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Characteristic classes and Ehresmann connections for Legendrian foliations

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Abstract. We find a necessary and sufficient condition for the (local) projectability of a Legendrian foliation of an almost $\mathcal{S}$-manifold onto a Lagrangian foliation of a symplectic manifold. In the context of Legendrian foliations, this result will be used for proving a Darboux theorem and some results about primary and secondary characteristic classes. Finally we show that, under suitable assumptions, every Legendrian foliation admits an Ehresmann connection.

1. Introduction

In recent years Legendrian foliations have been studied by several authors ([9], [18], [23], [24]) and from various points of view. In [10] the author introduced a linear connection, called *bi-Legendrian*, which is canonically attached to any Legendrian foliation of an almost $\mathcal{S}$-manifold $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$ and satisfies very natural properties with respect both to the Legendrian foliation and to the almost $\mathcal{S}$-structure: it preserves the foliation, and the 1-forms η_α and the fundamental 2-form Φ are parallel with respect to this connection. In this paper we present some applications of bi-Legendrian connections in the context of Legendrian foliations. We start with the following question: is it possible to project (of course, locally) a Legendrian foliation of an almost $\mathcal{S}$-manifold onto a Lagrangian foliation of a symplectic manifold? We find a necessary and sufficient condition

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in order to have this projection using in the proof the property that the parallel transport of bi-Legendrian connections preserves the Legendrian foliation in question. Then we prove a Darboux theorem for Legendrian foliations. This theorem was already proved by M. Y. Pang ([24]) for the case $r = 1$; however our proof uses completely different methods which are valid also for the case $r > 1$. Another consequence of the (local) projectability of Legendrian foliations is the possibility of characterizing bi-Legendrian connections as the lifts of bi-Lagrangian connections. We prove this fact in § 4.2 and then we will use it for proving a theorem which ensures that a Legendrian foliation, under some assumptions, admits an Ehresmann connection. This result is the analogue of a theorem of R. Wolak ([32]) about Ehresmann connections for Lagrangian foliations. Finally we present a basic theory of characteristic classes for Legendrian foliations, following the standard Bott's methods ([7]). We prove vanishing theorem for primary characteristic classes and then we draw conclusions of these theorems for secondary characteristic classes in the framework of Lehmann's theory ([22] or [28]).

2. Preliminaries

2.1. Almost $\mathcal{S}$-structures. An f-structure on a smooth manifold M of dimension m is defined by a non-vanishing tensor field ϕ of type $(1,1)$ and constant rank $2n$ which satisfies $\phi^3 + \phi = 0$. It is well known that $T(M)$ splits into two complementary subbundles $\mathrm{Im}(\phi)$ and $\ker(\phi)$. When $2n < m$ and $\ker(\phi)$ is parallelizable we say that we have an f-structure with parallelizable kernel, briefly an *$f \cdot pk$-structure*. In this case there exist global sections $\xi_1, \ldots, \xi_r$ of $\ker(\phi)$ which, together with their dual 1-forms $\eta_1, \ldots, \eta_r$, satisfy

$$\phi^2 = -I + \sum_{\alpha=1}^{r} \eta_\alpha \otimes \xi_\alpha, \quad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta},$$

from which it follows that $\phi(\xi_\alpha) = 0$ and $\eta_\alpha \circ \phi = 0$ for all $\alpha \in \{1, \ldots, r\}$. Almost complex and almost contact structures are $f \cdot pk$-structures according as $r = 0$ and $r = 1$, respectively. It is known that, given an $f \cdot pk$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha)$, there exist Riemannian metrics g on M such that

$$g\left(\phi(V), \phi(W)\right) = g(V, W) - \sum_{\alpha=1}^{r} \eta_\alpha(V)\eta_\alpha(W) \tag{1}$$

for all $V, W \in \Gamma(T(M))$. If g is any metric satisfying (1) we say that $(\phi, \xi_\alpha, \eta_\alpha, g)$ is a *metric $f \cdot pk$-structure*. Therefore, the tangent bundle of a metric $f \cdot pk$-manifold splits as complementary orthogonal sum of its subbundles $\mathrm{Im}(\phi)$ and $\ker(\phi)$. We denote their respective differentiable distributions by $\mathcal{H}$ and E. Let Φ be the 2-form defined by $\Phi(V,W) = g(V, \phi(W))$. A metric $f \cdot pk$-manifold M^{2n+r} with structure $(\phi, \xi_\alpha, \eta_\alpha, g)$ is called an *almost $\mathcal{S}$-manifold* if $d\eta_1 = \cdots = d\eta_r = \Phi$. This definition reduces to that one of contact metric manifold for $r = 1$. In the following lemma we recall some properties of an almost $\mathcal{S}$-structure:

Lemma 2.1. *Let $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$ be an almost $\mathcal{S}$-manifold. Then we have:*

(i) $\Phi(W, \xi_\alpha) = 0$ *for all* $W \in \Gamma(T(M))$ *and* $\alpha \in \{1, \ldots, r\}$;

(ii) $\mathcal{H} = \bigcap_{\alpha=1}^{r} \ker(\eta_\alpha)$ *and* $(\mathcal{H}, \Phi)$ *is a symplectic vector bundle;*

(iii) $[\xi_\alpha, \xi_\beta] = 0$ *and* $[Z, \xi_\alpha] \in \Gamma(\mathcal{H})$ *for all* $Z \in \Gamma(\mathcal{H})$ *and* $\alpha, \beta \in \{1, \ldots, r\}$;

(iv) $\mathcal{L}_{\xi_\alpha} \eta_\beta = \mathcal{L}_{\xi_\alpha} d\eta_\beta = 0$, *for all* $\alpha, \beta \in \{1, \ldots, r\}$;

(v) *for each* $\alpha \in \{1, \ldots, r\}$, ξ_α *is a Killing vector field if and only if* $\mathcal{L}_{\xi_\alpha} \phi = 0$.

Note that from (iii) it follows easily that the distribution $E = \mathrm{span}\{\xi_1, \ldots, \xi_r\}$ is integrable, hence it defines a flat r-dimensional foliation $\mathcal{E}$ of the $(2n+r)$-dimensional almost $\mathcal{S}$-manifold M. On the contrary, the $2n$-dimensional distribution $\mathcal{H}$ is not integrable, since it can be seen that $\eta_1 \wedge \cdots \wedge \eta_r \wedge (d\eta_\alpha)^n \neq 0$. For the proofs of these properties and more details on almost $\mathcal{S}$-manifolds, good references are, for instance, [1], [8] and [13].

2.2. Legendrian foliations. Let $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$ be an almost $\mathcal{S}$-manifold. A n-dimensional distribution L on M is called *Legendrian* if L is a subbundle of $\mathcal{H}$ and $\Phi(X, X') = 0$ for any $X, X' \in \Gamma(L)$. When L is involutive, the foliation $\mathcal{F}$ determined by L is called a *Legendrian foliation*. We denote by $L^\perp$ the orthogonal bundle of L. Then, setting $Q = \mathcal{H} \cap L^\perp$, we obtain another Legendrian distribution on M such that $\phi(L) = Q$ and we get the orthogonal decomposition $T(M) = L \oplus Q \oplus E_1 \oplus \cdots \oplus E_r = L \oplus Q \oplus E$, where E_α denotes the line bundle generated by ξ_α. Such a Legendrian distribution $Q = \phi(L)$ is called the *conjugate* Legendrian distribution of L. In general Q is not involutive, even if L is; precisely, for any $Y, Y' \in \Gamma(Q)$, $[Y, Y'] \in \Gamma(\mathcal{H})$. In [9] there are some results which ensure the integrability of Q under the assumption that L is integrable. When both L and $Q = \phi(L)$ are integrable, we have an example of a *bi-Legendrian structure*, that is a pair of two complementary Legendrian foliations of M. A Legendrian foliation is said to be *flat* (respectively, *strongly flat*) if $\overline{\xi} := \sum_{\alpha=1}^{r} \xi_\alpha$ (respectively, each $\xi_1, \ldots, \xi_r$) is a foliate vector field, i.e. if $[X, \overline{\xi}] \in \Gamma(L)$ whenever $X \in \Gamma(L)$. In the sequel we will make use of the following lemma, proven in [9].

Lemma 2.2. *Let $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ be an almost $\mathcal{S}$-manifold such that each ξ_α is a Killing vector field and $\mathcal{F}$ a Legendrian foliation on M. If $\mathcal{F}$ is strongly flat then also its conjugate Legendrian distribution is strongly flat.*

In [10] it has been proven that, given a pair of two complementary Legendrian distributions (L, Q) on the almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$, there exists a unique linear connection ∇ on M such that:

$$
\begin{aligned}
&\text{(i)} \quad \nabla L \subset L,\ \nabla Q \subset Q \ \text{ and, for all } \alpha \in \{1, \dots, r\},\ \nabla E_\alpha \subset E_\alpha;\\
&\text{(ii)} \quad \nabla \Phi = 0;\\
&\text{(iii)} \quad T(X, Y) = 2\Phi(X, Y)\bar{\xi}, \ \text{ for all } X \in \Gamma(L), Y \in \Gamma(Q),\\
&\qquad\ \ T(V, \xi_\alpha) = [\xi_\alpha, V_L]_Q + [\xi_\alpha, V_Q]_L, \ \text{ for all } V \in \Gamma(T(M)),\ \alpha \in \{1, \dots, r\},
\end{aligned}
\tag{2}
$$

where T denotes the torsion tensor of ∇ and V_L and V_Q, respectively, are the projections of V onto the subbundles L and Q of $T(M)$. Such a connection is called the *bi-Legendrian connection* associated to the pair (L, Q). In particular, to any Legendrian foliation $\mathcal{F}$ of M there is a canonically attached bi-Legendrian connection corresponding to the pair (L, Q), where $L = T(\mathcal{F})$ and $Q = \phi(L)$ is the conjugate Legendrian distribution of $\mathcal{F}$, and the leaves of $\mathcal{F}$ are totally geodesic submanifolds of M with respect to ∇. We recall the definition of this connection:

$$
\begin{aligned}
\nabla_V X &:= H(V_L, X)_L + [V_Q, X]_L + [V_E, X]_L,\\
\nabla_V Y &:= H(V_Q, Y)_Q + [V_L, Y]_Q + [V_E, Y]_Q,\\
\nabla_V Z &:= V(\eta_\alpha(Z))\xi_\alpha,
\end{aligned}
$$

for all $V \in \Gamma(T(M))$, $X \in \Gamma(L)$, $Y \in \Gamma(Q)$ and $Z \in \Gamma(E)$, where H denotes the operator such that, for all $V, W \in \Gamma(T(M))$, $H(V, W)$ is the unique section of $\mathcal{H}$ satisfying $i_{H(V,W)}\Phi|_{\mathcal{H}} = (\mathcal{L}_V i_W \Phi)|_{\mathcal{H}}$. Furthermore, we have the following results on the curvature of ∇ ([10]):

Proposition 2.3. *Let L and Q be two complementary strongly flat Legendrian distributions on the almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$. Then the curvature tensor of the bi-Legendrian connection associated to (L, Q) satisfies*

$$R(V, \xi_\alpha) = 0$$

for all $\alpha \in \{1, \dots, r\}$ and for all $V \in \Gamma(T(M))$.

Proposition 2.4. *Let $(\mathcal{F}, \mathcal{G})$ be a strongly flat bi-Legendrian structure on the almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$. Then the corresponding bi-Legendrian connection is flat along the leaves of the foliations $\mathcal{F}$ and $\mathcal{G}$.*

3. Projections of Legendrian foliations

Unless otherwise stated, $(M, \phi, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, \ldots, r\}$, will denote an almost $\mathcal{S}$-manifold of dimension $2n+r$, N a $2n$-dimensional manifold and $f : M \longrightarrow N$ a submersion with connected fibers such that $\mathcal{E}$ coincides with the foliation determined by the fibers of f. Then we have this first result:

Proposition 3.1. *Under the assumptions and the notation above, the* 2-*form* Φ *projects to a symplectic* 2-*form* Ω *on* N.

