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Classification of Frobenius Lie algebras of dimension ≤ 6

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Abstract. A Lie algebra $\mathfrak{g}$ over an arbitrary field is a Frobenius Lie algebra if there is a linear form $l \in \mathfrak{g}^*$ whose stabilizer with respect to the coadjoint representation of $\mathfrak{g}$, i.e. $\mathfrak{g}(l) = \{X \in \mathfrak{g} \mid l([X,Y]) = 0 \text{ for all } Y \in \mathfrak{g}\}$ is trivial. In the present paper we classify Frobenius Lie algebras of dimension 4 over arbitrary fields of characteristic $\neq 2$ and 6-dimensional Frobenius Lie algebras over algebraically closed fields of characteristic 0.

1. Introduction

A real or complex Lie group G and its Lie algebra $\mathfrak{g}$ are called Frobenius if the coadjoint representation of G has an open orbit. The Lie algebra of the stabilizer of $l \in \mathfrak{g}^*$ is $\mathfrak{g}(l) = \{X \in \mathfrak{g} \mid l([X,Y]) = 0 \text{ for all } Y \in \mathfrak{g}\}$, therefore the orbit of l is open if and only if $\mathfrak{g}(l) = 0$. In general, a Lie algebra over an arbitrary field is said to be Frobenius if there is a linear form $l \in \mathfrak{g}^*$ such that $\mathfrak{g}(l) = 0$.

The class of Frobenius Lie algebras was first introduced and studied by A. I. Ooms in [9], [10] and [11] in connection with the problem of Jacobson on the characterization of Lie algebras having a primitive universal enveloping algebra. Frobenius Lie groups appear in the theorem of Anh [1], saying that an exponential Lie group with trivial center has a square integrable representation if and only if it is a Frobenius Lie group. The study of Frobenius Lie algebras is motivated also by the fact that they possess a triangular Lie bialgebra structure and give solutions to the classical Yang–Baxter equation (see [5]).

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There are many interesting examples of Frobenius Lie algebras (see e.g. [6], [7], [12], [13]). The present paper focuses on the classification of low dimensional Frobenius Lie algebras. Frobenius Lie algebras are always even dimensional. There is a unique 2-dimensional Frobenius Lie algebra, the Lie algebra of the group of affine transformations of the line. Four-dimensional Frobenius Lie algebras over algebraically closed fields of characteristic 0 were first listed in [9]. This classification can be obtained from the classification of all 4-dimensional Lie algebras over algebraically closed fields of characteristic 0 (see [2]). In Section 3 we extend this classification for fields of characteristic $\neq 2$. In Section 4 we classify all 6-dimensional Frobenius Lie algebras over algebraically closed fields of characteristic 0. This yields an extension of the work of A. G. Elashvili [7], giving a list of all 6-dimensional almost algebraic Frobenius Lie algebras over such fields. Our results are summarized in Section 5. The embedding of A. G. Elashvili's list of almost algebraic Frobenius Lie algebras into the list of all Frobenius Lie algebras is given in Remark 5.3.

2. Definition and some properties of Frobenius Lie algebras

In this section we prove those general theorems on Frobenius Lie algebras that will be used during the classification process. We mention that in characteristic zero Propositions 2.1 and 2.3 were already obtained in [9].

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an arbitrary field $\mathbb{F}$, $\mathfrak{g}^*$ be its dual space. For $l \in \mathfrak{g}^*$, denote by $B_l : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$, $B_l(X,Y) = l([X,Y])$ the Kirillov form. The kernel of this form is

$$\mathfrak{g}(l) = \{X \in \mathfrak{g} \mid l([X,Y]) = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

Since $\mathfrak{g}(l) = \{X \in \mathfrak{g} \mid (\operatorname{ad} X)^* l = 0\}$, $\mathfrak{g}(l)$ is the stabilizator subalgebra of the linear form l with respect to the coadjoint representation. B_l induces a symplectic form on $\mathfrak{g}/\mathfrak{g}(l)$ and therefore $\dim\mathfrak{g} \equiv \dim\mathfrak{g}(l) \pmod 2$.

Definition 2.1. The index of a Lie algebra is the number

$$\operatorname{ind}\mathfrak{g} = \min\{\dim\mathfrak{g} - \operatorname{rk} B_l \mid l \in \mathfrak{g}^*\} = \min\{\dim\mathfrak{g}(l) \mid l \in \mathfrak{g}^*\}.$$

Linear forms for which $\dim\mathfrak{g}(l) = \operatorname{ind}\mathfrak{g}$ are called regular.

Definition 2.2. Lie algebras of index 0 are called Frobenius Lie algebras.

