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Optimal feedback control for impulsive systems on the space of finitely additive measures

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Abstract. In this paper we consider evolution equations and inclusions on the space of finitely additive measures arising naturally from differential equations and inclusions (on Banach spaces) containing non smooth vector fields or and multi functions possessing only measurable selections. We also consider applications of these results to control theory and min-max problems for uncertain dynamic systems.

1. Introduction

Let us consider the evolution equation,

$$\begin{aligned} dx &= Axdt + f(t,x)dt + g(t,x)\nu(dt), \quad t \geq 0 \\ x(0) &= \xi, \end{aligned} \tag{1}$$

in a Banach space E where A is the infinitesimal generator of a C_0-semigroup, $S(t)$, $t \geq 0$, on E and $f, g : [0,T] \times E \longrightarrow E$ are measurable maps to be clarified shortly and ν is a signed measure. Considering the special case ($g = 0$), if E is a finite dimensional space, continuity of f is sufficient for the existence of a continuous solution locally which may blow up in finite time. In contrast, it is well known that if E is infinite dimensional space, mere continuity of f in x does no longer guarantee the existence of a strong, mild or even a weak solution. Here we consider measurable vector fields; so the hope for path wise E valued

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solutions (that is, solutions defined on $I \equiv [0,T]$ with values in E) is completely dashed. Due to this fact in recent years significant interest has developed in the study of measure valued solutions see [Fattorini 7, Ahmed 1–6, 14]. In fact the notion of measure solution or generalized solution was already introduced by DiPerna [9] in his study of conservation laws described by nonlinear partial differential equations and later used by Slemrod and Roytburd [12] in the study of dynamic phase transitions. This was primarily based on the Young measure introduced by L. C. Young in the study of Calculus of variations and optimal control [13] where the controls (inputs) are measure valued functions called relaxed controls while the solutions are paths or curves in the original state space. The concept of measure valued solutions is substantially different. Here one deals with functions taking values from the space of measures on the original state space. One proves the existence of measure valued functions as solutions for evolution equations in situations where there is no solution in the usual sense such as strong, mild or weak.

Measure solutions introduced by Fattorini [7] and Ahmed [1–6] are substantial generalizations of Young measures covering semilinear and quasilinear evolution equations in abstract Banach spaces. Here we use the existence results from [1]–[3] in our study of control problems involving systems of the form (1) with measurable vector fields $\{f,g\}$ thus requiring the notion of measure solutions.

The rest of the paper is organized as follows. In Section 2, the notion of measure solution is introduced and existence of such solutions for the evolution equation (1) are presented. Based on these results, in Section 3 evolution equations on the space of measures considered as natural state space for such equations are introduced. In Section 4, we study the question of existence and regularity properties of measure solutions for differential inclusions. In Section 5, we study the question of existence of optimal feedback controls for several control problems involving such systems. The paper is concluded with some comments on applications and open problems.

2. Measure valued solutions

Let Z denote any regular topological space and $B_0(Z)$ the space of bounded scalar valued functions on Z with the topology of sup norm given by

$$\|f\| \equiv \sup\{|f(z)|,\ z \in Z\}.$$

This is a Banach space. However the elements of this space may not be Borel measurable. Let $\Sigma \equiv \Sigma_Z$ denote an algebra of subsets of the set Z generated by closed sets and let $B(Z) \equiv B(Z, \Sigma)$ denote the class of scalar valued functions defined on Z which are uniform limits of characteristic functions of sets from Σ. The space $B(Z)$ is furnished with the same topology as $B_0(Z)$. An element f of this space is said to be Σ measurable if for every Borel set Γ from the real line (the range space), the set

$$f^{-1}(\Gamma) \equiv \{z \in Z : f(z) \in \Gamma\} \in \Sigma.$$

The class of all bounded Σ measurable functions is dense in $B(Z)$. It is clear that $B(Z)$ is a closed subspace of $B_0(Z)$ and hence it is also a Banach space. Let $\mathcal{M}_{ba}(Z) \equiv \mathcal{M}_{ba}((Z, \Sigma))$ denote the class of all scalar valued bounded finitely additive measures (set functions) defined on the algebra Σ. Furnished with the total variation norm, $\mathcal{M}_{ba}(Z)$ is a Banach space.

The following Lemma characterizes the topological dual $B^*(Z)$ of the Banach space $B(Z)$.

Lemma 2.1. $B^*(Z) \cong \mathcal{M}_{ba}(Z)$. *Stated in words, the topological dual of $B(Z)$ is isometrically isomorphic to the space of finitely additive bounded measures on Z.*

PROOF. See DUNFORD and SCHWARTZ [8, Theorem IV.5.1, p. 258]. □

Let $\Pi_{ba}(Z) \subset \mathcal{M}_{ba}(Z)$ denote the class of finitely additive probability measures furnished with the relative topology. Recall that a Banach space X is said to satisfy Radon–Nikodym property (RNP) if, for every finite measure space $(\Omega, \mathcal{B}, \gamma)$ and every γ continuous bounded vector measure $\Xi : \mathcal{B} \longrightarrow X$, there exists a $g \in L_1(\Omega, X)$ such that

$$\Xi(\Gamma) = \int_\Gamma g(\omega)\gamma(dw), \ \forall\, \Gamma \in \mathcal{B}.$$

The Banach space $B(Z)$ and its dual $\mathcal{M}_{ba}(Z)$ do not satisfy (RNP). Therefore the dual of $L_1(I, B(Z))$ is not $L_\infty(I, \mathcal{M}_{ba}(Z))$. However, it follows from the theory of lifting [11] that the dual is given by $L^w_\infty(I, \mathcal{M}_{ba}(Z))$ which consists of weak* measurable $\mathcal{M}_{ba}(Z)$ valued functions in the sense that for every $\varphi \in B(Z)$,

$$t \longrightarrow \mu_t(\varphi) \equiv \int_Z \varphi(\xi)\mu_t(d\xi)$$

is measurable in the Lebesgue sense and the scalar valued function $t \longrightarrow \mu_t(\varphi)$ is essentially bounded in the usual sense. Note that $B(Z)$ is a nonseparable Banach space.

We consider the following measure driven Cauchy problem in a Banach space E,

$$\begin{aligned} dx(t) &= Ax(t)dt + f(t,x(t))dt + g(t,x(t-))\nu(dt), \quad t \in I \equiv [0,T], \\ x(0) &= x_0 \in E, \end{aligned} \tag{2}$$

where A is the generator of a C_0-semigroup $S(t)$, $t \geq 0$, in E, and f, g are suitable maps from $I \times E$ to E and ν is a signed measure on I.

It is known [1]–[3] that under the assumptions of mere measurability and local boundedness of $\{f, g\}$ this equation has finitely additive measure valued solutions. Let $\mathcal{B}$ denote the sigma algebra of Borel subsets of the interval I and $\Sigma \equiv \Sigma_E$ an algebra of subsets of the set E generated by closed subsets of E. Our general assumption is that f, g are $\mathcal{B} \times \Sigma$ measurable maps in the sense that for every Borel set Γ in the range space E,

$$h^{-1}(\Gamma) \equiv \{(t,\xi) \in I \times E : h(t,\xi) \in \Gamma\} \in \mathcal{B} \times \Sigma, \quad \text{for } h = f, g.$$

For non impulsive systems with f continuous and bounded on bounded sets, general notion of measure solutions was introduced by the author in [4]–[6], where the natural choice was regular bounded finitely additive measures $\mathcal{M}_{rba}(Z)$ instead of $\mathcal{M}_{ba}(Z) \supset \mathcal{M}_{rba}(Z) \cong (BC(Z))^*$. This is because the vector fields

$$e \longrightarrow h(t,e)$$

used there were assumed to be continuous from E to E. In [1]–[3], where measurable vector fields were admitted (permitting discontinuities), the most appropriate choice of the space of measures was found to be $\mathcal{M}_{ba}(Z)$, and this is what is used here.