Proof. Let X', $Y' \in \Gamma(T(N))$ and $X, Y \in \Gamma(T(M))$ be the basic vector fields f-related to X' and Y', respectively. Define

$$\Omega(X', Y') \circ f = \Phi(X, Y).$$

Note that the definition of Ω is well posed, i.e. $\Phi(X, Y)$ is constant on the fiber $f^{-1}(x)$, $x \in N$. In fact, by by Lemma 2.1, we get $\xi_\alpha(\Phi(X, Y)) - \Phi([\xi_\alpha, X], Y) - \Phi(X, [\xi_\alpha, Y]) = 0$, for all $\alpha \in \{1, \ldots, r\}$. Now, since X and Y are basic, $[\xi_\alpha, X]$ and $[\xi_\alpha, Y]$ are vertical vector fields, so $\Phi([\xi_\alpha, X], Y) = \Phi(X, [\xi_\alpha, Y]) = 0$ and we can conclude that $\xi_\alpha(\Phi(X, Y)) = 0$, for each $\alpha \in \{1, \ldots, r\}$. From the definition of Ω, we have that $\Phi = f^*(\Omega)$ and this implies that Ω is a symplectic form. Indeed, we have

$$f^*(d\Omega) = d(f^*(\Omega)) = d\Phi = 0$$

which implies $d\Omega = 0$. Moreover, $f^*(\Omega^n) = f^*(\Omega)^n = \Phi^n \neq 0$ and then $\Omega^n \neq 0$. □

Therefore the 2-form Φ projects to a symplectic 2-form Ω on N and (N, Ω) is a symplectic manifold. From the general theory of symplectic manifolds we know that there exist Riemannian metrics g' and almost complex structures J on N so that

$$g'(X', J(Y')) = \Omega(X', Y') \tag{3}$$

for every $X', Y' \in \Gamma(T(N))$. Now we want to find g' and J such that $f : (M, g) \longrightarrow (N, g')$ becomes a Riemannian submersion. We note that if such g' and J exist, then

$$J \circ f_* = f_* \circ \phi, \tag{4}$$

i.e. f is a (ϕ, J)-holomorphic map (cf. [13]). Indeed, take $X', Y' \in \Gamma(T(N))$, let $X, Y \in \Gamma(T(M))$ be the basic vector fields f-related to X' and Y', respectively, and call $Z \in \Gamma(T(M))$ the basic vector field f-related to $J(Y')$. Then, using $\Phi = f^*(\Omega)$ and $\eta_\alpha(X) = \eta_\alpha(Y) = 0$ we have $\Phi(X, \phi(Y)) = -g(X, Y) = -g'(X', Y') \circ f$

$= \Omega(X', J(Y')) \circ f = f^*(\Omega)(X,Y) = \Phi(X,Z)$, from which $Z = \phi(Y)$ and it follows

$$J(f_*(Y)) = J(Y') = f_*(Z) = f_*(\phi(Y)).$$

Note that if there exist a Riemannian metric g' and an almost complex structure J satisfying (3) and which make f a (ϕ, J)-holomorphic map, such g' and J are unique. Now we are going to construct such g' and J.

Proposition 3.2. *With the notation above, the following statements are equivalent:*

(a) *there exist a unique Riemannian metric g' and a unique almost complex structure J on N such that $\Omega(X',Y') = g'(X', J(Y'))$ for every $X', Y' \in \Gamma(T(N))$ and f is a (ϕ, J)-holomorphic Riemannian submersion;*

(b) *$\xi_1, \ldots, \xi_r$ are Killing vector fields.*

Moreover, if (a) *or, equivalently,* (b) *holds, then (N, J, g') is an almost Kählerian manifold.*

Proof. Note that, since $E = \mathrm{span}\{\xi_1, \ldots, \xi_r\}$, the condition (b) is equivalent to require that $\mathcal{E}$ is a Riemannian foliation, i.e. that f is a Riemannian submersion. Hence clearly (a) implies (b). For proving the vice versa we have only to define the almost complex structure J. Let $X' \in \Gamma(T(N))$ and let $X \in \Gamma(T(M))$ be the basic vector field f-related to X'. For any $p \in M$ we can define

$$J\big(X'_{f(p)}\big) = f_{*p}\big(\phi(X_p)\big).$$

since if p and q are points of M on the same fiber then we have

$$f_{*p}\big(\phi(X_p)\big) = f_{*q}\big(\phi(X_q)\big).$$

Namely, this is equivalent to require that $\phi|_{\mathcal{H}}$ is "foliated" with respect to the foliation $\mathcal{E}$, i.e. constant along the leaves of the foliation, and this follows from (b), applying (v) of Lemma 2.1. □

Now let $\mathcal{F}$ be a Legendrian foliation of $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$. We wonder whether $\mathcal{F}$ projects to a Lagrangian foliation $\mathcal{F}'$ on the symplectic manifold (N^{2n}, Ω) under the submersion f.

Theorem 3.3. *Let $f : M \longrightarrow N$ be a submersion from a $(2n+r)$-dimensional almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$, endowed with a Legendrian distribution L, onto a $2n$-dimensional symplectic manifold (N, Ω), such that $\Phi = f^*(\Omega)$ and the foliation determined by the fibers of f coincides with $\mathcal{E}$. Then the following statements are equivalent:*

(a) *L is a strongly flat Legendrian distribution, i.e. for all $\alpha \in \{1, \ldots, r\}$ ξ_α is an infinitesimal automorphism of the distribution L;*

(b) *L projects to a distribution L' on N under the submersion f.*

Moreover if (a) *or, equivalently,* (b) *holds, then L' is a Lagrangian distribution of the symplectic manifold (N, Ω), and L is integrable if and only if L' is integrable.*

Proof. Suppose that L projects to a distribution L' on N under the submersion f. Then there exists a local frame $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi_1, \ldots, \xi_r\}$, defined on an open subset $U \subset M$, where, for any $i \in \{1, \ldots, n\}$, $X_i \in \Gamma(L)$ are basic vector fields, and, for all $i \in \{1, \ldots, n\}$, $f_*([X_i, \xi_\alpha]) = 0$, since they are both vertical and horizontal vector fields by Lemma 2.1. Now, for a vector field $X \in \Gamma(L)$, locally written as $X|_U = \sum_{i=1}^n f_i X_i$, for some $f_i \in C^\infty(U)$, we have $[X, \xi_\alpha]|_U = -\sum_{i=1}^n \xi_\alpha(f) X_i \in \Gamma(L)$, so that L is strongly flat. Now we prove the converse. Take $x \in N$ and let $p \in M$ such that $x = f(p)$. Then we can define

$$L'_x := f_{*p}(L_p),$$

since the strongly flatness of L implies that for any $p, q \in M$ such that $f(p) = x = f(q)$, one has

$$f_{*p}(L_p) = f_{*q}(L_q). \tag{5}$$

Namely, let $\gamma : I \longrightarrow M$, I an open interval of $\mathbb{R}$ containing $[0, 1]$, be a vertical curve joining p with q, that is $\gamma(0) = p$, $\gamma(1) = q$ and, for all $t \in I$, $\gamma'(t) \in E_{\gamma(t)}$. Consider the parallel transport τ along γ with respect to the bi-Legendrian connection ∇ associated to (L, Q), where $Q = \phi(L)$. Then we prove that $f_{*q} \circ \tau = f_{*p}$ on L_p. Indeed let $v \in L_p$ and let $X : I \longrightarrow T(M)$ be the unique vector field along γ such that $\nabla_{\gamma'} X = 0$ and $X(0) = v$. Then $X(1) = \tau(v)$. Observe that, in fact, for all $t \in I$, $X(t) \in L_{\gamma(t)}$, since τ preserves the distribution L. Let Y' be any vector field on N and Y the corresponding basic vector field on M. Then we have

$$\begin{aligned}
\frac{d}{dt}\Big(\Omega\Big(f_{*\gamma(t)}\big(X(t), Y'_{f(\gamma(t))}\big)\Big)\Big) &= \frac{d}{dt}\big(\Phi(X(t), Y_{\gamma(t)})\big) \\
&= \Phi(\nabla_{\gamma'} X, Y)_{\gamma(t)} + \Phi(X, \nabla_{\gamma'} Y)_{\gamma(t)} \\
&= \Phi(X, \nabla_{\gamma'} Y_Q)_{\gamma(t)}
\end{aligned}$$

because X is ∇-parallel and the bi-Legendrian connection preserves the distributions L and Q. Now, since for all $t \in I$, $\gamma'(t) \in E_{\gamma(t)}$, for some functions a_α we have

$$\nabla_{\gamma'} Y_Q = \sum_{\alpha=1}^r a_\alpha \nabla_{\xi_\alpha} Y_Q = \sum_{\alpha=1}^r a_\alpha [\xi_\alpha, Y_Q]_Q$$

$$= \sum_{\alpha=1}^{r} a_\alpha \Big([\xi_\alpha, Y]_Q - [\xi_\alpha, Y_L]_Q - [\xi_\alpha, Y_E]_Q \Big) = \sum_{\alpha=1}^{r} a_\alpha [\xi_\alpha, Y]_Q = 0,$$

since L is strongly flat and Y is basic. Thus $\frac{d}{dt}(\Omega(f_{*\gamma(t)}(X(t)), Y'_{f(\gamma(t))})) = 0$ and this implies that $\Omega(f_{*p}(X(0)), Y'_x) = \Omega(f_{*q}(X(1)), Y'_x)$, from which $f_{*p}(v) = f_{*p}(\tau(v))$, Y' being arbitrary. Now, since $\tau(L_p) = L_q$, we have $f_{*p}(L_p) = f_{*q}(\tau(L_p)) = f_{*p}(L_p)$ and (5) holds. Therefore L projects to a subbundle L' of $T(N)$. Now we prove the last part of the theorem, i.e. that L' defines a foliation on N if and only if L is integrable. Indeed let $X', Y' \in \Gamma(L')$. Then there exist unique basic vector fields $X, Y \in \Gamma(L)$ such that $f_{*p}(X_p) = X'_{f(p)}$ and $f_{*p}(Y_p) = Y'_{f(p)}$ for all $p \in M$. Since for all $p \in M$, $[X', Y']_{f(p)} = f_{*p}([X, Y]_p)$, L is integrable if and only if L' is integrable. Finally, to conclude the proof it remains to prove that $L' = f_*(L)$ is a Lagrangian distribution of the symplectic manifold (N, Ω). First of all observe that, for all $p \in M$, $\dim(L'_{f(p)}) = \dim(L_p) = n = \frac{1}{2}\dim(T_{f(p)}N)$. Then, let $X', Y' \in \Gamma(L')$. Let $X, Y \in \Gamma(L)$ be the unique basic vector fields on M which project to X' and Y', respectively. Then, since L is a Legendrian distribution, we have $\Omega(X', Y') \circ f = \Phi(X, Y) = 0$, and L' is Lagrangian. $\square$

Corollary 3.4. *Let $f : M^{2n+r} \longrightarrow N^{2n}$ be a submersion from an almost $\mathcal{S}$-manifold $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$ onto a symplectic manifold (N^{2n}, Ω), such that $\Phi = f^*(\Omega)$ and the foliation determined by the fibers of f coincides with $\mathcal{E}$. Let (L, Q) be a pair of complementary Legendrian distributions on M (in particular a bi-Legendrian structure on M). Then the following statements are equivalent:*

- (a) *(L, Q) is strongly flat, that is each ξ_α is an infinitesimal automorphism with respect to both L and Q;*
- (b) *(L, Q) projects to a pair of transversal Lagrangian distributions on (N^{2n}, Ω) (in particular to a bi-Lagrangian structure on (N^{2n}, Ω)).*

Corollary 3.5. *Suppose that we are under the assumptions of Corollary 3.4 and, moreover, suppose that each ξ_α is a Killing vector field. Let $\mathcal{F}$ be a strongly flat Legendrian foliation of M and let $Q = \phi(L)$, $L = T(\mathcal{F})$, be the conjugate Legendrian distribution of $\mathcal{F}$. Then the pair (L, Q) projects to a pair (L', Q') of transversal Lagrangian distributions on N^{2n}.*

Proof. By Proposition 3.2, for all $p, q \in M^{2n+r}$, we have $f_{*p}(Q_p) = f_{*p}(\phi(L_p)) = J(f_{*p}(L_p)) = J(f_{*q}(L_q)) = f_{*q}(\phi(L_q)) = f_{*q}(Q_q)$ and Q projects to a Lagrangian distribution on (N^{2n}, Ω). $\square$

4. Applications

4.1. Darboux theorem for Legendrian foliations. It is known ([3]) that given an almost $\mathcal{S}$-manifold $(M^{2n+r},\phi,\xi_\alpha,\eta_\alpha,g)$ around each point of M it is possible to find coordinates $\{x'_1,\dots,x'_n,y'_1,\dots,y'_n,z'_1,\dots,z'_r\}$ such that, locally, $\eta_\alpha=dz'_\alpha-\sum_{k=1}^n y'_k dx'_k$ for all $\alpha\in\{1,\dots,r\}$. Moreover, if M is endowed with a Legendrian foliation $\mathcal{F}$, there exist local coordinates $\{x''_1,\dots,x''_n,y''_1,\dots,y''_n,z''_1,\dots,z''_r\}$ such that $\mathcal{F}$ is locally given by the equations $\{x''_i=\text{const.},z''_\alpha=\text{const.}\}$. These two kind of coordinate systems, in general, do not coincide, but they do if all the ξ_α's are foliate vector fields, as we prove now.