Let $\Lambda = (X_1, \dots, X_n)$ be a basis of $\mathfrak{g}$ and consider the skew symmetric matrix $M_\Lambda = ([X_i, X_j])_{i,j=1}^n$. The entries of M_Λ are in $\mathfrak{g}$ which is naturally embedded into the symmetric algebra $S(\mathfrak{g})$ generated by $\mathfrak{g}$. Thus we can compute the Pfaffian

$Q_\Lambda = \operatorname{Pf} M_\Lambda$ in $S(\mathfrak{g})$. If $C \in GL(\mathfrak{g})$ is an invertible linear transformation of $\mathfrak{g}$, $C\Lambda$ is the image of the basis Λ under C, C_Λ is the matrix of C with respect to the basis Λ, then $M_{C\Lambda} = C_\Lambda^T M_\Lambda C_\Lambda$, consequently

$$Q_{C\Lambda} = \operatorname{Pf}(M_{C\Lambda}) = \operatorname{Pf}(C_\Lambda^T M_\Lambda C_\Lambda) = \det C \cdot Q_\Lambda. \tag{1}$$

In particular, if C is a Lie algebra automorphism and S_C is the induced automorphism of the symmetric algebra $S(\mathfrak{g})$, then

$$\begin{aligned} S_C(Q_\Lambda) &= \operatorname{Pf}((C[X_i, X_j])_{i,j=1}^n) = \operatorname{Pf}(([CX_i, CX_j])_{i,j=1}^n) \\ &= Q_{C\Lambda} = \det C \cdot Q_\Lambda. \end{aligned} \tag{2}$$

According to (1), Q_Λ is uniquely determined by $\mathfrak{g}$ up to a nonzero multiplier. Q_Λ (and its nonzero multiples) will be called the Pfaffian(s) of $\mathfrak{g}$.

Since the elements of $\mathfrak{g}$ can be identified with linear functions on $\mathfrak{g}^*$, every element of $S(\mathfrak{g})$ yields a polynomial function on $\mathfrak{g}^*$ which will be denoted by the same symbol. Recall that the map from the ring of polynomials onto the space of polynomial functions on $\mathfrak{g}^*$ is always injective on the space of polynomials of degree $< |\mathbb{F}|$. The determinant of the matrix of B_l with respect to the basis Λ is $\det(l([X_i, X_j]))_{i,j=1}^n = Q_\Lambda^2(l)$, in particular, in the case of $\dim \mathfrak{g} < 2|\mathbb{F}|$, $\mathfrak{g}$ is Frobenius if and only if the polynomial $Q_\Lambda \neq 0$.

Every derivation δ of the Lie algebra $\mathfrak{g}$ extends uniquely to a derivation D_δ of $S(\mathfrak{g})$ whose restriction onto $\mathbb{F} \subset S(\mathfrak{g})$ is zero. An element F of $S(\mathfrak{g})$ is said to be a semi-invariant of weight $\lambda \in \mathfrak{g}^*$ if $D_{\operatorname{ad} X}(F) = \lambda(X)F$ for all $X \in \mathfrak{g}$. F is a characteristic semi-invariant of weight $\tilde{\lambda} \in (Der(\mathfrak{g}))^*$ if we have $D_\delta(F) = \tilde{\lambda}(\delta)F$ for every derivation $\delta \in Der(\mathfrak{g})$.

"Differentiating" equation (2) we obtain the following proposition.

Proposition 2.1. *The Pfaffian of a Lie algebra is a characteristic semi-invariant of weight* tr, *i.e.*

$$D_\delta(Q_\Lambda) = \operatorname{tr}(\delta) Q_\Lambda \quad \textit{for all } \delta \in \operatorname{Der}(\mathfrak{g}).$$

If V is a linear space, $W \leq V$ is a linear subspace, then the symmetric algebra $S(W)$ is embedded naturally into $S(V)$. For any element F of the symmetric algebra $S(V)$, there is a minimal subspace $W \leq V$ for which $F \in S(W)$. W will be called the support of F and will be denoted by $\operatorname{supp} F$.

The proof of the following proposition is straightforward.

Proposition 2.2. *If $P \in S(\mathfrak{g})$ is a characteristic semi-invariant and either* $\operatorname{char} \mathbb{F} = 0$ *or* $\deg P < \operatorname{char} \mathbb{F}$, *then the support of P is a characteristic ideal in $\mathfrak{g}$.* □

Proposition 2.3. *If* $\mathfrak{g} \neq 0$ *is a Frobenius Lie algebra and either* $\dim \mathfrak{g} < 2 \operatorname{char} \mathbb{F}$ *or* $\operatorname{char} \mathbb{F} = 0$, *then the weight of the Pfaffian, i.e. the linear form* $\operatorname{tr} \circ \operatorname{ad} \in \mathfrak{g}^*$ *is not* 0.

