\partial}{\partial x} m(x, y) > 0$ for each $x, y \in \mathbb{R}$. Now we show that this condition characterize the loop multiplications.

Proposition 4. *The multiplication $x \cdot y = \phi^{-1}\sigma_x\phi(y)$ associated with the left multiplication curve σ with respect to the Iwasava parametrization* (2) *determines a differentiable loop* $(\mathbb{R}, \cdot, 0)$ *if and only if the function $m(x, y) = x \cdot y$ satisfies* $\frac{\partial}{\partial x} m(x, y) > 0$.

Proof. If the inequality $\frac{\partial}{\partial x} m(x, y) > 0$ is satisfied for any $x, y \in \mathbb{R}$ then the maps $\rho_y : \mathbb{R} \to \rho_y(\mathbb{R})$ are monotone increasing diffeomorphisms of $\mathbb{R}$ onto the connected subset $\rho_y(\mathbb{R}) \subset \mathbb{R}$. We have to prove that the maps $\rho_y : \mathbb{R} \to \mathbb{R}$ are surjective. Let $y \in \mathbb{R}$ be fixed. We assume that there exists a real number a such that $\sup_{x \in \mathbb{R}}(x \cdot y) = \lim_{x \to \infty} \rho_y(x) = \lim_{x \to \infty} \lambda_x(y) = a$. We have up to a conjugation in $\widetilde{PSL_2}(\mathbb{R})$

$$\widetilde{\exp}(x\mathbf{U}) \cdot \widetilde{\exp}(\nu(x)(\mathbf{T} - \mathbf{U})) \cdot \widetilde{\exp}(\delta(x)\mathbf{H}) \cdot \widetilde{\exp}(y\mathbf{U}) \cdot \bar{H} = \widetilde{\exp}((x \cdot y)\mathbf{U}) \cdot \bar{H}.$$

Applying the continuous covering map $p : \widetilde{PSL_2}(\mathbb{R}) \to SL_2(\mathbb{R})$ we obtain

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} e^{\delta(x)} & 0 \\ 2\nu(x)e^{\delta(x)} & e^{-\delta(x)} \end{pmatrix} \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix} p(H) = \begin{pmatrix} \cos(x \cdot y) & \sin(x \cdot y) \\ -\sin(x \cdot y) & \cos(x \cdot y) \end{pmatrix} p(H).$$

Hence we have the decomposition

$$\begin{pmatrix} \cos(x - x \cdot y) & \sin(x - x \cdot y) \\ -\sin(x - x \cdot y) & \cos(x - x \cdot y) \end{pmatrix} \begin{pmatrix} e^{\delta(x)} & 0 \\ 2\nu(x)e^{\delta(x)} & e^{-\delta(x)} \end{pmatrix} = \begin{pmatrix} e^{d(x,y)} & 0 \\ 2n(x,y)e^{d(x,y)} & e^{-d(x,y)} \end{pmatrix} \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix},$$

where $d(x, y)$ and $n(x, y)$ are suitable continuous real functions. If $\sin y = 0$ then this equation implies $x = x \cdot y + 2k\pi$ for any $x \in \mathbb{R}$ and hence $y = 0$, in which case trivially $\rho_0(\mathbb{R}) = \mathbb{R}$. Assuming $\sin y \neq 0$ and comparing the first rows of these product matrices we obtain

$$\Big(e^{\delta(x)} \cos(x - x\cdot y) + 2\nu(x) e^{\delta(x)} \sin(x - x\cdot y), e^{-\delta(x)} \sin(x - x\cdot y)\Big) \\ = \Big(e^{d(x,y)} \cos y, -e^{d(x,y)} \sin y\Big).$$

From this we have $e^{2\delta(x)}(\cot(x - x\cdot y) + 2\nu(x)) = -\cot y$. Since the function $x \mapsto x - x\cdot y$ is continuous, its range is a connected subset of $\mathbb{R}$ such that $\sup_{x\in\mathbb{R}}(x - x\cdot y) = \infty$ because of $\lim_{x\to\infty} x\cdot y = a$. Hence there exists a real value $x_0 \in \mathbb{R}$ satisfying $\sin(x_0 - x_0\cdot y) = 0$ for which the equation

$$e^{2\delta(x_0)} (\cot(x_0 - x_0\cdot y) + 2\nu(x_0)) = -\cot y$$

gives a contradiction. □

We obtain the following characterization of the left multiplication curves of loop multiplications:

Theorem 5. *Let* $\sigma : x \mapsto \sigma_x : \mathbb{R} \to \widetilde{PSL_2}(\mathbb{R})$ *be a left multiplication curve of the shape* $\sigma_x = \widetilde{\exp}(x\mathbf{Z})\cdot\widetilde{\exp}(\nu(x)(\mathbf{Y}-\mathbf{Z}))\cdot\widetilde{\exp}(\delta(x)\mathbf{X})$, *where* $\nu(x)$ *and* $\delta(x)$ *are differentiable functions having the initial value* $\nu(0) = \delta(0) = 0$. *The multiplication* $(x,y) \mapsto x\cdot y$ *associated with the left multiplication curve* σ *with respect to the Iwasava parametrization* (2) *determines a differentiable loop on* $\mathbb{R}$ *if and only if the functions* $\delta(t)$ *and* $\nu(t)$ *satisfy the differential inequality*

$$g(\dot\sigma_x, \dot\sigma_x) = \dot\delta(t)^2 + 2\dot\nu(t) + 4\nu(t)\dot\delta(t) - 1 < 0,$$

where $\dot\delta = \frac{d\delta}{dx}$, $\dot\nu = \frac{d\nu}{dx}$ *and* $\dot\sigma_x = \frac{d\sigma_x}{dx}$.

Proof. The derivation of the multiplication given in Theorem 3 gives that

$$\frac{\partial}{\partial x} m(x,y) = 1 - \frac{2\dot\nu(x) - 2e^{-2\delta(x)}\dot\delta(x)\cot y}{1 + \big(2\nu(x) + e^{-2\delta(x)}\cot y\big)^2} > 0$$

if and only if $1 + (2\nu(x) + e^{-2\delta(x)}\cot y)^2 - 2\dot\nu(x) + 2e^{-2\delta(x)}\dot\delta(x)\cot y > 0$. Equivalently,

$$\big(1 + 4\nu^2(x) - 2\dot\nu(x)\big) + 2e^{-2\delta(x)}\big(2\nu(x) + \dot\delta(x)\big)\cot y + e^{-4\delta^2(x)}\cot^2 y > 0$$

for all $y \in \mathbb{R}$, which gives $\big(2\nu(x) + \dot\delta(x)\big)^2 - \big(1 + 4\nu^2(x) - 2\dot\nu(x)\big) < 0$ or

$$\dot\delta^2(x) + 2\dot\nu(x) + 4\nu(x)\dot\delta(x) - 1 < 0. \tag{3}$$

For the computation of the pseudo-Riemannian value $g(\dot\sigma_x, \dot\sigma_x)$ we apply the covering map $p : \widetilde{PSL_2}(\mathbb{R}) \to SL_2(\mathbb{R})$ to σ_x and derivate it:

$$\begin{aligned}\frac{d}{dx}p(\sigma_x) &= \begin{pmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{pmatrix} \begin{pmatrix} e^{\delta(x)} & 0 \\ 2\nu(x)e^{\delta(x)} & e^{-\delta(x)} \end{pmatrix} \\ &\quad + \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} \dot\delta(x)e^{\delta(x)} & 0 \\ 2\dot\nu(x) + 2\nu(x)\dot\delta(x))e^{\delta(x)} & -\dot\delta(x)e^{-\delta(x)} \end{pmatrix} \\ &= \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2\nu(x) & 1 \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} \dot\delta(x) & 0 \\ 2\dot\nu(x) + 2\nu(x)\dot\delta(x) & -\dot\delta(x) \end{pmatrix} \right\} \begin{pmatrix} e^{\delta(x)} & 0 \\ 0 & e^{-\delta(x)} \end{pmatrix}.\end{aligned}$$

Since $g(\dot\sigma_x, \dot\sigma_x) = \gamma({\sigma_x}^{-1}\dot\sigma_x) = -\det(p_*({\sigma_x}^{-1}\dot\sigma_x)) = -\det(p_*(\dot\sigma_x))$ we can compute

$$\begin{aligned} g(\dot\sigma_x, \dot\sigma_x) &= -\det \begin{pmatrix} 2\nu(x) + \dot\delta(x) & 1 \\ -1 + 2\dot\nu(x) + 2\nu(x)\dot\delta(x) & -\dot\delta(x) \end{pmatrix} \\ &= \dot\delta^2(x) + 2\dot\nu(x) + 4\nu(x)\dot\delta(x) - 1. \end{aligned}$$

Hence the condition (3) is equivalent to $g(\dot\sigma_x, \dot\sigma_x) < 0$ which proves the assertion. $\square$

Corollary 6. *The multiplication $(x, y) \mapsto x \cdot y$ associated with the left multiplication curve σ is a loop multiplication if and only if the left translation curve is time-like.*

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