Now we are prepared to introduce the relevant operators required for study of measure solutions. Let $D\phi$ denote the Frechet derivative of $\phi \in B(E)$ whenever it exists and introduce the class of test functions $\mathcal{F}$, given by

$$\mathcal{F} \equiv \{\phi \in B(E) : D\phi \text{ exists}, \ D\phi \in B(E, E^*)\}.$$

For given $\{A, f, g, \nu\}$, we introduce the operators $\mathcal{A}$ and $\mathcal{C}$ as follows. Define the operator $\mathcal{A}$ with domain given by

$$\mathcal{D}(\mathcal{A}) \equiv \{\phi \in \mathcal{F} : \mathcal{A}\phi \in B(E)\} \tag{3}$$

where

$$(\mathcal{A}\phi)(t,\xi) = \langle A^* D\phi(\xi), \xi \rangle_{E^*,E} + \langle D\phi(\xi), f(t,\xi) \rangle_{E^*,E} \tag{4}$$

for $\phi \in \mathcal{D}(\mathcal{A})$. The operator $\mathcal{C}$ is given by

$$(\mathcal{C}\phi)(t,\xi) \equiv \int_0^1 \langle D\phi(\xi + \theta g(t,\xi)\nu(\{t\})), g(t,\xi)\rangle_{E^*,E} d\theta. \tag{5}$$

Clearly if t is not an atom of the measure ν, the operator $\mathcal{C}$ reduces to

$$(\mathcal{C}\phi)(t,\xi) = (\mathcal{C}(t)\phi)(\xi) \equiv \langle D\phi(\xi), g(t,\xi)\rangle_{E^*,E}. \tag{6}$$

If ν is absolutely continuous with respect to the Lebesgue measure ℓ, the operator $\mathcal{C}$ will always take this form. These are the appropriate operators arising in the study of measure valued solutions for systems of the form (2).

For any positive measure β having bounded total variation on the interval I, we let $L_1(\beta, B(E))$ denote the Banach space of Lebesgue–Bochner integrable functions on I with values in $B(E)$. For $f \in L_1(\beta, B(E))$ we have

$$\|f\|_{L_1(\beta,B(E))} \equiv \int_I \|f(t)\|_{B(E)}\beta(dt) < \infty.$$

Because of the presence of measure ν which may contain atoms, the system (2) may be subjected to impulsive forces from time to time. Thus we find it necessary to use the vector space

$$L_1(\ell, B(E)) \oplus L_1(|\nu|, B(E))$$

where ℓ denotes the Lebesgue measure on I and $|\nu|$ denotes the positive measure induced by the variation of the signed measure ν. Since the component spaces are Banach spaces, this is also a Banach space. It is easy to see that the topological dual of this space is given by

$$L_\infty^w(\ell, \mathcal{M}_{ba}(E)) \cap L_\infty^w(|\nu|, \mathcal{M}_{ba}(E)).$$

We are particularly interested in its subset

$$L_\infty^w(\ell, \Pi_{ba}(E)) \cap L_\infty^w(|\nu|, \Pi_{ba}(E)),$$

where $\Pi_{ba}(E)$ denotes the space of bounded finitely additive probability measures on E, or more precisely, on Σ. Now we introduce the following definition.

Definition 2.1. A measure valued function

$$\lambda \in L_\infty^w(\ell, \Pi_{ba}(E)) \cap L_\infty^w(|\nu|, \Pi_{ba}(E))$$

is said to be a measure solution (or generalized solution) of equation (2) if, for every $\phi \in \mathcal{D}(\mathcal{A})$ with $D\phi$ having bounded supports, the following identity holds

$$\lambda_t(\phi) = \phi(x_0) + \int_0^t \lambda_s(\mathcal{A}(s)\phi)ds + \int_0^t \lambda_{s-}(\mathcal{C}(s)\phi)\nu(ds), \quad t \in I,$$
$$\lambda_0(\phi) = \phi(x_0), \tag{7}$$

where

$$\mu_t(\psi) \equiv \int_E \psi(\xi)\mu_t(d\xi), \quad t \in I.$$

Remark 2.1. In case the initial state x_0 has a distribution $\pi \in \Pi_{ba}(E)$, the first term on the right hand side of the expression (7) of the definition is replaced by $\pi(\phi)$. By the notation λ_{s-} we mean its left hand limit in the sense that for every test function φ, we have $\lambda_{s-}(\varphi) \equiv \lim_{t\uparrow s, t<s} \lambda_t(\varphi)$. If s is not an atom of the measure ν, we have $\lambda_{s-} = \lambda_s$.

For simplicity of notation we have used $\mathcal{D}(\mathcal{A})$ to denote the common domain of the operators $\mathcal{A}(t), t \in I$.

Now we are prepared to present the basic assumptions used in [1]–[3] for the proof of existence of (measure) solutions of the system (2):

(A1): E is a separable Banach space and A is the infinitesimal generator of a C_0-semigroup of operators $S(t), t \geq 0$, in E and ν is a countably additive bounded signed measure (possessing bounded variation on bounded sets) having no atom at $t = 0$.

(A2): the function $h : I \times E \longrightarrow E$, $(h = f, g)$ is a locally bounded $\mathcal{B} \times \Sigma$ measurable map satisfying the following approximation properties:

(i) there exists a sequence of locally bounded $\mathcal{B} \times \Sigma$ measurable maps $\{h_n\}$ such that for each $n \in N$, $h_n(t, x) \in D(A)$ for all $(t, x) \in I \times E$, and it is locally Lipschitz,

$$\|h_n(t, x) - h_n(t, y)\|_E \leq \alpha_{n,r}\|x - y\|_E, \quad \forall\, x, y \in B_r,$$

where $B_r \equiv \{\xi \in E : \|\xi\|_E \leq r\}$ and $\alpha_{n,r}$ are finite positive numbers for finite $r \geq 0$ and $n \in N$;

(ii): $h_n(t, x) \stackrel{w/s}{\longrightarrow} h(t, x)$ (weakly for f/strongly for g) for almost all $t \in I$ and uniformly on compact subsets of E.

Remark 2.2. Note that, under the assumption (A2), for every $0 < r < \infty$, we may have $\lim_{n\to\infty} \alpha_{n,r} \longrightarrow \infty$. Sufficient conditions guaranteeing the existence of such approximating sequence are given in [1, Proposition 3.2].

Remark 2.3. Also note that under the assumption (A2), f is locally bounded and hence, for every $\phi \in D(\mathcal{A})$, we have $D\phi(\xi) \in D(A^*)$. This follows from the fact that if $\phi \in D(\mathcal{A})$, by definition $\mathcal{A}\phi \in B(E)$. Thus for any ball $B_r(E)$ of E of radius r, $\sup\{|\mathcal{A}\phi(\xi)|,\ \xi \in B_r(E)\} < \infty$. Since f is locally bounded on E, and $D\phi(\xi) \in E^*$, we also have $\sup\{|\langle D\phi(\xi), f(t,\xi)\rangle_{E^*,E}|,\ \xi \in B_r(E)\} < \infty$. Thus it follows from the defining relation (4), that $\sup\{|\langle A^* D\phi(\xi), \xi\rangle_{E^*,E}|,\ \xi \in B_r(E)\} < \infty$. This implies that $A^* D\phi(\xi)$ must belong to E^* for every $\xi \in B_r(E)$. Since $r(> 0)$ is arbitrary, this implies that $D\phi(\xi) \in D(A^*)$ for every $\xi \in E$.