Theorem 4.1. *Let $\mathcal{F}$ be a strongly flat Legendrian foliation of an almost $\mathcal{S}$-manifold $(M^{2n+r},\phi,\xi_\alpha,\eta_\alpha,g)$. Then around each point of M there exist local coordinates $\{x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_r\}$ such that locally $\mathcal{F}$ is defined by the equations*

$$\{x_i=\text{const.},z_\alpha=\text{const.}\}$$

and, for all $\alpha\in\{1,\dots,r\}$, η_α are given by $\eta_\alpha=dz_\alpha-\sum_{i=1}^n y_i dx_i$.

Proof. Since $\mathcal{E}$ is a foliation, using the definition of foliations by means of cocycles, there exist a $2n$-dimensional manifold N and a cocycle $\{U_i,f_i,g_{ij}\}_{i,j\in I}$ modelled on N which define the foliation $\mathcal{E}$, that is: (i) $\{U_i\}_{i\in I}$ is an open covering of M, (ii) for all $i\in I$, $f_i:U_i\longrightarrow N$ are submersions with connected fibers which define $\mathcal{E}$, (iii) for all $i,j\in I$ such that $U_i\cap U_j\neq\emptyset$, $g_{ij}:f_j(U_i\cap U_j)\longrightarrow f_i(U_i\cap U_j)$ are local diffeomorphisms such that $g_{ij}\circ f_j=f_i$ on $U_i\cap U_j$. Now let $p\in M$. There exists $i\in I$ such that $p\in U_i$. From now on we work always on U_i, dropping, for simplicity of notation, the index to U_i and to the relative submersion $f_i:U_i\longrightarrow N$. Since $\mathcal{F}$ is strongly flat, we can use Theorem 3.3. Therefore the 2-form Φ projects to a symplectic 2-form Ω on N and $\mathcal{F}$ projects to a Lagrangian foliation $\mathcal{F}'$ on the symplectic manifold (N,Ω). Let $x=f(p)$. By a well-known theorem about Lagrangian foliations ([29]), around x there exist local coordinates $\{x'_1,\dots,x'_n,y'_1,\dots,y'_n\}$ such that $\mathcal{F}'$ is described by the equations $\{x'_i=\text{const.}\}$ and Ω is given by $\Omega=\sum_{k=1}^n dx'_k\wedge dy'_k$. As usual, let $L=T(\mathcal{F})$ and $L'=T(\mathcal{F}')$ be the tangent bundles of the foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. Consider each $\frac{\partial}{\partial y'_i}\in\Gamma(L')$, $i\in\{1,\dots,n\}$. There exist basic vector fields $Y_1,\dots,Y_n\in\Gamma(L)$ such that, for all $i\in\{1,\dots,n\}$, $f_*(Y_i)=\frac{\partial}{\partial y'_i}$. Analogously there exist basic vector fields $X_1,\dots,X_n$ such that $f_*(X_i)=\frac{\partial}{\partial x'_i}$, for all $i\in\{1,\dots,n\}$. Note that $[X_i,\xi_\alpha]=[Y_i,\xi_\alpha]=0$ because X_i and Y_i are basic vector fields. Moreover $f_*([Y_i,Y_j])=\left[\frac{\partial}{\partial y'_i},\frac{\partial}{\partial y'_j}\right]=0$, so $[Y_i,Y_j]$ is vertical. On the other hand, since L is integrable, $[Y_i,Y_j]\in\Gamma(L)$. Thus $[Y_i,Y_j]=0$. Finally, we examine the terms

$[X_i, Y_j]$ and $[X_i, X_j]$. For the latter we have $f_*([X_i, X_j]) = \left[\frac{\partial}{\partial x'_i}, \frac{\partial}{\partial x'_j}\right] = 0$, and, for all $\alpha \in \{1, \ldots, r\}$, since $\Phi = d\eta_\alpha$ and $\Phi = f^*(\Omega)$,

$$\eta_\alpha([X_i, X_j]) = -2\Omega\left(\frac{\partial}{\partial x'_i}, \frac{\partial}{\partial x'_j}\right) \circ f = -2\left(\sum_{k=1}^n dx'_k \wedge dy'_k\right)\left(\frac{\partial}{\partial x'_i}, \frac{\partial}{\partial x'_j}\right) \circ f = 0$$

so $[X_i, X_j]$ is both vertical and horizontal, hence vanishes. For $[X_i, Y_j]$ we have

$$f_*([X_i, Y_j]) = \left[\frac{\partial}{\partial x'_i}, \frac{\partial}{\partial y'_j}\right] = 0$$

and

$$\begin{aligned}\eta_\alpha([X_i, Y_j]) &= -2\Omega\left(\frac{\partial}{\partial x'_i}, \frac{\partial}{\partial y'_j}\right) \circ f \\ &= -2\left(\sum_{k=1}^n dx'_k \wedge dy'_k\right)\left(\frac{\partial}{\partial x'_i}, \frac{\partial}{\partial y'_j}\right) \circ f = -\delta_{ij}\end{aligned}$$

so $[X_i, Y_j] = -\sum_{\alpha=1}^r \delta_{ij}\xi_\alpha$. Thus, we have found $2n + r$ linearly independent vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi_1, \ldots, \xi_r$ such that

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, \xi_\alpha] = [Y_i, \xi_\alpha] = [\xi_\alpha, \xi_\beta] = 0,\ [X_i, Y_j] = -\delta_{ij}\sum_{\alpha=1}^r \xi_\alpha,$$

$i, j \in \{1, \ldots, n\}$, $\alpha, \beta \in \{1, \ldots, r\}$. Hence there exist coordinates $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_r\}$ such that

$$\xi_\alpha = \frac{\partial}{\partial z_\alpha},\ Y_i = \frac{\partial}{\partial y_i} \quad \text{and} \quad X_i = \frac{\partial}{\partial x_i} + y_i \sum_{\alpha=1}^r \frac{\partial}{\partial z_\alpha},$$

for all $\alpha \in \{1, \ldots, n\}$, $i \in \{1, \ldots, n\}$. Note that

$$(f^*(dy'_k))\left(\frac{\partial}{\partial x_i}\right) = dy'_k\left(f_*\left(X_i - y_i\sum_{\alpha=1}^r \xi_\alpha\right)\right) = dy'_k\left(\frac{\partial}{\partial x'_i}\right) = 0,$$

$$(f^*(dy'_k))\left(\frac{\partial}{\partial y_i}\right) = dy'_k\left(\frac{\partial}{\partial y'_i}\right) = \delta_{ki}$$

and

$$(f^*(dy'_k))\left(\frac{\partial}{\partial z_\alpha}\right) = dy'_k\left(f_*\left(\frac{\partial}{\partial z_\alpha}\right)\right) = 0,$$

so, for all $k \in \{1, \ldots, n\}$, $f^*(dy'_k) = dy_k$, and, analogously, we have $f^*(dx'_k) = dx_k$. Therefore

$$d\eta_\alpha = f^*(\Omega) = f^* \left(\sum_{k=1}^{n} dx'_k \wedge dy'_k \right) = \sum_{k=1}^{n} dx_k \wedge dy_k.$$

Thus, for all $\alpha \in \{1, \ldots, r\}$, $d\big(\eta_\alpha + \sum_{k=1}^{n} y_k dx_k\big) = 0$, hence $\eta_\alpha = dh_\alpha - \sum_{k=1}^{n} y_k dx_k$ for some functions h_α. Finally, $\eta_\alpha(X_i) = 0$, $\eta_\alpha\big(\frac{\partial}{\partial y_i}\big) = 0$, $\eta_\alpha\big(\frac{\partial}{\partial z_\beta}\big) = \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$ imply $\frac{\partial h_\alpha}{\partial x_i} = 0$, $\frac{\partial h_\alpha}{\partial y_i} = 0$ and $\frac{\partial h_\alpha}{\partial z_\beta} = \delta_{\alpha\beta}$, respectively, from which $dh_\alpha = dz_\alpha$ follows and so $\eta_\alpha = dz_\alpha - \sum_{k=1}^{n} y_k dx_k$. □

4.2. An interpretation of bi-Legendrian connections. Now we use the technique of projections of Legendrian foliations described in § 3 for showing another way of defining bi-Legendrian connections. As in Proposition 3.1 and Theorem 3.3, we suppose that there is a submersion $f : M \longrightarrow N$ from a $(2n+r)$-dimensional almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ to a $2n$-dimensional manifold N such that $\ker(f_{*p}) = E_p$ for all $p \in M$. Let (L, Q) be a pair of two complementary strongly flat Legendrian distributions on M. From Corollary 3.4 it follows that the 2-form Φ projects to a symplectic 2-form Ω and the Legendrian distributions L, Q onto the transversal Lagrangian distributions L', Q', respectively. Thus, according to [17], there exists a unique linear connection ∇' on N such that

1. $\nabla' L' \subset L'$, $\nabla' Q' \subset Q'$;
2. ∇' is symplectic, i.e. $\nabla'\Omega = 0$;
3. $T'(X', Y') = 0$ for $X' \in \Gamma(L')$ and $Y' \in \Gamma(Q')$,

where T' denotes the torsion tensor field of ∇'. The connection ∇' is called the *bi-Lagrangian connection* associated to the pair (L', Q'). Now we lift this connection to a linear connection ∇ on M^{2n+r} as follows. For any basic vector fields $X, Y \in \Gamma(T(M))$, f-related to $X', Y' \in \Gamma(T(N))$, respectively, we define $\nabla_X Y$ as the unique basic vector field on M f-related to $\nabla'_{X'} Y'$, i.e. the unique horizontal vector field such that

$$f_*(\nabla_X Y) = \nabla'_{f_*(X)} f_*(Y); \tag{6}$$

for each $\alpha \in \{1, \ldots, r\}$ and $W \in \Gamma(T(M))$, we put

$$\nabla \xi_\alpha = 0, \quad \nabla_{\xi_\alpha} W = [\xi_\alpha, W]. \tag{7}$$

In fact, (7) implies that, for any basic vector field $X \in \Gamma(T(M))$, $f_*(\nabla_X \xi_\alpha) = 0 = \nabla'_{f_*(X)} f_*(\xi_\alpha)$ and $f_*(\nabla_{\xi_\alpha} X) = f_*([\xi_\alpha, X]) = 0 = \nabla'_{f_*(\xi_\alpha)} f_*(X)$.

Proposition 4.2. *The above linear connection* ∇ *coincides with the bi-Legendrian connection associated to the pair of strongly flat bi-Legendrian distributions* (L, Q).