Proof. Suppose to the contrary that $D_{\operatorname{ad} X} Q_\Lambda = 0$ for all $X \in \mathfrak{g}$. Choosing a basis $\Lambda = (X_1, \dots, X_n)$ in $\mathfrak{g}$, $S(\mathfrak{g})$ can be identified with the algebra of polynomials $\mathbb{F}[X_1, \dots, X_n]$. With this identification, the derivation $D_{\operatorname{ad} X}$ can be expressed as

$$D_{\operatorname{ad} X} = \sum_{i=1}^n [X, X_i] \frac{\partial}{\partial X_i}.$$

In particular, we have

$$D_{\operatorname{ad} X} P(l) = \sum_{i=1}^n B_l(X, X_i) \frac{\partial P}{\partial X_i}(l) \tag{3}$$

for any $P \in S(\mathfrak{g})$ and $l \in \mathfrak{g}^*$. Select a regular form $l \in \mathfrak{g}^*$. Then $Q_\Lambda(l) \neq 0$ and B_l is non-singular. In view of equation (3), applying equation $D_{\operatorname{ad} X} Q_\Lambda = 0$ for the vectors of the B_l-dual basis of Λ we obtain that $\frac{\partial Q_\Lambda}{\partial X_i}(l) = 0$ for all i. However, Q_Λ is a homogeneous polynomial, thus, by Euler's formula

$$\deg Q_\Lambda \cdot Q_\Lambda(l) = \sum_{i=1}^n l(X_i) \frac{\partial Q_\Lambda}{\partial X_i}(l) = 0.$$

The assumptions on $\operatorname{char} \mathbb{F}$ guarantee that $\deg Q_\Lambda \neq 0$ in $\mathbb{F}$, therefore $Q_\Lambda(l) = 0$ and this contradicts the fact that l is regular. $\square$

Corollary 2.1. *Under the assumptions of Proposition 2.3,* $\mathfrak{g}$ *is not nilpotent and*

$$\operatorname{supp} Q_\Lambda \trianglelefteq [\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}.$$

Lemma 2.1. *Suppose that* $\operatorname{char} \mathbb{F} = 0$. *If* $\mathfrak{a} \trianglelefteq \mathfrak{g}$ *is a Frobenius ideal of a Lie algebra* $\mathfrak{g}$, *then the short exact sequence*

$$0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{a} \to 0$$

is splitting, i.e. $\mathfrak{g}$ *is the semidirect product of* $\mathfrak{a}$ *and* $\mathfrak{g}/\mathfrak{a}$. *In particular,* $\mathfrak{g}$ *is a Frobenius Lie algebra if and only if the factor* $\mathfrak{g}/\mathfrak{a}$ *is Frobenius.*

Proof. Choose a regular linear form $l \in \mathfrak{g}^*$ whose restriction $m = l|_\mathfrak{a}$ onto $\mathfrak{a}$ is also regular. Denote by $\mathfrak{b}$ the stabilizator of m in $\mathfrak{g}$, that is

$$\mathfrak{b} = \mathfrak{g}(m) = \{X \in \mathfrak{g} \mid l([X, Y]) = 0 \quad \forall Y \in \mathfrak{a}\}.$$

Then $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$; $\mathfrak{b} \cap \mathfrak{a} = \{0\}$, since $\mathfrak{a}$ is Frobenius and m is regular; and we have $\dim \mathfrak{a} \geq \dim \mathfrak{g} - \dim \mathfrak{b}$. From this follows that $\mathfrak{g}$ is the semidirect product of $\mathfrak{a}$ and $\mathfrak{b} \cong \mathfrak{g}/\mathfrak{a}$. □

Proposition 2.4. *Suppose that* $\operatorname{char} \mathbb{F} = 0$. *Let* $\mathfrak{g}$ *be a Frobenius Lie algebra,* $\mathfrak{g}_1 \leq \mathfrak{g}$ *be a subalgebra containing* $[\mathfrak{g}, \mathfrak{g}]$. *Then*

$$\operatorname{ind} \mathfrak{g}_1 = \dim \mathfrak{g} - \dim \mathfrak{g}_1.$$

Proof. Observe first that if $\mathfrak{g}$ is an arbitrary Lie algebra and $\mathfrak{g}_0 \geq [\mathfrak{g}, \mathfrak{g}]$ is a proper Lie subalgebra of $\mathfrak{g}$, then $\mathfrak{g}_0$ cannot be Frobenius since in that case the factor $\mathfrak{g}/\mathfrak{g}_0$ should be Frobenius as well by Lemma 2.1, however $\mathfrak{g}/\mathfrak{g}_0$ is commutative.

When $\mathfrak{g}$ is a Frobenius Lie algebra we can find a regular linear form l in $\mathfrak{g}^*$ whose restriction onto $\mathfrak{g}_1$ is also regular on $\mathfrak{g}_1$. Then B_l is a symplectic form on $\mathfrak{g}$ and its restriction onto $\mathfrak{g}_1$ has corank $k = \operatorname{ind} \mathfrak{g}_1$. This implies the existence of a k-dimensional linear subspace $\mathfrak{h}$ in $\mathfrak{g}$ complementary to $\mathfrak{g}_1$ such that B_l is a symplectic form on $\mathfrak{g}_0 = \mathfrak{g}_1 \oplus \mathfrak{h}$. However, in this case $\mathfrak{g}_0 \geq [\mathfrak{g}, \mathfrak{g}]$ is Frobenius, therefore $\mathfrak{g}_0 = \mathfrak{g}$. In particular,

$$\dim \mathfrak{g} = \dim \mathfrak{g}_0 = \dim \mathfrak{h} + \dim \mathfrak{g}_1 = \operatorname{ind} \mathfrak{g}_1 + \dim \mathfrak{g}_1. \qquad \square$$

Finally, we introduce some terminology and notation.