Theorem 2.2. *Suppose $\{A, f, g, \nu\}$ satisfy the assumptions* (A1) *and* (A2) *and that, for each $\xi \in E$, $t \longrightarrow f(t,\xi)$ is integrable (in Lebesgue–Bochner sense) with respect to the Lebesgue measure while $t \longrightarrow g(t,\xi)$ is integrable (Lebesgue–Bochner) with respect to the measure $|\nu|(\cdot)$ induced by the variation of the measure ν. Then for each $x_0 \in E$, the evolution equation*

$$\begin{aligned} dx(t) &= Ax(t)dt + f(t, x(t))dt + g(t, x(t-))\nu(dt), \quad t \in I, \\ x(0) &= x_0, \end{aligned} \tag{8}$$

has a measure valued solution $\lambda \in L^w_\infty(\ell, \Pi_{ba}(E)) \cap L^w_\infty(|\nu|, \Pi_{ba}(E))$ in the sense of Definition 2.1. This assertion remains valid also for $\mathcal{L}(x_0) = \pi \in \Pi_{ba}(E)$.

Proof. Proof is very similar to that of [1, Theorem 3.1, p. 471] and thus omitted. □

Remark 2.4. In case both f and g are uniformly bounded on $I \times E$, and $\{A, \nu\}$ satisfy the assumption (A1), and π has bounded support in E, the measure solution λ_t has bounded support for all finite t.

3. Differential equations on the space of measures

In view of the Definition 2.1 and the preceding result, we can reformulate our original Cauchy problem (8), which was defined on the Banach space E, into a Cauchy problem on the Banach space of finitely additive measures $\mathcal{M}_{ba}(E)$ as follows:

$$\begin{aligned} d\mu_t &= \mathcal{A}^*(t)\mu_t dt + \mathcal{C}^*(t)\mu_{t-}\nu(dt), \quad t \geq s, \\ \mu_s &= \pi, \end{aligned} \tag{9}$$

where $\mathcal{A}^*$ and $\mathcal{C}^*$ are the formal duals of the operators $\mathcal{A}$ and $\mathcal{C}$ respectively. This is the differential version of the functional equation (7) or equivalently

$$d\mu_t(\varphi) = \mu_t(\mathcal{A}(t)\varphi)dt + \mu_{t-}(\mathcal{C}(t)\varphi)\nu(dt), \quad t \geq s,$$
$$\mu_s(\varphi) = \pi(\varphi), \varphi \in D(\mathcal{A}). \tag{10}$$

This of course covers the original Cauchy problem as a special case for $s = 0$.

According to our existence result (Theorem 2.2), we have seen that this equation has a solution in the weak sense (Definition 2.1). Hence it follows from this result that, for each initial state $\pi \in \Pi_{ba}(E) \subset \mathcal{M}_{ba}(E)$, the evolution equation (9) has at least one solution $\mu \in L^w_\infty(\ell, \Pi_{ba}(E)) \cap L^w_\infty(|\nu|, \Pi_{ba}(E)) \subset L^w_\infty(\ell, \mathcal{M}_{ba}(E)) \cap L^w_\infty(|\nu|, \mathcal{M}_{ba}(E))$ which is piece wise weak* continuous. Consequently, there exists a piece wise weak* continuous transition operator $U^*(t, s)$, $t \geq s \geq 0$, which is a family of linear operators on the Banach space $\mathcal{M}_{ba}(E)$ defining the evolution of the measure solution,

$$\mu_t = U^*(t, s)\pi. \tag{11}$$

It is easy to verify that $|\mu_t(\varphi)| \leq \|\varphi\|_{B(E)}$ for all $\varphi \in B(E)$. Thus the operator $U^*(t, s), 0 \leq s \leq t < \infty$, is nonexpansive, that is, for arbitrary $\gamma \in \mathcal{M}_{ba}(E)$, we have

$$\|U^*(t, s)\gamma\|_{\mathcal{M}_{ba}(E)} \leq \|\gamma\|_{\mathcal{M}_{ba}(E)}, \ 0 \leq s \leq t \leq T$$

and so bounded on $\mathcal{M}_{ba}(E)$.

So far we have not discussed the question of uniqueness of solutions. In the time invariant case, a uniqueness result was proved in [1] under the assumption that $\mathcal{A}$ is a spectral operator. This assumption was dispensed with in [14]. Thus we may assume that the evolution operator $U^*(t, s)$, $0 \leq s \leq t < \infty$, is unique implying uniqueness of measure solutions.

Remark 3.1. The jump in the measure solution μ at any atom $\{t\}$ of the driving measure ν is given by the signed measure

$$\Delta\mu_t \equiv \mu_t - \mu_{t-} = \nu(\{t\})\mathcal{C}^*(t)\mu_{t-}.$$

If the operator $\mathcal{C}(t)$ is taken as that given by (6) which is independent of $\nu(\{t\})$, the impact is multiplicative only as seen in the above expression; while if (5) is used the operator itself is dependent on the atom and the impact is two fold one multiplicative and the other a translation by $\nu(\{t\})g(t, \cdot)$.

4. Differential inclusions

Results presented above can be extended to differential inclusions of the form:

$$dx \in Axdt + F(t,x)dt + G(t,x)\nu(dt), \mathcal{L}(x_0) = \pi_0 \in \Pi_{ba}(E), \tag{12}$$

where F, G are suitable multi functions. For technical reasons only, here we restrict ourselves to Borel measurable vector fields and multi functions. Let $\mathcal{B}_{I\times E}$ denote the Borel algebra of subsets of the set $I \times E$ and $BM(I \times E, E)$ denote the space of bounded $\mathcal{B}_{I\times E}$ measurable functions from $I \times E$ to E in the sense that the inverse image (with respect to $f \in BM(I \times E, E)$) of any Borel set in the range space E is an element of $\mathcal{B}_{I\times E}$. Since E is a Banach space, furnished with the sup norm topology

$$\sup\{\|f(t,x)\|_E,\ (t,x) \in I \times E\},$$

$BM(I \times E, E)$ is also a Banach space. Let $BMM(I \times E, 2^E \setminus \emptyset)$ denote the class of nonempty $\mathcal{B}_{I\times E}$ measurable multi functions in the sense that for every open set $\mathcal{O} \subset E$, the set

$$F^-(\mathcal{O}) \equiv \{(t,\xi) \in I \times E : F(t,\xi) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{B}_{I\times E},$$

and that

$$\|F(t,\xi)\|^o \equiv \sup\{\|e\|_E : e \in F(t,\xi)\} < \infty \ \forall\, (t,\xi) \in I \times E.$$

We are particularly interested in the following two classes of multi functions, $BMM(I \times E, wkc(E))$ and $BMM(I \times E, kc(E))$, where $wkc(E)$ $\big(kc(E)\big)$ denotes the class of nonempty weakly compact (compact) convex subsets of E.