Proof. We have to verify the relations in (2). The first property follows directly by the definition of ∇. Indeed, clearly, from $\nabla\xi_\alpha = 0$ we have that $\nabla E_\alpha \subset E_\alpha$. Next, let $X \in \Gamma(L)$ and $Y \in \Gamma(Q)$. We can suppose that X and Y are basic vector fields f-related to $X' \in \Gamma(L')$ and $Y' \in \Gamma(Q')$ respectively. Since $\nabla' L' \subset L'$ and $\nabla' Q' \subset Q'$, we have $f_*(\nabla_Z X) = \nabla'_{f_*(Z)} f_*(X) = \nabla'_{f_*(Z)} X' \in \Gamma(L')$ and $f_*(\nabla_Z Y) = \nabla'_{f_*(Z)} f_*(Y) = \nabla'_{f_*(Z)} Y' \in \Gamma(Q')$, for any basic vector field Z on M. Thus $\nabla_Z X \in \Gamma(L)$ and $\nabla_Z Y \in \Gamma(Q)$. Finally, for all $\alpha \in \{1, \dots, r\}$, as L and Q are strongly flat, $\nabla_{\xi_\alpha} X = [\xi_\alpha, X] \in \Gamma(L)$ and $\nabla_{\xi_\alpha} Y = [\xi_\alpha, Y] \in \Gamma(Q)$ for all $X \in \Gamma(L)$ and $Y \in \Gamma(Q)$. So (i) is verified. Now we can prove (ii). First of all, for all $\alpha \in \{1, \dots, r\}$, for all $V, W \in \Gamma(T(M))$

$$\begin{aligned}(\nabla_{\xi_\alpha}\Phi)(V, W) &= \xi_\alpha(\Phi(V, W)) - \Phi([\xi_\alpha, V], W) - \Phi(V, [\xi_\alpha, W]) \\ &= (\mathcal{L}_{\xi_\alpha}\Phi)(V, W) = 0\end{aligned}$$

and $(\nabla_V \Phi)(W, \xi_\alpha) = V(\Phi(W, \xi_\alpha)) - \Phi(\nabla_V W, \xi_\alpha) - \Phi(W, \nabla_V \xi_\alpha) = 0$. It remains to check that $(\nabla_Z \Phi)(X, Y) = 0$ for all $X \in \Gamma(L)$, $Y \in \Gamma(Q)$ and $Z \in \Gamma(\mathcal{H})$. It is sufficient to prove the property for basic vector fields. Assume that X, Y, Z are basic vector fields f-related to $X' \in \Gamma(L')$, $Y' \in \Gamma(Q')$ and $Z' \in \Gamma(T(N))$, respectively. Then, since $\Phi = f^*(\Omega)$, we get easily

$$(\nabla_Z \Phi)(X, Y) = (\nabla'_{Z'}\Omega)(X', Y') \circ f = 0.$$

To verify (iii), let $V \in \Gamma(T(M))$. Then

$$T(V, \xi_\alpha) = -[\xi_\alpha, V] - [V, \xi_\alpha] = 0 = [\xi_\alpha, V_L]_Q + [\xi_\alpha, V_Q]_L$$

since L and Q are strongly flat. Finally, we prove that

$$T(X, Y) = 2\Phi(X, Y)\overline{\xi} \tag{8}$$

for $X \in \Gamma(L)$ and $Y \in \Gamma(Q)$. As T is a tensor field it is sufficient to prove (8) when X and Y are basic vector fields. Then, denoting by $X' \in \Gamma(L')$ and $Y' \in \Gamma(Q')$ the vector fields on N f-related to X and Y, respectively, we have

$$T(X, Y) = \nabla_X Y - \nabla_Y X - h[X, Y] - v[X, Y]$$

where $h[X,Y]$ and $v[X,Y]$ denote the horizontal and the vertical component of $[X,Y]$. Since $h[X,Y]$ is the basic vector field f-related to $[X',Y']$, we have

$$\begin{aligned} f_*(T(X,Y)) &= f_*(\nabla_X Y - \nabla_Y X - h[X,Y]) \\ &= \nabla'_{f_*(X)} f_*(Y) - \nabla'_{f_*(Y)} f_*(X) - [f_*(X), f_*(Y)] \\ &= \nabla'_{X'} Y' - \nabla'_{Y'} X' - [X',Y'] = T'(X',Y') = 0 \end{aligned}$$

by the definition of bi-Lagrangian connections. So $T(X,Y)$ is vertical. Since $\nabla\mathcal{H} \subset \mathcal{H}$, we get $T(X,Y) = -v[X,Y] = -\sum_{\alpha=1}^{n} \eta_\alpha([X,Y]) = \sum_{\alpha=1}^{n} 2d\eta_\alpha(X,Y)\xi_\alpha = 2\Phi(X,Y)\overline{\xi}$. □

In general, given an almost $\mathcal{S}$-manifold M^{2n+r}, as in Theorem 4.1, since $\mathcal{E}$ is a foliation, there exists a $2n$-dimensional manifold N and a cocycle $\{U_i, f_i, g_{ij}\}_{i,j\in I}$ modelled on N which define $\mathcal{E}$. So we can locally construct the connection ∇ as in Proposition 4.2, and then, using the properties of cocycles, define a global connection on M which, by Proposition 4.2, is just the bi-Legendrian connection associated to the given pair (L,Q) of strongly flat Legendrian distributions on M. In more detail, we have the following

Theorem 4.3. *Let (L,Q) be a pair of strongly flat Legendrian distributions on the almost $\mathcal{S}$-manifold $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$ and $\{U_i, f_i, g_{ij}\}_{i,j\in I}$ a cocycle, modelled on a $2n$-dimensional manifold N, defining the foliation $\mathcal{E}$. Then (L,Q) projects to a pair of transversal Lagrangian distributions on N and the corresponding bi-Lagrangian connection lifts to the bi-Legendrian connection on M associated to (L,Q).*

Proof. From Corollary 3.4 it follows that, for each $i \in I$, the 2-form Φ projects to a symplectic 2-form Ω_i on N, and the pair of Legendrian distributions (L,Q) projects to a pair of transversal Lagrangian distributions (L'_i, Q'_i), under the submersion $f_i : U_i \longrightarrow N$. Note that for $i,j \in I$ such that $U_i \cap U_j \neq \emptyset$ the corresponding projections Ω_i and Ω_j of Φ satisfy

$$f_j^*(\Omega_j) = \Phi = f_i^*(\Omega_i) = (g_{ij} \circ f_j)^*(\Omega_i) = f^*(g_{ij}^*(\Omega_i))$$

from which, since f_j is a submersion, we get $\Omega_j = g_{ij}^*(\Omega_i)$, so g_{ij} are local symplectomorphisms between the symplectic manifolds (N,Ω_i) and (N,Ω_j). Moreover,

$$L'_i = f_{i*}(L) = g_{ij*}(f_{j*}(L)) = g_{ij*}(L'_j)$$

and

$$Q'_i = f_{i*}(Q) = g_{ij*}(f_{j*}(Q)) = g_{ij*}(Q'_j),$$

that is (L'_i, Q'_i) and (L'_j, Q'_j) are locally equivalent. Now, for each $i \in I$, let $\nabla'^{(i)}$ denote the bi-Lagrangian connection associated to (L'_i, Q'_i). Then we claim that g_{ij} are affine (local) diffeomorphisms between $(N, \nabla'^{(j)})$ and $(N, \nabla'^{(i)})$, that is

$$g_{ij*}\left(\nabla'^{(j)}_{X'} Y'\right) = \nabla'^{(i)}_{g_{ij*}(X')} g_{ij*}(Y').$$

Indeed, we define a connection $\widetilde{\nabla}$ on $f_j(U_i \cap U_j)$ as the pull-back of $\nabla'^{(i)}$ by means of the diffeomorphism $g_{ij} : f_j(U_i \cap U_j) \longrightarrow f_i(U_i \cap U_j)$, so setting

$$g_{ij*}\left(\widetilde{\nabla}_{X'} Y'\right) = \nabla'^{(i)}_{g_{ij*}(X')} g_{ij*}(Y').$$

If we prove that $\widetilde{\nabla}$ satisfies all the properties which characterize the bi-Lagrangian connection associated to (L'_j, Q'_j), then, from the uniqueness of such a connection, we conclude that $\widetilde{\nabla} = \nabla'_j$ and so g_{ij} are affine. So, first of all, $\widetilde{\nabla}\Omega_j = 0$, because g_{ij} are affine symplectomorphisms. Moreover, as $\nabla'^{(i)}$ preserves L'_i and $L'_i = g_{ij*}(L'_j)$, we have $g_{ij*}(\widetilde{\nabla} L'_j) = \nabla'^{(i)}(g_{ij*}(L'_j)) = \nabla'^{(i)} L'_i \subset L'^{(i)}$, from which it follows that $\widetilde{\nabla} L'_j \subset L'_j$. Analogously $\widetilde{\nabla} Q'_j \subset Q'_j$. Finally, we prove that $\widetilde{T}(X', Y') = 0$ for all $X' \in \Gamma(L'_j)$ and $Y' \in \Gamma(Q'_j)$. Indeed we get

$$g_{ij*}(\widetilde{T}(X', Y')) = T'^{(i)}(g_{ij*}(X'), g_{ij*}(Y')) = 0$$

because $g_{ij*}(X') \in \Gamma(L'_i)$ and $g_{ij*}(Y') \in \Gamma(L'_j)$. The last step is to lift the bi-Lagrangian connection ∇' on N to a connection ∇ on M. We can do it locally, on each $U_i \subset M$, using the submersion $f_i : U_i \longrightarrow N$. Let X_i, Y_i be basic vector fields on U_i f_i-related to $X', Y' \in \Gamma(T(N))$, respectively; then we define $\nabla^i_{X_i} Y_i$ as the unique vector field on U_i such that $f_{i*}(\nabla^i_{X_i} Y_i) = \nabla'^{(i)}_{f_{i*}(X_i)} f_{i*}(Y_i)$ and we check that this definition gives rise to a global connection ∇ on M. Indeed, consider $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$. On $U_i \cap U_j$ we have the connections ∇^i and ∇^j, defined by the formulas

$$f_{i*}(\nabla^i_{X_i} Y_i) = \nabla'^{(i)}_{f_{i*}(X_i)} f_{i*}(Y_i), \;\; f_{j*}(\nabla^j_{X_j} Y_j) = \nabla'^{(j)}_{f_{j*}(X_j)} f_{j*}(Y_j).$$

We prove that, on $U_i \cap U_j$, $\nabla^i = \nabla^j$. Note that if X_i is the basic vector field f_i-related to X', then X_i is also the basic vector field f_j-related to $g_{ij*}(X')$. In fact X_i is horizontal also for f_j, as $\ker(f_{i*}) = E = \ker(f_{j*})$, and

$$(f_{j*})_p((X_i)_p) = (g_{ij*})_{f_j(p)}(X'_{f_j(p)}),$$

since $f_i(p) = g_{ij}(f_j(p))$ and $g_{ij} = g_{ji}^{-1}$. Then, we get

$$f_{i*}(\nabla^i_{X_i} Y_i) = g_{ij*}(f_{j*}(\nabla^j_{X_i} Y_i)) = f_{i*}(\nabla^j_{X_i} Y_i),$$

which implies that $\nabla^i_{X_i} Y_i - \nabla^j_{X_i} Y_i$ is vertical. Since it is also horizontal, we deduce $\nabla^i_{X_i} Y_i = \nabla^j_{X_i} Y_i$. Moreover, clearly, $\nabla^i \xi_\alpha = 0 = \nabla^j \xi_\alpha$ and, on $U_i \cap U_j$, $\nabla^i_{\xi_\alpha} V = [\xi_\alpha, X] = \nabla^j_{\xi_\alpha} V$. So we obtain a global connection ∇ on M which, by Proposition 4.2, coincides with the bi-Legendrian connection associated to the pair (L, Q). $\square$

5. Characteristic classes for Legendrian foliations

As usual, given a Lie group G, we will denote by the same symbol $I^k(G)$ both the algebra of k-multilinear, symmetric, $\mathrm{ad}(G)$-invariant functions on the Lie algebra $\mathfrak{g}$ of G and the algebra of $\mathrm{ad}(G)$-invariant polynomials of degree k. Let $\mathcal{F}$ be a Legendrian foliation of the almost $\mathcal{S}$-manifold $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$ and Q a Legendrian distribution complementary to $L = T(\mathcal{F})$. This implies a reduction of the structure group to $G = O(n) \times O(n) \times I_r$. Hence $I(G) = I(O(n)) \otimes I(O(n))$. The algebra $I(O(n))$ has been computed (see, for example, [20]). It is generated by the $[\frac{n}{2}]$ $\mathrm{ad}(O(n))$-invariant polynomials $c'_2, c'_4, \ldots,$ where c'_i are given by the formula

$$\det(B - \lambda I_n) = \sum_{i=0}^{n} c'_i(B)\lambda^{n-i}, \quad B \in \mathfrak{gl}(n, \mathbb{R}),$$

with $c'_0(B) = 1$, $c'_1(B) = \mathrm{tr}(B), \ldots, c'_n(B) = \det(B)$. Let $A \in \mathfrak{g}$, $A = A_1 + A_2$, with $A_1, A_2 \in \mathfrak{o}(n)$. It is easy to see that, for $k \in \{1, \ldots, 2n\}$, $c_k(A) = \sum_{i=0}^{k} c'_i(A_1)c'_{k-i}(A_2)$ are the $ad(G)$-invariant polynomials determined by the formula

$$\det(A - \lambda I_{2n}) = \sum_{i=0}^{2n} c_k(A)\lambda^{2n-k}, \quad A \in \mathfrak{gl}(2n, \mathbb{R}),$$

and in particular the products $c'_i \otimes c'_j$ are generators for $I(G)$.

5.1. Primary characteristic classes. Let $L = T(\mathcal{F})$, as usual, be the tangent bundle of the Legendrian foliation $\mathcal{F}$ and suppose that (L, Q) is strongly flat. From Theorem 4.1 it follows that there exist local coordinates such that

(i) $L = \mathrm{span}\left\{\frac{\partial}{\partial y_j}\right\}_{j=1,\ldots,n}$,

(ii) $Q = \mathrm{span}\{X_i\}_{i=1,\ldots,n}$, where $X_i = \frac{\partial}{\partial x_i} - \sum_{l=1}^{n} t_i^l \frac{\partial}{\partial y_l} - \sum_{\alpha=1}^{r} s_i^\alpha \frac{\partial}{\partial z_\alpha}$, for some smooth functions t_i^l, s_i^α,

(iii) $\xi_\alpha = \frac{\partial}{\partial z_\alpha}$ for each $\alpha \in \{1, \ldots, r\}$.