Every 1-dimensional $\mathfrak{g}$-module spanned by a non-zero vector v induces a unique linear form $\lambda \in \mathfrak{g}^*$ such that $Xv = \lambda(X)v$ for all $X \in \mathfrak{g}$. λ is called the weight of the module. Every solvable Lie algebra $\mathfrak{g}$ has a sequence of ideals $\mathfrak{g} = \mathfrak{n}_n \rhd \mathfrak{n}_{n-1} \rhd \cdots \rhd \mathfrak{n}_0$ with $\dim \mathfrak{n}_i = i$. The roots of $\mathfrak{g}$ are the weights of the 1-dimensional $\mathfrak{g}$-modules $\mathfrak{n}_i/\mathfrak{n}_{i-1}$.

If A is a linear transformation of a linear space V of dimension n, and λ is an eigenvalue of A, then the generalized eigenspace corresponding to λ is the kernel of the operator $(A - \lambda I)^n$.

The linear hull of the vectors $X_1, X_2, \dots$ will be denoted by $\langle X_1, X_2, \dots \rangle$.

3. Classification of 4-dimensional Frobenius Lie algebras over fields of characteristic $\neq 2$

Let $\mathfrak{g}$ be a 4-dimensional Frobenius Lie algebra over a field $\mathbb{F}$ of characteristic $\neq 2$. The Pfaffian Q_Λ of $\mathfrak{g}$ is a quadratic form and thus, choosing a suitable basis $\Lambda = (X_1, X_2, X_3, X_4)$ it takes the form $Q_\Lambda = X_1^2 + \sum_{i=2}^4 \varepsilon_i X_i^2$, where $\varepsilon_i \in \mathbb{F}$. It is clear that the support of Q_Λ is the linear space spanned by $\{X_1\} \cup \{X_i \mid \varepsilon_i \neq 0, i \in \{2,3,4\}\}$. Since $\operatorname{supp} Q_\Lambda \neq \mathfrak{g}$ by Corollary 2.1, one of the numbers ε_i, say ε_4 is equal to 0.

Suppose first that $\varepsilon_2 \cdot \varepsilon_3 \neq 0$. Then from equation

$$D_{\operatorname{ad} X_1} Q_\Lambda = 2([X_1, X_2]\varepsilon_2 X_2 + [X_1, X_3]\varepsilon_3 X_3)$$

we see that $D_{\operatorname{ad} X_1} Q_\Lambda = \alpha \cdot Q_\Lambda$ can hold only if $\alpha = 0$ and then there is a $\lambda \in \mathbb{F}$ such that

$$[X_1, X_2] = \lambda\varepsilon_3 X_3, \qquad [X_1, X_3] = -\lambda\varepsilon_2 X_2.$$

A similar argument for $D_{\operatorname{ad} X_2} Q_\Lambda$ shows that $[X_2, X_3] = \lambda X_1$.

λ is not zero, since otherwise we would have $Q_\Lambda = 0$. This implies that the ideal $\operatorname{supp} Q_\Lambda$ is semisimple. Over fields of characteristic 0 semisimplicity of a Lie algebra implies that the Lie algebra has no outer derivations, however, this is not true in general over fields of prime characteristic. Nevertheless, in our special case, a direct computation shows that if $\operatorname{char} \mathbb{F} \neq 2$, then $\operatorname{supp} Q_\Lambda$ has no outer derivations. In particular, there is an element $X \in \operatorname{supp} Q_\Lambda$ such that $\operatorname{ad} X$ and $\operatorname{ad} X_4$ coincide on $\operatorname{supp} Q_\Lambda$. However, in that case $X - X_4$ is a non-zero central element of $\mathfrak{g}$ contradicting the fact that Frobenius Lie algebras have trivial center. The contradiction shows that $\dim \operatorname{supp} Q_\Lambda$ is either 1 or 2.

Consider the case when $\dim \operatorname{supp} Q_\Lambda = 1$. Then the basis Λ can be chosen in such a way that $Q_\Lambda = X_1^2$, $[X_2, X_1] = [X_3, X_1] = 0$, $[X_4, X_1] = X_1$. The commutation table of $\mathfrak{g}$ with respect to Λ has the form