Theorem 4.1. *Suppose the following assumptions hold.* (a1)*: E is a separable Banach space and A is the infinitesimal generator of a C_0 semigroup of operators in E,* (a2)*: $F \in BMM(I \times E, wkc(E))$ and $G \in BMM(I \times E, kc(E))$,* (a3)*: ν is a bounded signed measure having bounded variation on bounded sets. Then, for every $\pi_0 \in \Pi_{ba}(E)$, the system* (12) *has a nonempty set of measure solutions $\Lambda(F,G,\pi_0) \subset L^w_\infty(\ell, \Pi_{ba}(E)) \cap L^w_\infty(|\nu|, \Pi_{ba}(E))$ which is w^*-sequentially compact.*

Proof. Clearly, under the assumption (a2), the multi functions F and G take values from $c(E)$, the class of nonempty closed subsets of E, and that they are measurable with respect to the Borel field of sets $\mathcal{B}_{I\times E}$. Since $(I \times E, \mathcal{B}_{I\times E})$

is a measurable space and E a separable Banach space (hence a Polish space), by the well known Kuratowski–Ryll Nardzewski selection theorem [10, Theorem 2.1, p. 154], both F and G have $\mathcal{B}_{I\times E}$ measurable selections. We choose any pair of such measurable selections $\{f,g\}$ of the multi functions $\{F,G\}$. Then we use Theorem 2.2 of this paper to conclude that the system

$$dx = Axdt + f(t,x(t))dt + g(t,x(t-))\nu(dt), \mathcal{L}(x_0) = \pi_0 \tag{13}$$

has a measure solution $\lambda \equiv \lambda(f,g) \in L^w_\infty(\ell,\Pi_{ba}(E)) \cap L^w_\infty(|\nu|,\Pi_{ba}(E))$ in the sense of Definition 2.1. In other words, the evolution equation,

$$d\lambda_t = \mathcal{A}^*_f(t)\lambda_t dt + \mathcal{C}^*_g(t)\lambda_{t-}\nu(dt), \lambda_0 = \pi_0, \tag{14}$$

has a weak solution $\lambda = \lambda(f,g) \in L^w_\infty(\ell,\Pi_{ba}(E)) \cap L^w_\infty(|\nu|,\Pi_{ba}(E))$, where the operators $\{\mathcal{A}_f,\mathcal{C}_g\}$ are precisely the operators as defined by (4) and (5) corresponding to f and g respectively. In fact, it follows from Theorem 2.2 that every pair $\{f,g\}$ of measurable selections of $\{F,G\}$ determines a unique weak solution of the evolution equation (14) in the sense of Definition 2.1. Thus the solution set denoted by $\Lambda(F,G,\pi_0)$ is a nonempty subset of $L^w_\infty(\ell,\Pi_{ba}(E)) \cap L^w_\infty(|\nu|,\Pi_{ba}(E))$. Since the set

$$L^w_\infty(\ell,\Pi_{ba}(E)) \cap L^w_\infty(|\nu|,\Pi_{ba}(E))$$

is a weak* compact subset of $L^w_\infty(\ell,\mathcal{M}_{ba}(E)) \cap L^w_\infty(|\nu|,\mathcal{M}_{ba}(E))$ we conclude that the set $\Lambda(F,G,\pi_0)$ is relatively weak* compact, that is, its weak* closure is weak* compact. Thus for weak* sequential compactness, it suffices to prove that $\Lambda(F,G,\pi_0)$ is weak* sequentially closed. Let $\{\lambda^n\}$ be any sequence from the set $\Lambda(F,G,\pi_0)$ and suppose it converges in the weak* topology to an element $\lambda^0 \in L^w_\infty(\ell,\Pi_{ba}(E)) \cap L^w_\infty(|\nu|,\Pi_{ba}(E))$. We must show that $\lambda^0 \in \Lambda(F,G,\pi_0)$. Since $\lambda^n \in \Lambda(F,G,\pi_0)$ there exists a sequence $\{f_n,g_n\}$ of $\mathcal{B}_{I\times E}$ measurable selections of $\{F,G\}$ so that $\lambda^n = \lambda(f_n,g_n)$. Again we use the assumption (a2), in particular the properties that F takes values from $wkc(E)$ and G takes values from $kc(E)$. Using this we can prove that there exists a subsequence of the sequence $\{f_n,g_n\}$, relabeled as the original sequence, and a pair of $\mathcal{B}_{I\times E}$ measurable selections $\{f_0,g_0\}$ of the multis F and G respectively, so that

$$f_n(t,e) \xrightarrow{w} f_0(t,e) \text{ in } E$$
$$g_n(t,e) \xrightarrow{s} g_0(t,e) \text{ in } E$$

point wise on $I\times E$ and hence on compact subsets of $I\times E$. Then using the Definition 2.1, we write equation (14) in its weak form using the pairs of vector fields $\{f_n, g_n\}$ and $\{f_0, g_0\}$ respectively for any choice of $\varphi \in D(\mathcal{A})$ with $D\varphi$ having compact support. Taking the difference of the corresponding expressions and using the convergence properties stated above, we can then verify relatively easily that λ^0 coincides with the solution $\lambda(f_0, g_0)$ of equation (14) corresponding to $f = f_0$ and $g = g_0$ in the weak sense. Thus we have $\lambda^0 \in \Lambda(F, G, \pi_0)$ thereby proving that the set $\Lambda(F, G, \pi_0)$ is sequentially weak* closed and hence sequentially weak* compact. This completes the proof. □

The result of Theorem 4.1 has interesting applications. One such application is given in the following corollary.

Corollary 4.2. *Suppose the assumptions of Theorem 4.1 hold. Let $t \to \Gamma(t)$, considered as the target, be a multi function on I with values in $c(E)$ (class of nonempty closed subsets of E) and continuous in the Hausdorff metric. Let Ψ be an upper semi continuous (possibly nonnegative, nondecreasing) real valued function. Then the functional,*

$$J(\lambda) \equiv \int_I \Psi(\lambda_t(\Gamma(t)))dt, \tag{15}$$

attains its maximum on $\Lambda(F, G, \pi_0)$.

Proof. Let $\lambda^n \subset \Lambda(F, G, \pi_0)$ be a maximizing sequence. Since this set is weak* sequentially compact, there exists a subnet, relabeled as λ^n and a $\lambda^o \in \Lambda(F, G, \pi_0)$ so that $\lambda^n \xrightarrow{w^*} \lambda^o$. By virtue of our assumption that ν has bounded variation on bounded sets, it has at most a countable set of atoms. Hence both λ^n and λ^o are weak* continuous a.e on I. Thus $\lambda^n_t(\Gamma(t)) \longrightarrow \lambda^o_t(\Gamma(t))$ a.e on I. Then by upper semi continuity of Ψ we have

$$\limsup_{n\to\infty} \Psi(\lambda^n_t(\Gamma(t))) \le \Psi(\lambda^o_t(\Gamma(t))), \text{ a.e on } I.$$

Integrating this over the interval I, it follows from Fatou's Lemma that

$$\limsup_{n\to\infty} \int_I \Psi(\lambda^n_t(\Gamma(t)))dt \le \int_I \limsup_{n\to\infty} \Psi(\lambda^n_t(\Gamma(t)))dt \le \int_I \Psi(\lambda^o_t(\Gamma(t)))dt,$$

and hence

$$\limsup_{n\to\infty} J(\lambda^n) \le J(\lambda^o).$$

Thus J is weak* upper semi continuous on $\Lambda(F, G, \pi_0)$. Since this set is weak* sequentially compact, J attains its maximum at $\lambda^o \in \Lambda(F, G, \pi_0)$. This completes the proof. □

Remark 4.1. Note that maximizing the functional (15) can be interpreted in physical terms as maximizing the probability of close pursuit of the target $\Gamma(t)$ during the entire time interval I. In case $\Gamma(t)$ is considered as a pursuer, one wants to evade it and the functional (15) must be minimized. In this case Ψ is assumed to be lower semi continuous and bounded away from $-\infty$. This leads to the conclusion that the functional (15) attains its minimum.