Moreover, with respect to these coordinates, Φ is given by $\Phi = \sum_{k=1}^{n} dx_k \wedge dy_k$.

Consider now the bi-Legendrian connection ∇ associated to (L, Q).

Proposition 5.1. *With respect to the coordinates stated in Theorem 4.1 the bi-Legendrian connection ∇ associated to (L,Q) has the following local expression:*

(a) $\nabla_{\frac{\partial}{\partial y_j}}\frac{\partial}{\partial y_i}=0$, $\nabla_{\frac{\partial}{\partial z_\alpha}}\frac{\partial}{\partial y_i}=0$, $\nabla_{X_j}\frac{\partial}{\partial y_i}=\sum_{k=1}^{n}\frac{\partial t_j^k}{\partial y_i}\frac{\partial}{\partial y_k}$,

(b) $\nabla_{\frac{\partial}{\partial y_j}}X_i=0$, $\nabla_{\frac{\partial}{\partial z_\alpha}}X_i=0$, $\nabla_{X_j}X_i=-\sum_{k=1}^{n}\frac{\partial t_j^i}{\partial y_k}X_k$,

(c) $\nabla_{\frac{\partial}{\partial y_i}}\frac{\partial}{\partial z_\alpha}=\nabla_{X_i}\frac{\partial}{\partial z_\alpha}=\nabla_{\frac{\partial}{\partial z_\beta}}\frac{\partial}{\partial z_\alpha}=0$,

and the torsion tensor T of ∇ is given by:

$$T\left(\frac{\partial}{\partial y_i},\frac{\partial}{\partial y_j}\right)=0,\ T\left(\frac{\partial}{\partial y_i},X_j\right)=-\delta_{ij}\sum_{\alpha=1}^{r}\frac{\partial}{\partial z_\alpha},\ T\left(\frac{\partial}{\partial y_i},\frac{\partial}{\partial z_\alpha}\right)=0,$$

$$T\left(X_i,\frac{\partial}{\partial z_\alpha}\right)=0,\ T\left(\frac{\partial}{\partial z_\alpha},\frac{\partial}{\partial z_\beta}\right)=0,\ T(X_i,X_j)=-\sum_{k=1}^{n}\tau_{ijk}\frac{\partial}{\partial y_k},$$

where

$$\tau_{ijk}=\frac{\partial t_i^k}{\partial x_j}-\frac{\partial t_j^k}{\partial x_i}+\sum_{h=1}^{n}t_i^h\frac{\partial t_j^k}{\partial y_h}-\sum_{h=1}^{n}t_j^h\frac{\partial t_i^k}{\partial y_h}$$

and Q is integrable if and only if $\tau_{ijk}\equiv 0$.

Proof. Obviously, as $\nabla\xi_\alpha=0$, we have (c). Then, by a straightforward computation we have $H\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=H\left(\frac{\partial}{\partial y_h},\frac{\partial}{\partial y_k}\right)=H\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_h}\right)=H\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial z_\alpha}\right)=H\left(\frac{\partial}{\partial y_h},\frac{\partial}{\partial z_\alpha}\right)=H\left(\frac{\partial}{\partial z_\alpha},\frac{\partial}{\partial z_\beta}\right)=0$, for all $i,j,h,k\in\{1,\dots,n\}$ and $\alpha,\beta\in\{1,\dots,r\}$. Then, we have

$$\nabla_{\frac{\partial}{\partial y_j}}\frac{\partial}{\partial y_i}=H\left(\frac{\partial}{\partial y_j},\frac{\partial}{\partial y_i}\right)_L=0$$

and, by a long computation,

$$\nabla_{X_j}X_i=H(X_i,X_j)_Q=-H\left(t_j^i\frac{\partial}{\partial y_i},\frac{\partial}{\partial x_i}\right)_Q.$$

Now we prove

$$H\left(t_j^i\frac{\partial}{\partial y_i},\frac{\partial}{\partial x_i}\right)_Q=\sum_{k=1}^{n}\frac{\partial t_j^i}{\partial y_k}X_k \tag{9}$$

and we get $\nabla_{X_j}X_i=-\sum_{k=1}^{n}\frac{\partial t_j^i}{\partial y_k}X_k$. Indeed, for each $h\in\{1,\dots,n\}$, we have

$$\Phi\left(\sum_{k=1}^{n}\frac{\partial t_j^i}{\partial y_k}X_k,\frac{\partial}{\partial y_h}\right)=\frac{1}{2}\frac{\partial t_j^i}{\partial y_h}=\Phi\left(H\left(t_j^i\frac{\partial}{\partial y_i},\frac{\partial}{\partial x_i}\right)_Q,\frac{\partial}{\partial y_h}\right).$$

Since Φ is non-degenerate on $\mathcal{H}$, (9) follows. The other relations are easy to prove. In fact, since $\xi_\alpha = \frac{\partial}{\partial z_\alpha}$,

$$\nabla_{\frac{\partial}{\partial z_\alpha}} \frac{\partial}{\partial y_i} = \left[\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial y_i}\right]_L = 0$$

and

$$\nabla_{\frac{\partial}{\partial z_\alpha}} X_i = \sum_{j=1}^{n} \left(\frac{\partial t_i^j}{\partial z_\alpha} \frac{\partial}{\partial y_i}\right)_Q + \sum_{\beta=1}^{r} \left(\frac{\partial s_i^\beta}{\partial z_\alpha} \frac{\partial}{\partial z_\beta}\right)_Q = 0.$$

Finally,

$$\nabla_{X_j} \frac{\partial}{\partial y_i} = \sum_{k=1}^{n} \left(\frac{\partial t_j^k}{\partial y_i} \frac{\partial}{\partial y_k}\right)_L + \sum_{\alpha=1}^{r} \left(\frac{\partial s_j^\alpha}{\partial y_i} \frac{\partial}{\partial z_\alpha}\right)_L = \sum_{k=1}^{n} \frac{\partial t_j^k}{\partial y_i} \frac{\partial}{\partial y_k},$$

and

$$\nabla_{\frac{\partial}{\partial y_j}} X_i = -\sum_{k=1}^{n} \left(\frac{\partial t_i^k}{\partial y_j} \frac{\partial}{\partial y_k}\right)_Q - \sum_{\alpha=1}^{r} \left(\frac{\partial s_i^\alpha}{\partial y_j} \frac{\partial}{\partial z_\alpha}\right)_Q = 0$$

and this proves the first part of the proposition. Some direct computations, finally, prove the relations about the torsion tensor. In particular, since $T(Y, Y') = -p_{Q^\perp}([Y, Y'])$ for $Y, Y' \in \Gamma(Q)$ (cf. [10]), it follows that Q is integrable if and only if the functions τ_{ijk} vanish identically. $\square$

Concerning the curvature of the bi-Legendrian connection ∇ corresponding to the pair (L, Q), Proposition 2.3, 2.4 and 5.1 imply the following result.

Proposition 5.2. *With respect to the coordinates stated in Theorem 4.1, the curvature tensor of ∇ has the following local expression:*

(i) $R\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = 0$;

(ii) $R\left(\frac{\partial}{\partial y_i}, X_j\right) \frac{\partial}{\partial y_k} = \sum_{h=1}^{n} \frac{\partial^2 t_j^h}{\partial y_i \partial y_k} \frac{\partial}{\partial y_h}$, $R\left(\frac{\partial}{\partial y_i}, X_j\right) X_k = -\sum_{h=1}^{n} \frac{\partial^2 t_j^k}{\partial y_i \partial y_h} X_h$,
$R\left(\frac{\partial}{\partial y_i}, X_j\right) \frac{\partial}{\partial z_\alpha} = 0$;

(iii) $R(X_i, X_j)\frac{\partial}{\partial y_k} = -\sum_{h=1}^{n} \frac{\partial \tau_{ijk}}{\partial y_h} \frac{\partial}{\partial y_h}$, $R(X_i, X_j)X_k = \sum_{h=1}^{n} \frac{\partial \tau_{ijk}}{\partial y_h} X_h$,
$R(X_i, X_j)\frac{\partial}{\partial z_\alpha} = 0$;

(iv) $R\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_\alpha}\right) = R\left(X_i, \frac{\partial}{\partial z_\alpha}\right) = R\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right) = 0.$

According to [32] we give the following definition.

Definition 5.3. Let $\mathcal{F}$ be a Legendrian foliation on the almost $\mathcal{S}$-manifold $(M,\phi,\xi_\alpha,\eta_\alpha,g)$ and let Q be a Legendrian distribution complementary to $L = T(\mathcal{F})$. Then the bi-Legendrian connection ∇ associated to (L,Q) is called *tangential* if, for all $X \in \Gamma(L)$,

(i) $R(X,Y)=0$ for all $Y \in \Gamma(Q)$,

(ii) $R(X,\xi_\alpha)=0$ for all $\alpha \in \{1,\dots,r\}$.

In particular, when $\mathcal{F}$ is strongly flat, the corresponding bi-Lagrangian connection is tangential if and only if $R(X,Y)=0$ for all $X \in \Gamma(L)$, $Y \in \Gamma(Q)$. When ∇ is tangential, from Proposition 5.2 we deduce that $\frac{\partial^2 t^i_j}{\partial y_h \partial y_k} = 0$ for all $h,k \in \{1,\dots,n\}$, that is t^i_j are leafwise affine functions and, in analogy with Lagrangian foliations ([26], [27]), we say that Q is a *Legendrian affine transversal distribution* for L. When this happens and under the assumption of strong flatness of (L,Q), the curvature 2-form of ∇ has a very simple expression. Indeed locally

$$\Omega = \sum_{1\le i<j\le n} \Omega_{ij} dx_i \wedge dx_j + \sum_{\substack{1\le i\le n\\ 1\le h\le n}} \Omega_{ih} dx_i \wedge dy_h + \sum_{\substack{1\le i\le n\\ 1\le \alpha\le r}} \Omega_{i\alpha} dx_i \wedge dz_\alpha$$
$$+ \sum_{1\le h<k\le n} \Omega_{hk} dy_h \wedge dy_k + \sum_{\substack{1\le h\le n\\ 1\le \alpha\le r}} \Omega_{h\alpha} dy_h \wedge dz_\alpha + \sum_{1\le \alpha<\beta\le r} \Omega_{\alpha\beta} dz_\alpha \wedge dz_\beta.$$

(Throughout all this work, if no confusion is feared, we identify forms on M with their lifts to principal bundle of linear frames $L(M)$.)

Now, the hypothesis of tangentiality and Corollary 5.2 yield

$$\Omega_{\alpha\beta} = \Omega\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right) = 0, \quad \Omega_{hk} = \Omega\left(\frac{\partial}{\partial y_h}, \frac{\partial}{\partial y_k}\right) = 0,$$
$$\Omega_{h\alpha} = \Omega\left(\frac{\partial}{\partial y_h}, \frac{\partial}{\partial z_\alpha}\right) = 0,$$

which imply

$$\Omega_{ih} = \Omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_h}\right) = \Omega\left(X_i, \frac{\partial}{\partial y_h}\right) = 0$$

and

$$\Omega_{i\alpha} = \Omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial z_\alpha}\right) = \Omega\left(X_i, \frac{\partial}{\partial z_\alpha}\right) = 0.$$

So Ω can be written as

$$\Omega = \sum_{1\le i<j\le n} \Omega_{ij} dx_i \wedge dx_j,$$

from which we deduce that Ω^k vanishes for $k > \left[\frac{n}{2}\right]$. So, if $f \in I^k(G)$ is an $\mathrm{ad}(G)$-invariant polynomial of degree k, we have that $f(\Omega) = 0$ for $k = \deg(f) > [\frac{n}{2}]$. This proves the following strong vanishing theorem for characteristic classes of Legendrian foliations:

Theorem 5.4. *Let $\mathcal{F}$ be a strongly flat Legendrian foliation on an almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ of dimension $2n+r$. A necessary condition for $\mathcal{F}$ to admit a strongly flat Legendrian affine transversal distribution is that* $\mathrm{Pont}^j(N(\mathcal{F}))$ *vanishes for $j > n$, where $N(\mathcal{F})$ denotes the normal bundle of the foliation $\mathcal{F}$ and* $\mathrm{Pont}(N(\mathcal{F}))$ *denotes the Pontryagin algebra of the bundle $N(\mathcal{F})$.*

This theorem points out an obstruction to the existence of a Legendrian affine transversal distribution for the strongly flat Legendrian foliation $\mathcal{F}$. In the general case, when $\mathcal{F}$ does not admit such a transversal distribution, the above arguments allow to get a weaker vanishing theorem:

Theorem 5.5. *Let $\mathcal{F}$ be a strongly flat Legendrian foliation of the almost $\mathcal{S}$-manifold $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$ such that there exists a strongly flat Legendrian distribution complementary to the tangent bundle of $\mathcal{F}$. Then* $\mathrm{Pont}^j(N(\mathcal{F}))$ *vanishes for $j > 2n$.*

Proof. Indeed in this case Ω can be written as

$$\Omega = \sum_{1 \leq i < j \leq n} \Omega_{ij} dx_i \wedge dx_j + \sum_{1 \leq i \leq n, 1 \leq h \leq n} \Omega_{ih} dx_i \wedge dy_h$$

hence Ω^k vanishes for $k > n$. □

In particular when all the ξ_α are Killing vector fields, we get an obstruction to the existence of a strongly flat Legendrian foliation on $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$, because by Lemma 2.2 also $Q = \phi(L)$ is strongly flat.