$[\,,\,]$	X_1	X_2	X_3	X_4
X_1	0	0	0	$-X_1$
X_2	0	0	$[X_2, X_3]$	$*$
X_3	0	$[X_3, X_2]$	0	$*$
X_4	X_1	$*$	$*$	0

,

hence $X_1^2 = Q_\Lambda = \operatorname{Pf} M_\Lambda = X_1[X_3, X_2]$ and therefore $[X_3, X_2] = X_1$. Set $[X_4, X_2] = \alpha X_1 + \beta X_2 + \gamma X_3$ and $[X_4, X_3] = \alpha' X_1 + \beta' X_2 + \gamma' X_3$. Replacing X_4 by $X_4 - \alpha X_3 + \alpha' X_2$ we may assume without loss of generality that $\alpha = \alpha' = 0$. Furthermore, the Jacobi identity for the triple (X_2, X_3, X_4) yields the relation $\gamma' + \beta = 1$. It is not difficult to check that the determinant

$$\Delta = \det \begin{pmatrix} \beta & \gamma \\ \beta' & 1 - \beta \end{pmatrix}$$

is an invariant of the isomorphism class of $\mathfrak{g}$. Indeed, Δ can be derived from the Lie algebra structure of $\mathfrak{g}$ in the following natural way. The support $\mathfrak{g}_1$ of Q_Λ and its centralizer $\mathfrak{g}_2$ are characteristic ideals of $\mathfrak{g}$. The adjoint action of $\mathfrak{g}$ induces a nontrivial representation ρ of the one-dimensional factor algebra $\mathfrak{g}/\mathfrak{g}_2$ on the 2-dimensional factor $\mathfrak{g}_2/\mathfrak{g}_1$. Δ can be expressed as $\Delta = \det(\rho(X))/\operatorname{tr}^2(\rho(X))$, where $X \in \mathfrak{g}/\mathfrak{g}_2$ is an arbitrary nonzero element. If the operator $\operatorname{ad} X_4$ acts on the linear subspace $\langle X_2, X_3\rangle$ as a scalar multiplication, then it must be a multiplication by $1/2$ since $\operatorname{tr}((\operatorname{ad} X_4)|_{\langle X_2,X_3\rangle}) = 1$. On the other hand, if $\operatorname{ad} X_4$ is not a scalar multiplication on $\langle X_2, X_3\rangle$, then the basis (X_2, X_3) of this subspace can be modified in such a way that $X_3 = \operatorname{ad} X_4(X_2)$ takes place. Then the Cayley–Hamilton theorem gives

$$[X_4, X_3] = (\operatorname{ad} X_4)^2 X_2 = (\operatorname{ad} X_4 - \Delta)X_2 = X_3 - \Delta X_2.$$

Thus we obtain the following pairwise non-isomorphic Frobenius Lie algebras:

Φ' :

[,]	X_1	X_2	X_3	X_4
X_1	0	0	0	$-X_1$
X_2	0	0	$-X_1$	$-\frac{1}{2}X_2$
X_3	0	X_1	0	$-\frac{1}{2}X_3$
X_4	X_1	$\frac{1}{2}X_2$	$\frac{1}{2}X_3$	0

,

and

$\Phi''(\Delta)$:

[,]	X_1	X_2	X_3	X_4
X_1	0	0	0	$-X_1$
X_2	0	0	$-X_1$	$-X_3$
X_3	0	X_1	0	$-X_3 + \Delta X_2$
X_4	X_1	X_3	$X_3 - \Delta X_2$	0

, $\Delta \in \mathbb{F}$.

If $\mathbb{F}$ is algebraically closed, then it is more natural to transform the matrix of $\operatorname{ad} X_4|_{\langle X_2,X_3\rangle}$ to a Jordan normal form. This operator is either diagonalizable with diagonal elements $(\xi, 1-\xi)$, and then $\mathfrak{g}$ is isomorphic to $\Phi_{4,1}(\xi) \cong \Phi_{4,1}(1-\xi)$ of [9], or it has a 2×2 Jordan cell with eigenvalue $1/2$, and then it is isomorphic to $\Phi_{4,2}$ of [9]. We remark that $\Phi_{4,1}(\xi) \cong \Phi''(\xi(1-\xi))$ when $\xi \neq 1/2$, $\Phi_{4,1}(1/2) \cong \Phi'$ and $\Phi_{4,2} \cong \Phi''(1/4)$.

Suppose that $\dim \operatorname{supp} Q_\Lambda = 2$. Then there is a basis $\Lambda = (X_1, \ldots, X_4)$ such that $Q_\Lambda = X_1^2 + \varepsilon X_2^2$, where $0 \neq \varepsilon \in \mathbb{F}$. Since $D_{\operatorname{ad} X_1}(Q_\Lambda) = 2\varepsilon[X_1, X_2]X_2$ must be proportional to Q_Λ, $[X_1, X_2] = 0$. Suppose that $[X_3, X_1] = \alpha X_1 + \beta X_2$ and

$[X_3, X_2] = \gamma X_1 + \delta X_2$. Then $D_{\mathrm{ad}\, X_3} Q_\Lambda = 2(\alpha X_1^2 + \varepsilon\delta X_2^2 + (\beta + \varepsilon\gamma)X_1X_2)$, and the semi-invariance of Q_Λ implies $\alpha = \delta$ and $\beta = -\varepsilon\gamma$. Similarly, $[X_4, X_1] = \alpha' X_1 - \gamma'\varepsilon X_2$ and $[X_4, X_2] = \gamma' X_1 + \alpha' X_2$ for some $\alpha', \gamma' \in \mathbb{F}$. Computing the Pfaffian of $\mathfrak{g}$ we obtain $X_1^2 + \varepsilon X_2^2 = Q_\Lambda = (\alpha\gamma' - \gamma\alpha')(X_1^2 + \varepsilon X_2^2)$, i.e. $\alpha\gamma' - \gamma\alpha' = 1$.