5. Applications to optimal control

Let Ξ be a real separable Banach space and $f : I \times E \times \Xi \longrightarrow E$ be a Borel measurable map. Throughout the rest of this section, the vector field f is assumed to satisfy the following regularity condition:

(**A3**:) The vector field $f : I \times E \times \Xi \longrightarrow E$ is Borel measurable in the first two variables and continuous in the third argument.

Consider the control system

$$dx(t) = Ax(t)dt + f(t, x(t), u)dt + g(t, x(t-))\nu(dt), x(0) = x_0, \qquad (16)$$

with the control $u \in \mathcal{U}_{ad}$ where $\mathcal{U}_{ad}$ denotes the class of admissible controls as defined below.

Admissible Controls. Let Ξ be a separable Banach space and $BM(I \times E, \Xi)$ the vector space of bounded Borel measurable functions defined on $I \times E$ and taking values from the Banach space Ξ. Furnished with the sup norm topology, this is a Banach space. We give this space a weaker topology. Let τ_{wu} denote the topology of weak convergence in Ξ, uniformly on compact subsets of $I \times E$. That is, a sequence $\{u^n\}$ from $BM(I \times E, \Xi)$ is τ_{wu} convergent to $u^o \in BM(I \times E, \Xi)$ written, $u^n \xrightarrow{\tau_{wu}} u^o$, if and only if

$$\lim_{n\to\infty} (\eta(t, \xi), u^n(t, \xi) - u^o(t, \xi))_{\Xi^*,\Xi} = 0$$

uniformly on $I \times E$, for every $\eta \in BM(I \times E, \Xi^*)$ having compact support in $I \times E$. Let $\mathcal{U}$ be a weakly compact subset of Ξ and $U : I \times E \longrightarrow cc(\mathcal{U})$ a Borel measurable multi function with nonempty closed convex values. For admissible controls we take the set $\mathcal{U}_{ad}$ given by the family of Borel measurable selections of the multi function U,

$$\mathcal{U}_{ad} \equiv \big\{u \in BM(I \times E, \Xi) : u(t, x) \in U(t, x) \;\forall\, (t, x) \in I \times E\big\}. \qquad (17)$$

Note that by virtue of Kuratowski–Ryll Nardzewski selection theorem [10, Theorem 2.1, p. 154] this is a nonempty set. We consider $\mathcal{U}_{ad}$ to be endowed with the relative τ_{wu} topology.

Let λ^u denote the measure solution of equation (16) corresponding to an admissible control $u \in \mathcal{U}_{ad}$. The first problem we consider is as described below.

Problem P1. The problem is to find a control that maximizes the probability of closely following a moving target, $\Gamma(t), t \in I$. In other words find u that maximizes the functional

$$J(u) = \int_I \Psi(\lambda_t^u(\Gamma(t)))dt, \tag{18}$$

where Ψ is any nonnegative nondecreasing real valued function.

Define the multi function

$$F(t,x) \equiv f(t,x,U(t,x)), \quad (t,x) \in I \times E.$$

Since f satisfies assumption (**A3**) and U is a measurable multi function, the composition map F is a measurable multi function [15, Theorem 8.2.8, p. 314]. Clearly, the system (16) can be considered as a particular realization of the differential inclusion

$$dx \in Axdt + F(t,x)dt + g(t,x-)\nu(dt), x(0) = x_0. \tag{19}$$

Since $\{F,g\}$ are only measurable maps, in view of (14), equation (19) should be recast as a differential inclusion on the space $\mathcal{M}_{ba}(E)$ as follows

$$d\lambda_t \in \mathcal{A}_F^*(t)\lambda_t dt + \mathcal{C}_g^*(t)\lambda_{t-}\nu(dt), \ \pi_0 = \delta_{x_0}, \quad t \in I, \tag{20}$$

where $\mathcal{A}_F \equiv \{\mathcal{A}_f, f \in \mathcal{S}_F\}$ with $\mathcal{S}_F$ denoting the set of measurable selections of the multi function F.

Theorem 5.1. *Suppose $\{A,F,\nu\}$ satisfy the assumptions of Theorem 4.1, $\{\Gamma,\Psi\}$ satisfy the assumptions of Corollary 4.2, and g is a single valued locally bounded Borel measurable map and $\mathcal{L}(x_0) = \pi_0 \in \Pi_{ba}(E)$. Then the optimal control problem (P1) with the objective functional* (18) *has a solution.*

Proof. Note that the functional (18) is equivalent to (15) written as

$$\tilde{J}(\lambda) \equiv \int_I \Psi(\lambda_t(\Gamma(t)))dt \tag{21}$$

for $\lambda \in \Lambda(F,g,\pi_0)$. Thus it follows from Corollary 4.2 that there exists a $\lambda^o \in \Lambda(F,g,\pi_0)$ at which (21) attains its maximum. Letting $\mathcal{S}_F$ denote the Borel measurable selections of the multi function F, we conclude that there exists an $f^o \in \mathcal{S}_F$ so that $\lambda^o = \lambda(f^o, g)$. Clearly the multi function H given by

$$H(t,x) \equiv \{v \in U(t,x) : f^o(t,x) = f(t,x,v)\}, \quad (t,x) \in I \times E$$

is nonempty and it is also measurable since $f : I \times E \times \Xi \longrightarrow E$ satisfies assumption (**A3**) and the multi function U is assumed to be measurable with values in $cc(E)$. Thus the graph of the multi function H given by

$$\mathcal{G}_r(H) \equiv \{(t,\xi,v) \in \mathcal{G}_r(U) : f^o(t,\xi) = f(t,\xi,v)\}$$

is an element of $\mathcal{B}_{I\times E} \times \mathcal{B}_\Xi$. Then it follows from Yankov–Von Neumann–Aumann [10, Theorem 2.14, p. 158] selection theorem that there exists a measurable selection u^o of H such that $u^o(t,x) \in H(t,x)$ for all $(t,x) \in I \times E$. Hence $f^o(t,x) = f(t,x,u^o(t,x))$ for all $(t,x) \in I \times E$. Thus we have proved that $J(u^o) = \tilde{J}(\lambda^o)$. This completes the proof. □

Another interesting problem, similar to (P1) arises in applications requiring obstacle avoidance strategies.

Problem P2. Let $\Gamma(t) \subset E$ be a nonempty possibly closed bounded measurable set valued function. The concern is to stay away from this obstacle as far as possible. The problem can be formulated as follows: Let $d(\xi, \Gamma(t))$ denote the distance of ξ from the set $\Gamma(t)$ given by

$$d(\xi, \Gamma(t)) \equiv \inf_{\gamma \in \Gamma(t)} \|\xi - \gamma\|_E.$$

For any positive number r, define the multi function

$$\mathcal{Q}_r(t) \equiv \{\xi \in E : d(\xi, \Gamma(t)) > r\}, \quad t \in I,$$

and, for any Borel measurable set $K \subset E$, let χ_K denote the characteristic function of the set K. The problem is to find a control that maximizes the functional

$$J(u) \equiv \int_{I\times E} \chi_{Q_r(t)}(\xi)\ \lambda_t^u(d\xi)dt. \tag{22}$$

Maximizing this functional is equivalent to avoiding r-neighborhood of the obstacle. In this regard we have the following result.

Theorem 5.2. *Suppose $\{A, F, g, \nu\}$ satisfy the assumptions of Theorem 5.1. Let $t \longrightarrow \Gamma(t)$ be a nonempty measurable set valued function with values in $c(E)$. Then there exists an optimal control $u^o \in \mathcal{U}_{ad}$ that maximizes the functional* (22).