5.2. Secondary characteristic classes. One of the consequences of the vanishing theorems for characteristic classes of foliations is the possibility of constructing secondary (or exotic) characteristic classes, which we now define. We will assume definitions and notation of [22]. Let $\mathcal{F}$ be a strongly flat Legendrian foliation on the almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ and let Q be a strongly flat Legendrian distribution complementary to $L = T(\mathcal{F})$. We suppose that the bi-Legendrian connection ∇ corresponding to (L, Q) is tangential, whereas ∇' will denote any metric connection on (M, g).

If $\overline{\nabla}$ is any connection on M, $\lambda_{\overline{\nabla}} : I(G) \longrightarrow \Lambda(M)$ will denote the Chern–Weil homomorphism, defined by $\lambda_{\overline{\nabla}}(f) = f(\overline{\Omega}, \ldots, \overline{\Omega}) \in \Lambda^{2k}(M)$, for any $f \in I^k(G)$, where $\overline{\Omega}$ denotes the curvature 2-form of $\overline{\nabla}$. $\lambda_{\overline{\nabla}}$ induces a homomorphism of graded algebras $\lambda : I(G) \longrightarrow H(M, \mathbb{R})$ which does not depend on the connection, that is, if $\overline{\nabla}'$ is any other connection on M, then, for every $f \in I(G)$, $\lambda_{\overline{\nabla}}(f) - \lambda_{\overline{\nabla}'}(f)$ is an exact form. In particular this is true for ∇ and ∇', and for the construction of secondary characteristic classes it is useful to indicate explicitly what $\lambda_{\nabla}(f) - \lambda_{\nabla'}(f)$ is equal to. Namely, let

$$\int_0^1 : \Lambda^s(M \times [0,1]) \longrightarrow \Lambda^{s-1}(M)$$

be the integration along the fibers of the projection $M \times [0,1] \longrightarrow M$ and let $\widetilde{\nabla}$ denote the connection on $M \times [0,1]$ defined by $\widetilde{\nabla}\big(\frac{\partial}{\partial t}\big) = 0$ and $\widetilde{\nabla}|_{M\times\{t\}} = t\nabla' + (1-t)\nabla$. Set $\Delta_{\nabla,\nabla'} : I^k(G) \longrightarrow \Lambda^{2k-1}(M)$ be the composition

$$\Delta_{\nabla,\nabla'} = \int_0^1 \circ \lambda_{\widetilde{\nabla}}.$$

Actually, it can be shown ([20]) that $\lambda_{\nabla'} - \lambda_{\nabla} = d \circ \Delta_{\nabla,\nabla'}$.

Now, let $J \subset I^+(G) = \bigoplus_{k\geq 1} I^k(G)$ be a homogenous ideal of $I(G)$. We recall that a connection on M is called a *J-connection* if $\lambda_{\nabla}(f) = 0$ for every $f \in J$. We adopt the following notation. If P denotes a property on the degree of homogenous polynomials on G, we will denote by $J(P)$ the homogenous ideal generated by homogenous polynomials whose degree verifies P. Moreover, if $f_1, \ldots, f_q$ are homogenous polynomials, we will denote by $\{f_1, \ldots, f_q\}$ the homogenous ideal generated by $f_1, \ldots, f_q$. So, in particular, ∇' is a J'-connection, where $J' = \{c_{\mathrm{odd}}\}$ and ∇ a $J\big(>\big[\frac{n}{2}\big]\big)$-connection. Consider the quotient algebras $I(G)/J\big(>\big[\frac{n}{2}\big]\big)$ and $I(G)/J'$, and denote by $\overline{f}$ and $\overline{\overline{f}}$ the equivalence classes of $f \in I^k(G)$ modulo $J\big(>\big[\frac{n}{2}\big]\big)$ and modulo J', respectively. Then the algebra

$$W\left(J\left(>\left[\frac{n}{2}\right]\right), J'\right) = I(G)/J\left(>\left[\frac{n}{2}\right]\right) \otimes_{\mathbb{R}} I(G)/J' \otimes_{\mathbb{R}} \Lambda(I^+(G))$$

will be called a *secondary universal algebra* of G. Furthermore, we attribute degrees to the elements of $W\big(J\big(>\big[\frac{n}{2}\big]\big), J'\big)$ setting $\deg\big(\overline{f}\big) = \deg\big(\overline{\overline{f}}\big) = 2k$ and $\deg\big(\widehat{f}\big) = 2k-1$, for any $f \in I^k(G)$, where $\widehat{f}$ denotes the image of f under the isomorphism $I^+(G) \longrightarrow \Lambda^1(I^+(G))$. We define a differentiation by setting $d\overline{f} = d\overline{\overline{f}} = 0$, $d\widehat{f} = \overline{\overline{f}} - \overline{f}$ and by their natural algebraic extensions of these operations. It can be seen that d raises the degrees by 1 and $d^2 = 0$. Thus, $W\big(J\big(>\big[\frac{n}{2}\big]\big), J'\big)$ becomes a differential graded algebra, with corresponding cohomology algebra $H\big(W\big(J\big(>\big[\frac{n}{2}\big]\big), J'\big)\big)$. Now, we can define a homomorphism of graded algebras

$\rho_{\nabla,\nabla'} : W\big(J\big(> \big[\frac{n}{2}\big]\big), J'\big) \longrightarrow \Lambda(M)$ in the following way. For any $f \in I(G)$ and for any $f_1, \ldots, f_s \in I^+(G)$ we set

$$\rho_{\nabla,\nabla'}\big(\overline{f}\,\big) = \lambda_{\nabla}(f), \rho_{\nabla,\nabla'}\big(\overline{\overline{f}}\,\big) = \lambda_{\nabla'}(f)$$

and $\rho_{\nabla,\nabla'}(f_1 \wedge \cdots \wedge f_s) = \Delta_{\nabla,\nabla'}(f_1) \wedge \cdots \wedge \Delta_{\nabla,\nabla'}(f_s)$. The homomorphism $\rho_{\nabla,\nabla'}$ is called the *secondary Chern–Weil homomorphism*. It can be easily shown that the homomorphism $\rho_{\nabla,\nabla'}$ is degree preserving and commutes with the differentials of the algebras $W\big(J\big(> \big[\frac{n}{2}\big]\big), J'\big)$ and $\Lambda(M)$. Hence it induces a homomorphism in cohomology $\rho^*_{\nabla,\nabla'}$, called the *cohomological secondary Chern–Weil homomorphism*.

Definition 5.6. The cohomology classes in $\mathrm{Im}(\rho^*_{\nabla,\nabla'}) - \mathrm{Im}\,\lambda)$ are called *secondary characteristic classes* of (∇, ∇').

Unlike primary characteristic classes, secondary characteristic classes depend on the connections. However, according to the Lehmann theory, we can state the following

Theorem 5.7. *The cohomological secondary Chern–Weil homomorphism $\rho^*_{\nabla,\nabla'}$ remains unchanged if the connections ∇, ∇' are replaced by ∇_1, ∇'_1, where ∇ is $J\big(> \big[\frac{n}{2}\big]\big)$-homotopic to ∇_1 and ∇' is J'-homotopic to ∇'_1.*

We explain the meaning of "J-homotopy" used in the previous theorem. Let ∇_0, ∇_1 be two J-connections on M. Then ∇_0 and ∇_1 are said to be *differentiably J-homotopic* if there exists a J-connection $\widetilde{\nabla}$ on $M \times [0,1]$ such that $\widetilde{\nabla}|_{M\times\{0\}} = \nabla_0$ and $\widetilde{\nabla}|_{M\times\{1\}} = \nabla_1$. More generally, ∇_0 and ∇_1 are said *J-homotopic* if there exists a finite sequence $\nabla_0 = \nabla_{s_0}, \nabla_{s_1}, \ldots, \nabla_{s_k} = \nabla_1$ of J-connections such that, for all $i \in \{0, \ldots, k-1\}$, ∇_{s_i} and $\nabla_{s_{i+1}}$ are differentiably J-homotopic. The relation of J-homotopy is an equivalence relation on the set of the connections on M into disjoint J-homotopy classes of connections. A set $C \neq \emptyset$ of J-connections is said to be *J-connected* if any two connections of C are J-homotopic. So Theorem 5.7 says that if ∇ and ∇' vary, respectively, in two sets C and C' which are, respectively, $J\big(> \big[\frac{n}{2}\big]\big)$-connected and J'-connected, then we obtain secondary characteristic classes which do not depend on the choice of the connection within the sets C and C'. One has the following result:

Proposition 5.8 ([22])**.** *With the notation above, if $C \cap C' \neq \emptyset$ all secondary characteristic classes vanish.*

Now we prove that, indeed, our construction of secondary characteristic classes for Legendrian foliations does not depend on the choice of a metric connection on M. But, before to prove this, we need the following preliminary result:

Lemma 5.9. *Let $(M^{2n+r}, \phi, \xi_\alpha, \eta_\alpha, g)$ be an almost $\mathcal{S}$-manifold. Then any two almost $\mathcal{S}$-structures $(\phi_0, \xi_\alpha, \eta_\alpha, g_0)$ and $(\phi_1, \xi_\alpha, \eta_\alpha, g_1)$ (same ξ_α and η_α) on M^{2n+r} are homotopic, that is they can be joined by a differentiable curve $(\phi_t, \xi_\alpha, \eta_\alpha, g_t)$, $0 \leq t \leq 1$, of almost $\mathcal{S}$-structures on M^{2n+r}.*

Proof. We set $J_0 := \phi_0|_{\mathcal{H}}$ and $J_1 := \phi_1|_{\mathcal{H}}$, so obtaining two almost complex structures J_0 and J_1 on $\mathcal{H} = \bigcap_{\alpha=1}^{r} \ker(\eta_\alpha)$ and by Theorem 3.1.2 of [28] there exists a homotopy $\{J_t\}_{0 \leq t \leq 1}$ between J_0 and J_1. Then define ϕ_t putting $\phi_t|_{\mathcal{H}} := J_t$ and $\phi_t(\xi_1) = \cdots = \phi_t(\xi_r) = 0$. Moreover we define a metric g_t putting, for all $Z, Z' \in \Gamma(\mathcal{H})$, $g_t(Z, Z') := -\Phi(Z, \phi_t(Z'))$ and $g_t(V, \xi_\alpha) := \eta_\alpha(V)$ for all $\alpha \in \{1, \ldots, r\}$ and $V \in \Gamma(T(M))$. Clearly $(\phi_t, \xi_\alpha, \eta_\alpha, g_t)$ are almost $\mathcal{S}$-structures on M^{2n+r} (note that $\Phi_t = \Phi$, where $\Phi_t(V, W) = g_t(V, \phi_t(W))$) which realize the homotopy between $(\phi_0, \xi_\alpha, \eta_\alpha, g_0)$ and $(\phi_1, \xi_\alpha, \eta_\alpha, g_1)$. □

Proposition 5.10. *The set of all connections on M which are metric with respect to an associated metric is J'-connected.*

Proof. Let ∇^0 and ∇^1 be two metric connections with respect to the associated metrics g_0 and g_1, respectively. By Lemma 5.9 there exists a homotopy $\{g_t\}_{t \in [0,1]}$ between g_0 and g_1. Let $\widetilde{g}$ be a metric on $M \times [0,1]$ such that $\widetilde{g}|_{M \times \{t\}} = g_t$ and let $\widetilde{\nabla}$ be a connection on $M \times [0,1]$ such that $\widetilde{\nabla}\widetilde{g} = 0$. We denote by ∇'_t the connection induced by $\widetilde{\nabla}$ on $M \times \{t\}$. Then $\widetilde{\nabla}$ defines a homotopy between ∇'_0 and ∇'_1. So for ending the proof we have to prove that ∇^0 is homotopic to ∇'_0 and ∇^1 is homotopic to ∇'_1. Indeed, $\nabla^t := (1-t)\nabla^0 + t\nabla'_0$ is a homotopy between ∇'_0 and ∇^1 and as $\nabla^0 g = \nabla'_0 g = 0$, ∇^t is a metric connection with respect to g_0 for all $t \in [0,1]$. The same arguments work for ∇_1 and ∇'_1. □