This means that the basis vectors X_3, X_4 can be modified in such a way that we have $(\alpha, \gamma) = (1, 0)$ and $(\alpha', \gamma') = (0, 1)$. Since $\mathrm{ad}[X_3, X_4]$ acts on $\langle X_1, X_2\rangle$ trivially, $[X_3, X_4]$ must be a linear combination of X_1 and X_2, say $[X_3, X_4] = \xi_1 X_1 + \xi_2 X_2$. In this case replacing X_3 by $X_3 - (1/\varepsilon)\xi_2 X_1 + \xi_1 X_2$ we obtain a basis with respect to which the commutation table has the form

$\Phi'''(\varepsilon)$:

[,]	X_1	X_2	X_3	X_4
X_1	0	0	$-X_1$	εX_2
X_2	0	0	$-X_2$	$-X_1$
X_3	X_1	X_2	0	0
X_4	$-\varepsilon X_2$	X_1	0	0

,

where $0 \neq \varepsilon \in \mathbb{F}$. If $0 \neq \lambda \in \mathbb{F}$, then the commutation table of $\Phi'''(\varepsilon)$ with respect to the basis $(X_1,\ X_2/\lambda,\ X_3,\ \lambda X_4)$ has the same form as the commutation table defining $\Phi'''(\varepsilon\lambda^2)$, thus, $\Phi'''(\varepsilon)$ is isomorphic to $\Phi'''(\varepsilon\lambda^2)$. Conversely, if $\Phi'''(\varepsilon)$ and $\Phi'''(\bar{\varepsilon})$ are isomorphic, then both ε and $\bar{\varepsilon}$ can be obtained as the discriminant of a multiple of the quadratic form $Q_\Lambda \in S(\mathrm{supp}\, Q_\Lambda)$ with respect to a basis of $\mathrm{supp}\, Q_\Lambda$, therefore $\varepsilon = \bar{\varepsilon}\lambda^2$ for some nonzero $\lambda \in \mathbb{F}$. This proves that the Lie algebras $\Phi'''(\varepsilon_1)$ and $\Phi'''(\varepsilon_2)$ corresponding to the nonzero parameters $\varepsilon_1, \varepsilon_2 \in \mathbb{F}$ are isomorphic if and only if $\sqrt{\varepsilon_1/\varepsilon_2} \in \mathbb{F}$. In particular, if $\mathbb{F}$ is algebraically closed, then each of these Lie algebras are isomorphic to $\Phi'''(1) \cong \mathfrak{aff}(1, \mathbb{F}) \oplus \mathfrak{aff}(1, \mathbb{F})$.

4. Classification of 6-dimensional Frobenius Lie algebras

In this section $\mathbb{F}$ is an algebraically closed field of characteristic 0. The classification of 6-dimensional Frobenius Lie algebras over $\mathbb{F}$ goes by the following scheme. First we consider solvable Frobenius Lie algebras. The nilradical of such a Lie algebra has dimension ≤ 5. It is known that there is only a finite number of isomorphism classes of nilpotent Lie algebras of dimension ≤ 6 (see [8]). With the help of Proposition 2.4, we can select those 6 nilpotent Lie algebras, that can serve for the nilradical of a Frobenius Lie algebra. After this step, classification is split into cases depending on the isomorphism class of the nilradical. As for the non-solvable case, the semisimple part of a 6-dimensional Frobenius

Lie algebra can only be $\mathfrak{sl}(2,\mathbb{F})$, since any other semisimple Lie algebra has too large dimension. In this case, the 3-dimensional radical is either an irreducible $\mathfrak{sl}(2,\mathbb{F})$-module, or it can be split into the direct sum of a one-dimensional and a 2-dimensional irreducible submodule. It turns out that only the second case is possible, therefore, the only nonsolvable Frobenius Lie algebra of dimension 6 is the Lie algebra $\mathfrak{aff}(2,\mathbb{F})$ of the group of affine transformations of the plane $\mathbb{F}^2$.

Recall the description of the set N_n of the isomorphism classes of nilpotent Lie algebras of dimension $n \leq 5$ (see [4], [8]). The numeration given here is due to Dixmier [4]. Every such Lie algebra contains a basis $\{Y_i \mid i = 1, \ldots, n\}$ the elements of which commute according to the rule

$$[Y_i, Y_j] = a_{ij} Y_{i+j}, \tag{4}$$

where $a_{ij} \in \{0,1\}$ if $i < j$ and $Y_{i+j} = 0$ if $i + j > n$. Thus, the isomorphism class of such a Lie algebra is uniquely determined by the set of vanishing a_{ij}'s.