PROOF. Measurability of the multi function Γ implies measurability of the function $t \longrightarrow d(\xi, \Gamma(t))$. This in turn implies that Q_r is also a measurable multi function. Thus the function $\psi(t,\xi) \equiv \chi_{Q_r(t)}(\xi)$ is a Borel measurable function and it belongs to $L_1(I, B(E))$. Hence the functional C defined by,

$$C(\lambda) \equiv \int_{I\times E} \psi(t,\xi)\lambda_t(d\xi)dt,$$

is a weak* continuous bounded linear functional on $L^w_\infty(I, \mathcal{M}_{ba}(E))$. This is precisely the cost functional given by the expression (22) with λ^u replaced by $\lambda \in L^w_\infty(I, \mathcal{M}_{ba}(E))$. Since $\Lambda(F, g, \pi_0)$ is a weak* compact subset of $L^w_\infty(I, \Pi_{ba}(E))$, the functional C as defined above attains its maximum at some point $\lambda^o \in \Lambda(F, g, \pi_0)$. The rest of the proof based on measurable selection theorem is entirely similar to that of the preceding theorem. This completes our proof. □

Next we consider the Bolza problem.

Problem (P3). Consider the control system

$$dx = Axdt + f(t,x)dt + K(t,x)u(t,x)dt + g(t,x)\nu(dt) \tag{23}$$

with the cost functional given by

$$J(u) = \int_0^T \{\ell_0(t,x(t)) + \rho(t,x(t))|u(t,x(t))|_\Xi\}dt + \Psi(x(T)), \tag{24}$$

where $\{A, f, g, \nu\}$ are as before, $K \in BM(I \times E, \mathcal{L}(\Xi, E))$ and the control $u \in BM(I \times E, \Xi)$. The objective is to find a control u that minimizes the functional (24). Clearly this is how the original problem would be stated though equation (23) may have no solution in the classical sense (weak, mild, strong). This is because we admit measurable (non smooth) vector fields and so (23) may not posses any E-valued path wise solutions x. Thus again the appropriate formulation of this problem is: find an admissible control that minimizes the functional

$$J(u) \equiv \int_{I\times E} \{\ell_0(t,\xi) + \rho(t,\xi)|u(t,\xi)|_\Xi\}\lambda^u_t(d\xi)dt + \lambda^u_T(\Psi), \tag{25}$$

where λ^u is the measure solution of equation (23) or equivalently the weak solution of the evolution equation

$$d\lambda_t = \mathcal{A}^*(t)\lambda_t dt + \mathcal{B}^*_u(t)\lambda_t dt + \mathcal{C}^*(t)\lambda_{t-}\nu(dt), \quad \lambda_0 = \pi_0 \tag{26}$$

where the operators $\mathcal{A}$ and $\mathcal{C}$ are as given by the expressions (4) and (5) respectively and the operator $\mathcal{B}_u$ is given by

$$(\mathcal{B}_u\varphi)(t,\xi) \equiv \langle K^*(t,\xi)D\varphi(\xi), u(t,\xi)\rangle_{\Xi^*,\Xi}.$$

For admissible controls, we choose a subset $\mathcal{U}_{ad}$ of the space $BM(I\times E,\Xi)$ which is endowed with the τ_{wu} topology as described above. We prove the following result.

Theorem 5.3. *Consider the control problem* (P3) *and suppose that* $\mathcal{U}_{ad}$ *is a* τ_{wu} *compact subset of* $BM(I\times E,\Xi)$ *and the dual* Ξ^* *is a uniformly convex Banach space. Suppose* $\{A,f,g,\nu\}$ *satisfy the assumptions of Theorem 2.2 with* $T \notin a(\nu)$ *and* $K : I\times E \longrightarrow \mathcal{L}(\Xi,E)$ *is a bounded Borel measurable map. The function* ℓ_0 *is Borel measurable on* $I\times E$ *bounded from below by an integrable function and* ρ *is a nonnegative function having bounded support in* E *and* Ψ *is a Borel measurable function bounded on bounded sets satisfying* $\Psi(\xi) > -\infty$ *on* E. *Then there exists an optimal control for the problem* (P3).

Proof. Consider the cost functional $J(u)$ given by (25). If $J(u) \equiv +\infty$ (identically) there is nothing to prove. So we may assume the contrary and, without loss of generality, we may assume that it is bounded on all of $\mathcal{U}_{ad}$. We prove that J is τ_{wu} lower semi continuous on $\mathcal{U}_{ad}$. Let $u^n \xrightarrow{\tau_{wu}} u^o$. Then by virtue of the assumption on the operator valued function K, it can be shown that the weak solution $\{\lambda^n\}$, corresponding to the sequence of controls $\{u^n\}$ of the Cauchy problem (26), has a weak* convergent subnet having the limit λ^o which is the unique weak solution of (26) corresponding to the control u^o. Clearly by definition the corresponding cost functional is given by

$$J(u^o) \equiv \int_{I\times E} \big\{\ell_0(t,\xi) + \rho(t,\xi)|u^o(t,\xi)|_\Xi\big\}\lambda_t^o(d\xi)dt + \lambda_T^o(\Psi). \tag{27}$$

Let $\beta : \Xi \longrightarrow \Xi^*$ denote the normalized duality map, that is, for each $z \in \Xi$,

$$\beta(z) \equiv \{z^* \in \Xi^* : (z^*,z) = |z|_\Xi\}.$$

By Hahn–Banach theorem this set is nonempty. Since Ξ^* is uniformly convex, the duality map is single valued and uniformly continuous. Using this fact we write

$$|u^o(t,\xi)|_\Xi = (\beta(u^o(t,\xi)), u^o(t,\xi)) \equiv (\eta^o(t,\xi), u^o(t,\xi)),$$

where we have defined $\eta^o \equiv \beta(u^o)$. Since β is continuous and u^o is a Borel measurable Ξ valued function, $\eta^o(t,\xi)$ is a bounded Ξ^* valued Borel measurable function on $I \times E$ satisfying $|\eta^o(t,\xi)|_{\Xi^*} = 1$ for all $(t,\xi) \in I \times E$. Using the function η^o in equation (27), it is easy to verify that for all $n \in N$,

$$J(u^o) \leq I_{1,n} + I_{2,n} + J(u^n). \tag{28}$$

where

$$I_{1,n} \equiv \int_{I\times E} \big(\ell_0(t,\xi) + \rho(t,\xi)|u^o(t,\xi)|_{\Xi}\big)(\lambda_t^o(d\xi) - \lambda_t^n(d\xi))dt + \lambda_T^o(\Psi) - \lambda_T^n(\Psi),$$

$$I_{2,n} \equiv \int_{I\times E} \big\{\rho(t,\xi)(\eta^o(t,\xi), u^o(t,\xi) - u^n(t,\xi))\big\}\lambda_t^n(d\xi)dt.$$

Since λ^n converges to λ^o in the weak* topology of

$$L_\infty^w(\ell, \mathcal{M}_{ba}(E)) \cap L_\infty^w(|\nu|, \mathcal{M}_{ba}(E)),$$

the first component of $I_{1,n}$ converges to zero as $n \longrightarrow \infty$. Recall that by our assumption (A1) of Section 2, ν is a bounded signed measure having bounded variation on bounded sets. Thus ν can have at most a countable set of atoms on I and hence the measure solution is piece wise weak* continuous on I with at most a countable set of discontinuities arising from the atoms $a(\nu)$ of the measure ν. By our assumption $T \notin a(\nu)$ and hence $\{\lambda_T^n, \lambda_T^o\}$ are well defined and $\lambda_T^n(\Psi) \longrightarrow \lambda_T^o(\Psi)$ also. Thus we have $\lim_{n\to\infty} I_{1,n} = 0$. Recalling that ρ has compact support, and $u^n \xrightarrow{\tau_{wu}} u^o$, it follows from Lebesgue dominated convergence theorem that the integrand of the second term $I_{2,n}$ converges strongly in $L_1(I, B(E))$ while λ^n converges in the weak* topology of its dual. Hence $\lim_{n\to\infty} I_{2,n} = 0$ also. Thus it follows from (28) that

$$J(u^o) \leq \liminf_{n\to\infty} J(u^n),$$

proving that J is τ_{wu} lower semi continuous on $\mathcal{U}_{ad}$. Since $\mathcal{U}_{ad}$ is τ_{wu} compact, J attains its minimum on $\mathcal{U}_{ad}$. This proves the existence of an optimal control. □