Corollary 5.11. *For all $\alpha \in H(W(J > [\frac{n}{2}], J'))$ the cohomology class $\rho^*_{\nabla, \nabla'}(\alpha)$ is independent on the choice of the metric connection ∇'.*

Furthermore, from Proposition 5.8 we get an obstruction for the bi-Legendrian connection being a metric connection:

Corollary 5.12. *If the bi-Legendrian connection ∇ is a metric connection-with respect to the associated metric g then all secondary characteristic classes vanish.*

By Vey Theorem (Theorem 6.3 of [22]) $c_j h_i := \overline{c_{j_1} \cdots c_{j_a}} \otimes h_{i_1} \wedge \cdots \wedge h_{i_b}$ is a basis for the cohomology complex $H(W(J(> [\frac{n}{2}]), J'))$, where j and i are any sequences of integers having the properties

$$1 \leq j_1 \leq \cdots \leq j_a \leq 2n, 1 \leq i_1 < \cdots < i_b < 2n, i_k \text{ an odd integer,}$$

$$j_1 + \cdots + j_a + i_0 > \left[\frac{n}{2}\right], \quad i_0 \leq j_0,$$

h_q denoting the image of c_q under the canonical isomorphism $I^+(G) \to \Lambda^1(I^+(G))$ and where we set $i_0 = i_1$ if $i \neq \emptyset$, $i_0 = +\infty$ if $i = \emptyset$ and $j_0 = j_1$ if $j \neq \emptyset$, $j_0 = +\infty$ if $j = \emptyset$. We conclude by formulating the following "rigidity theorem" ([16]).

Theorem 5.13. *With the previous notation, if $j_1 + \cdots + j_a + i_0 > [\frac{n}{2}] + 1$, then $\rho^*_{\nabla,\nabla'}([c_j h_i])$ depends only on the arc-component of the connection ∇ in the space of connections satisfying $\Omega^{[\frac{n}{2}]+1} = 0$.*

6. Ehresmann connections for Legendrian foliations

In this section we will see another application of the projectability of Legendrian foliations. Namely, we will show that, under certain assumptions, every strongly flat Legendrian foliation admits an Ehresmann connection. We begin with some preliminaries on Ehresmann connections. Let $(M, \mathcal{F})$ be a foliated manifold and D a distribution on M which is supplementary to the tangent bundle L of the foliation $\mathcal{F}$. A *horizontal curve* is a piecewise smooth curve $\beta : [0, b] \longrightarrow M$, $b \in \mathbb{R}$, such that $\beta'(t) \in D_{\beta(t)}$ for all $t \in [0, b]$. A *vertical curve* (or a *leaf curve*) is a piecewise smooth curve $\alpha : [0, a] \longrightarrow M$, $a \in \mathbb{R}$, which lies entirely in one leaf of $\mathcal{F}$. A *rectangle* is a piecewise smooth map $\sigma : [0, a] \times [0, b] \longrightarrow M$ such that for every fixed $s \in [0, b]$ the curve $\sigma_s := \sigma|_{[0,a]\times\{s\}}$ is vertical and for every fixed $t \in [0, a]$ the curve $\sigma^t := \sigma|_{\{t\}\times[0,b]}$ is horizontal. The curves $\sigma_0 = \sigma(\cdot, 0)$, $\sigma_b = \sigma(\cdot, b)$, $\sigma^0 = \sigma(0, \cdot)$ and $\sigma^a = \sigma(a, \cdot)$ are called, respectively, the *initial vertical edge*, the *final vertical edge*, the *initial horizontal edge* and the *final horizontal edge* of σ.

Definition 6.1 ([5]). A complementary distribution D to the foliation $\mathcal{F}$ is called an *Ehresmann connection* for $\mathcal{F}$ if for every vertical curve α and horizontal curve β with the same initial point, there exists a rectangle whose initial edges are α and β. This rectangle is unique and is called the *rectangle associated to α and β.*

Now we state the main result of the section.

Theorem 6.2. *Let $\mathcal{F}$ be a strongly flat Legendrian foliation on a compact connected almost $\mathcal{S}$-manifold. Let Q be a strongly flat Legendrian distribution, transversal to $\mathcal{F}$, for which the bi-Legendrian connection ∇ is tangential. If the leaves of $\mathcal{F}$ are complete affine manifolds then the subbundle $D := Q \oplus E$ is an Ehresmann connection for the foliation $\mathcal{F}$ and preserves ∇.*

The last statement of Theorem 6.2 needs some explanations. Given a foliated manifold $(M, \mathcal{F})$ and a supplementary subbundle D to $L = T(\mathcal{F})$, any horizontal curve $\tau : [0,1] \longrightarrow M$ defines a family of diffeomorphisms $(\varphi_t : V_0 \longrightarrow V_t)_{t\in[0,1]}$ such that

1. each V_t is a neighborhood of $\tau(t)$ in the leaf of $\mathcal{F}$ through $\tau(t)$, for all $t \in [0,1]$,
2. $\varphi_t(\tau(0)) = \tau(t)$ for all $t \in [0,1]$,
3. for any fixed $p \in V_0$ the curve $t \mapsto \varphi_t(p)$ is horizontal,
4. $\varphi_0 : V_0 \longrightarrow V_0$ is the identity map.

This family of diffeomorphisms is called an *element of holonomy* along τ ([5]). It is shown in [19] and in [4] that an element of holonomy along τ exists and is unique, in the sense that any two elements of holonomy must agree on some neighborhood of $\tau(0)$ in the leaf through $\tau(0)$. When the leaves of the foliation have a geometric structure, we say that D preserves the geometry of the leaves if the element of holonomy along any horizontal curve is a local isomorphism of the particular geometric structure. Returning to Theorem 6.2, we say that D *preserves* ∇ if the elements of holonomy along horizontal curves are affine transformations with respect to the connection induced on the leaves by the bi-Legendrian connection ∇. On a manifold M with a torsion free linear connection and endowed with a totally geodesic foliation $\mathcal{F}$, Blumenthal and Hebda studied conditions for a complementary distribution D to be an Ehresmann connection for $\mathcal{F}$ preserving the linear connection on the leaves. Their arguments work also for bi-Legendrian connections and, in particular, from Proposition 5.3 of [5] we deduce the following

Proposition 6.3. *Let $\mathcal{F}$ be a Legendrian foliation on an almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, \dots, r\}$. Let Q be any Legendrian distribution complementary to $\mathcal{F}$ such that the corresponding bi-Legendrian connection ∇ is tangential. Then the subbundle $Q \oplus E$ preserves the connection ∇.*

Proposition 6.3 proves the last part of Theorem 6.2. Now we need a number of preliminary lemmas.

Lemma 6.4. *Let $(M, \phi, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, \dots, r\}$, be a $(2n+r)$-dimensional almost $\mathcal{S}$-manifold endowed with a strongly flat Legendrian foliations $\mathcal{F}$ and a strongly flat Legendrian distribution Q complementary to $L = T(\mathcal{F})$. Denote by ∇ the bi-Legendrian connection associated to (L, Q) and suppose that there exists a submersion with connected fibers $f : M \longrightarrow N$ defining the foliation $\mathcal{E}$. Let $\mathcal{F}'$ and Q' be the Lagrangian foliation and the Lagrangian distribution on N which are projections of $\mathcal{F}$ and Q, respectively, according to Corollary 3.4. Then* *the corresponding bi-Lagrangian connection ∇' is tangential if and only if ∇ is tangential.*

Proof. Let $X' \in \Gamma(L')$, $Y' \in \Gamma(Q')$ and $Z' \in \Gamma(T(N))$ and consider the unique basic vector fields $X \in \Gamma(L)$, $Y \in \Gamma(Q)$ and $Z \in \Gamma(T(M))$ such that $f_*(X) = X'$, $f_*(Y) = Y'$ and $f_*(Z) = Z'$. Then

$$\begin{aligned} f_*(R(X,Y)Z) &= \nabla'_{X'} f_*(\nabla_Y Z) - \nabla'_{Y'} f_*(\nabla_X Z) - f_*(\nabla_{h[Z,Y]} Z) \\ &= R'(X',Y')Z' \end{aligned}$$

since $\nabla_{v[X,Y]} Z = \sum_{\alpha=1}^{r} \eta_\alpha([X,Y])[\xi_\alpha, Z] = 0$. Hence one has immediately that ∇ is tangential if and only if ∇' is tangential. □

Lemma 6.5. *Under the same assumptions of Lemma 6.4, if the leaves of $\mathcal{F}$ are complete affine manifolds then also the leaves of $\mathcal{F}'$ are complete affine manifolds.*

Proof. Suppose that the leaves of $\mathcal{F}$ are complete affine manifolds. Let $\overline{\gamma}$ be a geodesic (with respect to the bi-Lagrangian connection ∇') lying on a leaf $\mathcal{L}'$ of $\mathcal{F}'$ and defined by the initial conditions $\overline{\gamma}(0) = x$, $\overline{\gamma}'(0) = v' \in L'_x$, where, as usual, L' is the tangent bundle of the foliation $\mathcal{F}'$. Let p be any point on the fiber over x and consider the leaf $\mathcal{L}$ through p. As f maps leaves of $\mathcal{F}$ onto leaves of $\mathcal{F}'$, we can lift $\overline{\gamma}$ to a geodesic γ lying on $\mathcal{L}$. Indeed, let v be the unique vector of L_p such that $f_{*p}(v) = v'$. Then there exists a unique geodesic on $\mathcal{L}$, say γ, such that $\gamma(0) = p$ and $\gamma'(0) = v$. Note that, by hypothesis, γ is defined for all $t \in \mathbb{R}$. Consider the projection of γ on $\mathcal{L}'$, $\widetilde{\gamma} := f \circ \gamma$. We prove that $\widetilde{\gamma}$ is still a geodesic (for the bi-Lagrangian connection ∇'). Indeed $\nabla'_{\widetilde{\gamma}'}\widetilde{\gamma}' = \nabla'_{f_*\widetilde{\gamma}'} f_*\widetilde{\gamma}' = f_*(\nabla_{\gamma'}\gamma') = 0$ since γ is a geodesic for the bi-Legendrian connection ∇. Moreover, $\widetilde{\gamma}(0) = f(\gamma(0)) = f(p) = x$ and $\widetilde{\gamma}'(0) = f_{*\gamma(0)}(\gamma'(0)) = f_{*p}(v) = v'$. So for the uniqueness of the geodesic with given initial conditions we get $\overline{\gamma} = \widetilde{\gamma}$. In particular, $\overline{\gamma}$ can be extended to a geodesic defined for all $t \in \mathbb{R}$. □

Proof of Theorem 6.2. The proof is divided into two steps: firstly we suppose that there exists a submersion f whose fibers define the foliation $\mathcal{E}$ and then we drop this hypothesis using the definition of $\mathcal{E}$ by means of cocycles.

Step 1

Let $f : M \longrightarrow N$ be a submersion as in Lemma 6.4. Let $\alpha : [0,a] \longrightarrow M$ be a vertical curve and $\beta : [0,b] \longrightarrow M$ a horizontal curve such that $\alpha(0) = \beta(0)$. We look for a map $\sigma : [0,a] \times [0,b] \longrightarrow M$ such that:

1. $\sigma_s : [0,a] \longrightarrow M$, $\sigma_s := \sigma|_{[0,a]\times\{s\}}$, is a vertical curve,

2. $\sigma_0 = \alpha$,
3. $\sigma^t : [0,b] \longrightarrow M$, $\sigma^t := \sigma|_{\{t\}\times[0,a]}$, is a horizontal curve,
4. $\sigma^0 = \beta$.