Below we give a complete list of the elements of N_n for $n \leq 5$. Following the notation of an isomorphism class, we list in curly brackets those coefficients a_{ij} which have to be made 0 if we want to define the given class with the commutation rule (4).

$$N_1 : \mathbb{F};$$
$$N_2 : \mathbb{F}^2;$$
$$N_3 : \Gamma_3\,\{\},\ \mathbb{F}^3\,\{a_{12}\};$$
$$N_4 : \Gamma_4\,\{\},\ \Gamma_3 \oplus \mathbb{F}\,\{a_{12} \text{ or } a_{13}\},\ \mathbb{F}^4\,\{a_{12}, a_{13}\};$$

N_5 contains the decomposable Lie algebras $\Gamma_4 \oplus \mathbb{F}$, $\Gamma_3 \oplus \mathbb{F}^2$, $\mathbb{F}^5$ and the indecomposable Lie algebras $\Gamma_{5,1}\,\{a_{12}, a_{13}\}$, $\Gamma_{5,2}\,\{a_{13}, a_{23}\}$, $\Gamma_{5,3}\,\{a_{12}\}$ or $\{a_{13}\}$, $\Gamma_{5,4}\,\{a_{14}\}$, $\Gamma_{5,5}\,\{a_{23}\}$, $\Gamma_{5,6} = \Gamma_5\,\{\}$.

By Proposition 2.4, if $\mathfrak{n}$ is the nilradical of the 6-dimensional Frobenius Lie algebra $\mathfrak{g}$, then it satisfies

$$\operatorname{ind}\mathfrak{n} = 6 - \dim\mathfrak{n}. \tag{5}$$

Among the nilpotent Lie algebras of dimension ≤ 5, the following six Lie algebras satisfy condition (5):

$$\mathbb{F}^3,\ \Gamma_4,\ \Gamma_3 \oplus \mathbb{F},\ \Gamma_{5,1},\ \Gamma_{5,3},\ \Gamma_{5,6}. \tag{6}$$

Case 1: $\mathfrak{n} \cong \mathbb{F}^3$. Since the nilradical of a solvable Lie algebra is the intersection of the kernels of its roots, in our case, $\mathfrak{g}$ has 3 linearly independent roots, say λ_1, λ_2, λ_3. Choose an element $\bar{X} \in \mathfrak{g}$ such that the numbers $\lambda_1(\bar{X})$, $\lambda_2(\bar{X})$, $\lambda_3(\bar{X})$ are different, and denote by Y_i the (unique up to a scalar multiplier) eigenvector of the operator $\operatorname{ad}\bar{X}$ corresponding to the eigenvalue $\lambda_i(\bar{X})$. It is clear that

$\{Y_1, Y_2, Y_3\}$ is a basis of the nilradical. Since $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}] = 0$, we have

$$[\bar{X}, [X, Y_i]] = [[\bar{X}, X], Y_i] + [X, [\bar{X}, Y_i]] = \lambda_i(\bar{X})[X, Y_i] \quad i = 1, 2, 3, \tag{7}$$

showing that the vectors Y_i are eigenvectors for the operator $\operatorname{ad} X$ for any $X \in \mathfrak{g}$. Now choose linearly independent vectors $\{X_1', X_2', X_3'\}$ in a subspace complementary to $\mathfrak{n}$ such that $\lambda_i(X_j') = \delta_{ij}$ for $1 \leq i, j \leq 3$, (where δ_{ij} is the Kronecker δ), then set $X' = X_1' + X_2' + X_3'$ and $X_i = X_i' - [X', X_i']$, $(i = 1, 2, 3)$. Since $[X', Y_i] = Y_i$, $\operatorname{ad} X'|_{\mathfrak{n}} = \operatorname{id}_{\mathfrak{n}}$ and the vectors X_i, Y_i obey the following commutation rules:

$$\begin{aligned}
[Y_i, Y_j] &= 0;\\
[X_i, Y_j] &= \lambda_j(X_i)Y_j = \lambda_j(X_i' - [X', X_i'])Y_j = \lambda_j(X_i')Y_j = \delta_{ij}Y_j;\\
[X_i, X_j] &= [X_i' - [X', X_i'], X_j' - [X', X_j']]\\
&= [X_i', X_j'] - [[X', X_i'], X_j'] - [X_i', [X', X_j']]\\
&= [X_i', X_j'] - \operatorname{ad} X'([X_i', X_j']) = 0.
\end{aligned}$$

This implies that $\mathfrak{g}$ is the direct sum of three copies of the Lie algebra $\mathfrak{aff}(1, \mathbb{F})$, i.e.

$$\mathfrak{g} \cong \mathfrak{aff}(1, \mathbb{F}) \oplus \mathfrak{aff}(1, \mathbb{F}) \oplus \mathfrak{aff}(1, \mathbb{F}). \tag{8}$$