Remark 5.1. The assumption of uniform convexity of the Banach space Ξ^* is probably not necessary. All that is required is the existence of a measurable selection of the multi function $(t,\xi) \longrightarrow \beta(u^o(t,\xi))$. It is easy to verify that the multi function $\beta : \Xi \longrightarrow 2^{B_1(\Xi^*)} \setminus \emptyset$ is monotone with values from the class of nonempty weak* compact convex subsets of the unit ball $B_1(\Xi^*)$ of the dual Ξ^*.

Remark 5.2. The assumption of ρ having compact support on $I \times E$ seems to be unnatural. This is very much interwound with the topology of the space of admissible controls. Thus if one is weakened the other must be strengthened. For example, if one can accept a stronger topology for the space of controls requiring weak convergence in Ξ on bounded subsets of $I \times E$, then ρ can be chosen as any nonnegative (uniformly) bounded measurable function on $I \times E$.

Control of Uncertain Systems. So far we have not exploited the full potential of Theorem 4.1 in applications. We consider this partially below. Consider the control system:

$$dx = Axdt + f(t, x, u)dt + g(t)\nu(dt), \quad x(0) = x_0, \tag{29}$$

subject to uncertain impulsive forces. The uncertainty arises from the fact that the function g is unknown. However it is known that it takes values from the set of uncertainty determined by the multi function G with values $G(t, x(t-))$ which may depend on current time and state. In other words the intensity of impulsive forces are both time and state dependent. We will assume that $G : I \times E \longrightarrow kc(E)$, the class of nonempty compact convex subsets of E. The system (29) can be formulated as a differential inclusion as follows:

$$dx \in Axdt + f(t, x, u)dt + G(t, x)\nu(dt), \quad x(0) = x_0. \tag{30}$$

Our problem is to control this system in the presence of uncertainty. Adopting the pessimistic viewpoint one tries to minimize the maximum risk or losses. Thus the optimal control problem may be stated as follows:

Problem P4. Let $L : I \times E \times \Xi \longrightarrow R$ and $\Psi : E \longrightarrow R$ be Borel measurable maps satisfying certain properties stated later. The problem is to find u^o that minimizes the functional

$$J_0(u) = \sup\left\{\int_I L(t, x(t), u)dt + \Psi(x(T)) : x \in \mathcal{X}(u, x_0)\right\}, \tag{31}$$

where $\mathcal{X}(u, x_0)$ denotes the family of solutions of the differential inclusion (30) for a given control policy u and initial state x_0. In case there is also uncertainty in the initial state and we know the range of values it may assume, giving $x_0 \in X_0 \subset E$, the supremum is taken over the set $\mathcal{X}(u, X_0) \equiv \{\mathcal{X}(u, \xi) : \xi \in X_0\}$. This is how the problem would be stated if the vector fields $\{f, g\}$ were at least locally Lipschitz. In our formulation, we do not make such regularity assumptions. Thus we must use measure formulation as we have done above.

Suppose we know the distribution of the initial state given by $\mathcal{L}(x_0) = \pi_0$. For the contingent function F we take

$$F(t, \xi) \equiv f(t, \xi, U(t, \xi)), \quad (t, \xi) \in I \times E. \tag{32}$$

Accordingly we reformulate the control problem as follows. The system is governed by the evolution inclusion

$$d\lambda_t \in \mathcal{A}_F^*(t)\lambda_t dt + \mathcal{C}_G^*(t)\lambda_{t-}\nu(dt),$$
$$\lambda_0 = \pi_0, \quad t \in I, \tag{33}$$

on the Banach space $\mathcal{M}_{ba}(E)$ where $\mathcal{A}_F \equiv \{\mathcal{A}_f, f \in \mathcal{S}_F\}$, $\mathcal{C}_G \equiv \{\mathcal{C}_g, g \in \mathcal{S}_G\}$ with $\mathcal{S}_F$, $\mathcal{S}_G$ denoting the class of Borel measurable selections of the multi functions F and G respectively. The objective functional (31) is replaced by the following functional

$$J_0(u) = \sup\Big\{ \int_{I\times E} L(t,\xi,u(t,\xi))\lambda_t(d\xi)dt + \int_E \Psi(\xi)\lambda_T(d\xi),\ \lambda \in \Lambda(f^u, G, \pi_0)\Big\} \tag{34}$$

where $\Lambda(f^u, G, \pi_0)$ denotes the set of measure solutions of the system (33) corresponding to a choice of an admissible control u and hence a measurable selection $f^u \equiv f(\cdot,\cdot,u(\cdot,\cdot))$ of F. Given that $T \notin a(\nu)$, there is no particular difficulty dealing with the terminal cost. Hence we consider only the Lagrange problem

$$J_0(u) = \sup\Big\{C(u,\lambda) \equiv \int_{I\times E} L(t,\xi,u(t,\xi))\lambda_t(d\xi)dt,\ \lambda \in \Lambda(f^u, G, \pi_0)\Big\}. \tag{35}$$

Our objective is to prove the existence of an optimal control for this problem.

Theorem 5.4. *Suppose $\{A, F, G, \pi_0\}$ satisfy the assumptions of Theorem 4.1, the cost integrand $L : I \times E \times \Xi \longrightarrow R$ is Borel measurable and bounded on bounded sets, and the map*

$$v \longrightarrow (L(t,\xi,v), f(t,\xi,v))$$

is continuous from Ξ to $R \times E$ for every $(t,\xi) \in I \times E$, and that the set valued function

$$Q(t,\xi) \equiv \{(z,\eta) \in R \times E : z \geq L(t,\xi,v),\ \eta = f(t,\xi,v),\ v \in U(t,\xi)\} \tag{36}$$

defined on $I \times E$ is closed convex valued. Further, suppose there exists a $\psi \in L_1(I, B(E))$ such that $L(t,\xi,v) \geq \psi(t,\xi)$ for all $v \in U(t,\xi)$. Then there exists a control $u^o \in \mathcal{U}_{ad}$ such that $J_0(u^o) \leq J_0(u)$, $u \in \mathcal{U}_{ad}$.