Initially we suppose that α is a geodesic with respect to the bi-Legendrian connection ∇. In this case, $\overline{\alpha} := f \circ \alpha$ is a geodesic with respect to the bi-Lagrangian connection ∇' corresponding to the pair of transversal Lagrangian distributions (L', Q'), projection of (L, Q). So, if we set $\overline{\beta} := f \circ \beta$, we obtain a leaf curve $\overline{\alpha}$ and a horizontal curve $\overline{\beta}$ such that $\overline{\alpha}(0) = \overline{\beta}(0)$. Note that from Lemma 6.4 it follows that ∇' is tangential and, by Lemma 6.5, the leaves of $\mathcal{F}'$ are complete affine manifolds. Moreover, $N = f(M)$ is compact and connected. So we can apply the results of R. Wolak ([32]) and we find a unique rectangle $\overline{\sigma} : [0,a] \times [0,b] \longrightarrow N$ whose initial vertical edge is $\overline{\alpha}$ and whose initial horizontal edge is $\overline{\beta}$. Now we lift this rectangle to a map $\sigma : [0,a] \times [0,b] \longrightarrow M$. More precisely, for any fixed $s \in [0,b]$, consider the leaf curve $\overline{\sigma}_s : [0,a] \longrightarrow N$. We show that $\overline{\sigma}_s$ is a geodesic. Indeed, consider, for any $s \in [0,b]$, the geodesic $\overline{\tau}_s$ determined by the initial conditions $\overline{\tau}_s(0) = \overline{\sigma}_s(0)$ and $\overline{\tau}'_s(0) = \overline{\sigma}'_s(0)$. As the leaves of $\mathcal{F}$ are affine complete manifolds, applying Lemma 6.5, $\overline{\tau}_s$ is defined for all values of the parameter t and in this way we obtain a rectangle $\overline{\tau} : [0,a] \times [0,b] \longrightarrow N$ whose initial vertical edge is $\overline{\alpha}$ and whose initial horizontal edge is $\overline{\beta}$. By the uniqueness of such a rectangle, we get $\overline{\sigma} = \overline{\tau}$ and so, for all $s \in [0,b]$, $\overline{\sigma}_s = \overline{\tau}_s$. Since each $\overline{\sigma}_s$ is a geodesic, as in Lemma 6.5, we can lift it to a geodesics σ_s on the corresponding leaf of $\mathcal{F}$, namely the leaf through $\beta(s)$. In this way we find a rectangle $\sigma : [0,a] \times [0,b] \longrightarrow M$ given by $\sigma(t,s) := \sigma_s(t)$, which is the rectangle we are looking for. Indeed, by definition each σ_s is a leaf curve and, since α is a geodesic, $\sigma_0 = \alpha$. So conditions (1) and (2) are verified. Then, for every fixed $t \in [0,a]$, we have $f_{*\sigma^t(s)}({\sigma^t}'(s)) = \overline{\sigma^t}'(s) \in Q'_{\overline{\sigma}^t(s)} = f_{*\sigma^t(s)}(Q_{\sigma^t(s)})$, from which we have that, for all $s \in [0,b]$, ${\sigma^t}'(s) \in Q_{\sigma'(s)} \oplus E_{\sigma^t(s)}$, so that each σ^t is a horizontal curve. Finally $\sigma^0(s) = \sigma_s(0) = \beta(s)$, and so also (3) and (4) are satisfied.

Now we suppose that α is not a geodesic. As the leaves of $\mathcal{F}$ are complete affine manifolds, there exist $\epsilon > 0$ such that for any $p \in M$ the ball $B(p,\epsilon)$ is convex. Thus, since the leaves are totally geodesic, the ϵ-balls $B_{\mathcal{L}}(p,\epsilon)$ in any leaf $\mathcal{L}$ of $\mathcal{F}$ coincide with the corresponding connected component of $B(p,\epsilon) \cap \mathcal{L}$. Therefore there exist $\epsilon > 0$ such that the $B_{\mathcal{L}}(p,\epsilon)$ are convex. Suppose now that $\alpha : [0,a] \longrightarrow M$ is a vertical curve contained in $B_{\mathcal{L}}(p,\epsilon)$, with $p = \alpha(0)$. Let α_t denote the geodesic on $\mathcal{L}$ joining p with $\alpha(t)$, for any fixed $t \in [0,a]$. Then we define

$$\sigma(t,s) := \sigma_{\alpha_t, \beta|_{[0,s]}}(t,s),$$

for any $(t,s) \in [0,a] \times [0,b]$, where $\sigma_{\alpha_t,\beta|_{[0,s]}}$ denotes the rectangle associated to the curves α_t and $\beta|_{[0,s]}$. By the first part of the proof, σ is just the rectangle whose initial edges are α and β. Finally, if α is any leaf curve on M, not necessarily contained in $B_{\mathcal{L}}(p,\epsilon)$, then we can always find a partition of $[0,a]$, $0 = t_0 < t_1 < \cdots < t_m = a$, with the property that, for any $i \in \{0,\ldots,m-1\}$, $\alpha(t_i), \alpha(t_{i+1}) \in B(\alpha(t_i),\epsilon)$. Let $\sigma_{(0)}$ be the rectangle corresponding to $\alpha|_{[0,t_1]}$ and β. The curve $\beta_1 := \sigma_{(0)}|_{\{t_1\}\times[0,b]}$ is horizontal and $\beta_1(0) = \alpha(t_1)$, so we can find a rectangle $\sigma_{(1)}$ whose edges are $\alpha_{|_{[t_1,t_2]}}$ and β_1. After m steps we have m rectangles $\sigma_{(0)}, \sigma_{(1)}, \ldots, \sigma_{(m-1)}$ and we can define $\sigma := \sigma_{(0)} \cup \sigma_{(1)} \cup \cdots \cup \sigma_{(m-1)}$ obtaining the rectangle whose edges are α and β.

Step 2

In the general case we have a family of submersions $(f_i : U_i \longrightarrow N)_{i\in I}$ whose fibers define the foliation $\mathcal{E}$, where $\{U_i\}_{i\in I}$ is an open covering of M. We can find $U_{i_0}, \ldots, U_{i_m}$ which cover α and we can choose this covering in such a way that $\alpha(0) \in U_{i_0}$. Then let t_1 be the infimum of all $t \in [0,a]$ such that $\alpha(t)$ does not belong to U_{i_0}. Up to renumbering the open sets U_{i_j}, we may suppose that $\alpha(t_1) \in U_{i_1}$. Then let t_2 be infimum of all $t \in [t_1,a]$ such that $\alpha(t)$ does not belong to U_{i_1}. As before we can suppose that $\alpha(t_2) \in U_{i_2}$. After m steps, setting $t_0 := 0$ and $t_m := a$, we get a partition of $[0,a]$, $0 = t_0 < t_1 < \cdots < t_m = a$, with the property that, for each $j \in \{0,\ldots,m\}$, $\alpha(t_j) \in U_{i_j}$. Then, for all $t \in [0,a]$, define:

$$\overline{\alpha}(t) := \begin{cases} f_{i_0}(\alpha(t)), & \text{if } \ t_0 \leq t < t_1; \\ f_{i_1}(\alpha(t)), & \text{if } \ t_1 \leq t < t_2; \\ \vdots & \quad \vdots \\ f_{i_m}(\alpha(t)), & \text{if } \ t_{m-1} \leq t \leq t_m. \end{cases}$$

A similar construction can be repeated for β and we can choose the corresponding finite covering $\{U_{j_k}\}$ of β in such a way that $U_{i_0} = U_{j_0}$. So we have projected α and β onto two piecewise smooth curves $\overline{\alpha}$ and $\overline{\beta}$ on N such that $\overline{\alpha}$ is a vertical curve, $\overline{\beta}$ a horizontal curve and $\overline{\alpha}(0) = f_{i_0}(\alpha(0)) = f_{i_0}(\beta(0)) = \overline{\beta}(0)$. Using again [32] we can find a rectangle $\overline{\sigma} : [0,a] \times [0,b] \longrightarrow N$ whose initial edges are $\overline{\alpha}$ and $\overline{\beta}$ and lifting this rectangle, as we have seen in Step 1, by means of the submersions f_i we obtain a smooth piecewise rectangle $\sigma : [0,a] \times [0,b] \longrightarrow M$ whose initial edges are α and β. $\square$

Now we suppose that the Legendrian foliation $\mathcal{F}$ admits the Ehresmann connection $D = Q \oplus E$ and examine some consequences of the existence of an Ehresmann connection for $\mathcal{F}$.

Corollary 6.6 ([5])**.** *Any two leaves of $\mathcal{F}$ can be joined by a horizontal curve.*

Corollary 6.7 ([5])**.** *The universal covers of any two leaves of $\mathcal{F}$ are isomorphic.*

In general, to each leaf $\mathcal{L}$ of a foliation admitting an Ehresmann connection D it is attached a group $H_D(\mathcal{L}, p)$, $p \in \mathcal{L}$, defined as follows ([5]). Let Ω_p be the set of all horizontal curves $\beta : [0,1] \longrightarrow M$ with starting point p. Then there is an action of the fundamental group $\pi_1(\mathcal{L}, p)$ of $\mathcal{L}$ on Ω_p given in the following way: for any $\delta = [\tau] \in \pi_1(\mathcal{L}, p)$ and for any $\beta \in \Omega_p$, $\tau \cdot \beta$ is the final horizontal edge of the rectangle corresponding to τ and β. It can be proved that this definition does not depend on the vertical loop τ in p representing δ. Let $K_D(\mathcal{L}, p) = \{\delta \in \pi_1(\mathcal{L}, p) : \tau \cdot \beta = \beta \text{ for all } \beta \in \Omega_p\}$. Then $K_D(\mathcal{L}, p)$ is a normal subgroup of $\pi_1(\mathcal{L}, p)$ and we define

$$H_D(\mathcal{L}, p) := \pi_1(\mathcal{L}, p)/K_D(\mathcal{L}, p).$$

It is known that $H_D(\mathcal{L}, p)$ does not depend on the Ehresmann connection D, thus it is an invariant of the foliation. Concerning this group $H_D(\mathcal{L}, p)$ for a Legendrian foliation $\mathcal{F}$ admitting the Ehresmann connection $D = Q \oplus E$, we can state the following

Corollary 6.8. *If $\mathcal{F}$ has a compact leaf $\mathcal{L}_0$ with finite $H_D(\mathcal{L}_0, p_0)$, then every leaf $\mathcal{L}$ of $\mathcal{F}$ is compact with finite $H_D(\mathcal{L}, p)$.*

The proof follows by [6] and by Theorem 6.2. Another consequence of Theorem 6.2 is the following

Corollary 6.9 ([5])**.** *If Q is integrable (and so, since Q is strongly flat also $D = Q \oplus E$ is integrable), then the universal cover $\widetilde{M}$ of M is topologically a product $\widetilde{\mathcal{L}} \times \widetilde{\mathcal{D}}$, where $\widetilde{\mathcal{L}}$ is the universal cover of the leaves of $\mathcal{F}$ and $\widetilde{\mathcal{D}}$ the universal cover of the leaves of the foliation D.*

In [33] R. Wolak studied the relations between Ehresmann connections, vanishing cycles and graphs of a foliation. A *vanishing cycle* for a foliation $\mathcal{F}$ is a mapping $c : S^1 \times [0,1] \longrightarrow M$ such that for any $t \in [0,1]$ $c_t = c|_{S^1 \times \{t\}}$ is a loop on a leaf of $\mathcal{F}$ and c_0 is not homotopic to the constant loop in the leaf but for $t > 0$ the loops c_t are (cf. [25]). Among other things, R. Wolak has demonstrated that a foliation admitting an Ehresmann connection has no vanishing cycles. In particular this implies the following

Corollary 6.10. *Let $\mathcal{F}$ be a Legendrian foliation of an almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ such that the assumptions of Theorem 6.2 are satisfied. Then $\mathcal{F}$ has no vanishing cycles.*

We conclude with another corollary of Theorem 6.2. Recall that the *homotopy groupoid* of a foliation $\mathcal{F}$ is the space of equivalence classes of triples (x, α, y), where x and y are points of the same leaf $\mathcal{L}$ of $\mathcal{F}$ and α is a path in $\mathcal{L}$ linking x to y. Two triples (x, α, y) and (x', α', y') are equivalent if and only if $x = x'$, $y = y'$ and the leaf curves α and α' are homotopic relative to their ends in the corresponding leaf. One of the question about the homotopy groupoid is whether it is Hausdorff. Indeed, in general, the homotopy groupoid is a manifold, but not necessarily Hausdorff. In [12] it is proved that the non-existence of vanishing cycles is equivalent to the Hausdorfness of the homotopy groupoid of the foliation. This result, together with Corollary 6.10, implies the following

Corollary 6.11. *Let $\mathcal{F}$ be a Legendrian foliation of an almost $\mathcal{S}$-manifold $(M, \phi, \xi_\alpha, \eta_\alpha, g)$ such that the assumptions of Theorem 6.2 are satisfied. Then the homotopy groupoid of $\mathcal{F}$ is a Hausforff manifold.*

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