PROOF. By theorem 4.1, for every $u \in \mathcal{U}_{ad}$ the set $\Lambda(f^u, G, \pi_0)$ is weak* compact. Hence for a fixed but arbitrary $u \in \mathcal{U}_{ad}$, the functional $\lambda \longrightarrow C(u, \lambda)$ given by the expression within the parenthesis of equation (35) attains its supremum on $\Lambda(f^u, G, \pi_0)$. Thus the functional $J_0(u)$ is well defined possibly taking values from the extended real line. Since $\psi \in L_1(I, B(E))$, we have $J_0(u) > -\infty$ for all $u \in \mathcal{U}_{ad}$. If $J_0(u) = +\infty$ for all $u \in \mathcal{U}_{ad}$, there is nothing to prove. So we may assume the contrary. In this case there exists an $m \in (-\infty, +\infty)$ such that

$$\inf\{J_0(u), u \in \mathcal{U}_{ad}\} = m.$$

We must prove the existence of at least one admissible control u^o at which $J_0(u^o) = m$. Let $\{u^n\} \subset \mathcal{U}_{ad}$ be a minimizing sequence so that

$$\lim_{n \to \infty} J_0(u^n) = m. \tag{37}$$

Corresponding to the sequence $u^n \in \mathcal{U}_{ad}$, define the sequence $\{\ell^n, f^n\}$ by $\{\ell^n(t,\xi) \equiv L(t, \xi, u^n(t, \xi))\}$, $\{f^n(t,\xi) \equiv f(t, \xi, u^n(t, \xi))\}$. Let $\lambda^n \in \Lambda(f^n, G, \pi_0) \subset \Lambda(F, G, \pi_0)$ be an element at which the function $\lambda \longrightarrow C(u^n, \lambda)$ attains its supremum. Since the set $\Lambda(f^n, G, \pi_0)$ is weak* compact, the supremum is attained for every $n \in N$. Thus by construction we have

$$J_0(u^n) = \int_{I \times E} \ell^n(t, \xi) \lambda_t^n(d\xi) dt \tag{38}$$

and

$$(\ell^n(t, \xi), f^n(t, \xi)) \in Q(t, \xi), \ (t, \xi) \in I \times E, \tag{39}$$

for all $n \in N$. Since L is bounded on bounded sets the function $\hat{\ell}(t, \xi) \equiv \sup\{L(t, \xi, v), \ v \in U(t, \xi)\}$ is well defined Borel measurable function on $I \times E$. Thus without any change of the original problem we may replace the multi function Q by $\hat{Q}$ given by

$$\begin{aligned} \hat{Q}(t, \xi) \equiv \big\{(z, \eta) \in R \times E : \hat{\ell}(t, \xi) \geq z \geq L(t, \xi, v), \\ \eta = f(t, \xi, v), \ v \in U(t, \xi)\big\}. \end{aligned} \tag{40}$$

By our assumption the multi function F is $wkc(E)$ valued and hence it follows from (40) that the multi function $\hat{Q}$ is also $wkc(R \times E)$ valued. By Theorem 4.1, $\Lambda(F, G, \pi_0)$ is weak* sequentially compact. Thus there exists a subnet of the net $\{\lambda^n\}$ and a corresponding subnet of the net $\{\ell^n, f^n\}$, relabeled as the original nets, and $\lambda^o \in \Lambda(F, G, \pi_0)$ and Borel measurable functions $\{\ell^o, f^o\}$ defined on $I \times E$ and taking values from $R \times E$ such that

$$\lambda^n \xrightarrow{w^*} \lambda^o \text{ in } L_\infty^w(I, \Pi_{ba}(E))$$

and

$$(\ell^n(t,\xi), f^n(t,\xi)) \xrightarrow{w} (\ell^o(t,\xi), f^o(t,\xi)) \text{ in } R \times E.$$

In fact this convergence is also uniform on compact subsets of $I \times E$. Since $Q(t,\xi)$ is closed convex valued we have

$$(\ell^o(t,\xi), f^o(t,\xi)) \in Q(t,\xi) \quad \text{for all } (t,\xi) \in I \times E.$$

Here we have used the well known result (Mazur's theorem) that states that a convex set in a locally convex topological vector space is weakly closed if and only if it is strongly (norm) closed. Define the set valued function B on $I \times E$ with values

$$B(t,\xi) \equiv \{v \in U(t,\xi) : \ell^o(t,\xi) \geq L(t,\xi,v),\ f^o(t,\xi) = f(t,\xi,v)\}.$$

By virtue of measurability of the defining functions, B is a nonempty measurable multi function. Further, for any fixed $(t,\xi) \in I \times E$, it follows from continuity of the map $v \longrightarrow (L(t,\xi,v), f(t,\xi,v))$ from Ξ to $R \times E$, that $B(t,\xi)$ has closed values in $\mathcal{U} \subset \Xi$. Thus again by Kuratowski–Ryll Nardzewski selection theorem there exists a measurable selection u^o of B such that

$$\ell^o(t,\xi) \geq L(t,\xi,u^o(t,\xi)) \text{ and } f^o(t,\xi) = f(t,\xi,u^o(t,\xi)) \,\forall\, (t,\xi) \in I \times E. \tag{41}$$

Since $\ell^n(t,\xi) \longrightarrow \ell^o(t,\xi)$ uniformly on compact subsets of $I \times E$, for any nonnegative bounded continuous function ρ defined on $I \times E$ having compact support we have $\ell^n \rho \xrightarrow{s} \ell^o \rho$ in $L_1(I, B(E))$. Thus for every such ρ

$$\int_{I \times E} (\ell^n(t,\xi)\rho(t,\xi))\lambda_t^n(d\xi)dt \longrightarrow \int_{I \times E} (\ell^o(t,\xi)\rho(t,\xi))\lambda_t^o(d\xi)dt.$$

Since this holds for every bounded continuous $\rho \geq 0$ having compact support, where the support may vary with the choice of ρ, and $|m| < \infty$, we have

$$m \equiv \lim_{n\to\infty} \int_{I \times E} \ell^n(t,\xi)\lambda_t^n(d\xi)dt = \int_{I \times E} \ell^o(t,\xi)\lambda_t^o(d\xi)dt.$$

From this and the first inequality of (41) we may conclude that

$$m \geq \int_{I \times E} \ell^o(t,\xi)\lambda_t^o(d\xi)dt \geq \int_{I \times E} L(t,\xi,u^o(t,\xi))\lambda_t^o(d\xi)dt \equiv J_0(u^o).$$

Since m is the infimum of $J_0(\cdot)$ on $\mathcal{U}_{ad}$ which is closed and $u^o \in \mathcal{U}_{ad}$, we have $J_0(u^o) \geq m$. Thus it follows from the above expression that $J_0(u^o) = m$ proving the existence of an optimal control. $\square$

Some Comments on Applications. The notion of measure solution is rather recent and has not yet captured the attention of the main stream control community. However we have seen some applications of Measure solutions in Fluid dynamics involving Navier–Stokes equation with boundary controls. Existence of optimal controls including necessary conditions of optimality have been developed in [16]. In fact it is believed that the notion of measure solution may have a significant impact on the study and control of hydro-dynamic turbulence. Concept of measure solution has also found application in the study of nonlinear conservation laws [9] and problems in fluid dynamic phase transitions [12]. We believe that many of the nonlinear wave equations and Schrödinger equations with potentials having polynomial or even exponential growth which do not admit any (strong, mild,weak) solution may posses global measure solutions.

Some Open Questions. [1]: It is not necessary to restrict ν to a (scalar valued) signed measure. In fact the results presented can be easily extended to vector measure ν with values in another Banach space, say, F. In that case g is an operator valued map

$$g : E \longrightarrow \mathcal{L}(F, E)$$

and the vector measure ν is a countably additive bounded vector measure having bounded total variation on I. This requires that g admits an approximating sequence $\{g_n\}$ such that $g_n \in \mathcal{L}(F, [D(A)])$ where $[D(A)] \subset E$ denotes the Banach space with respect to the norm topology induced by the graph norm, $\|\xi\|_{D(A)} \equiv \|\xi\|_E + \|A\xi\|_E$. [2]: For some applications it may be necessary to admit vector fields $\{f, g\}$ mapping (E, Σ_E) to $(E, \mathcal{B}_E)$, where Σ_E is an algebra rather than Borel algebra. It is not clear to the author if the selection theorems due to Kuratowski–Ryll Nardzewski or Yankov–Von Neumann–Aumann hold in this general case?

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