Case 2: $\mathfrak{n} \cong \Gamma_4$. We have the following chain of characteristic ideals in the Lie algebra Γ_4:

$$\Gamma_4 \rhd \bar{\mathfrak{z}} \rhd [\Gamma_4, \Gamma_4] \rhd \mathfrak{z}, \tag{9}$$

where $\mathfrak{z}$ is the center of Γ_4, $\bar{\mathfrak{z}}$ is the centralizer of the commutator algebra $[\Gamma_4, \Gamma_4]$. It is easy to check that if we denote by α and β the roots of the Lie algebra $\mathfrak{g}$ induced on the 1-dimensional $\mathfrak{g}$-modules $\Gamma_4/\bar{\mathfrak{z}}$ and $\bar{\mathfrak{z}}/[\Gamma_4, \Gamma_4]$, respectively, then the 1-dimensional $\mathfrak{g}$-modules $[\Gamma_4, \Gamma_4]/\mathfrak{z}$ and $\mathfrak{z}$ induce the roots $\alpha + \beta$ and $2\alpha + \beta$, respectively. Since the nilradical of $\mathfrak{g}$ has codimension 2, α and β are linearly independent. Thus, one can find an element $X \in \mathfrak{g}$ such that $\alpha(X) = 1$ and $\beta(X) = 2$. The restriction of the operator $\operatorname{ad} X$ onto the kernel of α has eigenvalues 0, 1, 2, 3, 4. Let X', Y_1, Y_2, Y_3, Y_4 be the eigenvectors corresponding to these eigenvalues. The commutator $[Y_i, Y_j]$ is either 0 or an eigenvector of $\operatorname{ad} X$ with eigenvalue $i + j$, therefore

$$[Y_i, Y_j] = \begin{cases} a_{ij} Y_{i+j} & \text{if } (i, j) = (1, 3) \text{ or } (1, 2),\\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Since the commutator algebra $[\Gamma_4, \Gamma_4]$ is 2-dimensional, the coefficients a_{12} and a_{13} are different from zero, therefore, by a suitable choice of the eigenvectors Y_1, Y_2, Y_3, Y_4 they will be equal to 1. If we choose a basis $\{X_1, X_2\}$ in the linear hull of the vectors X and X' such that $\alpha(X_1) = \beta(X_2) = 1$ and $\alpha(X_2) = \beta(X_1) = 0$, then the commutation table of $\mathfrak{g}$ with respect to the basis $\{X_1, X_2, Y_1, Y_2, Y_3, Y_4\}$ takes the form

$[\,,\,]$	X_1	X_2	Y_1	Y_2	Y_3	Y_4
X_1	0	0	Y_1	0	Y_3	$2Y_4$
X_2	0	0	0	Y_2	Y_3	Y_4
Y_1	$-Y_1$	0	0	Y_3	Y_4	0
Y_2	0	$-Y_2$	$-Y_3$	0	0	0
Y_3	$-Y_3$	$-Y_3$	$-Y_4$	0	0	0
Y_4	$-2Y_4$	$-Y_4$	0	0	0	0

.

It is not difficult to check that the obtained Lie algebra $\Phi_{6,1}$ is the Borel subalgebra of the simple Lie algebra of type B_2.

Case 3: $\mathfrak{n} \cong \Gamma_3 \oplus \mathbb{F}$. The Lie algebra $\mathfrak{n} = \Gamma_3 \oplus \mathbb{F}$ has two characteristic ideals: the 2-dimensional center $\mathfrak{z}$ and the 1-dimensional commutator subalgebra $[\mathfrak{n}, \mathfrak{n}]$. Let α_1, α_2 be the roots of $\mathfrak{g}$ induced on the $\mathfrak{g}$-module $\mathfrak{n}/\mathfrak{z}$ and β be the root induced on the $\mathfrak{g}$-module $\mathfrak{z}/[\mathfrak{n}, \mathfrak{n}]$. Then the 1-dimensional $\mathfrak{g}$-module $[\mathfrak{n}, \mathfrak{n}]$ induces the root $\alpha_1 + \alpha_2$. The geometric picture of the roots is an invariant of the isomorphism class of $\mathfrak{g}$, therefore, we have to consider all possible arrangements of the roots α_1, α_2, β satisfying the condition that they span a 2-dimensional linear space.

Case 3.1: Assume that α_1 and α_2 are linearly independent. Then the pair of numbers $\{\lambda, \mu\}$ for which $\beta = \lambda\alpha_1 + \mu\alpha_2$ is an invariant of the isomorphism class of $\mathfrak{g}$.

Case 3.1.1: Suppose that $\beta \notin \{0, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. Under this assumption, we can choose $X \in \mathfrak{g}$ such that the numbers $0, \alpha_1(X), \alpha_2(X), \alpha_1(X) + \alpha_2(X)$ are different eigenvalues of the restriction of the operator $\operatorname{ad} X$ onto the kernel of α_1. Let X', Y_1, Y_2, Y_3, Y_4 be the corresponding (unique up to scalar multiplier) eigenvectors of the restricted operator. Since $[X, X'] = 0$, Y_1, Y_2, Y_3, Y_4 are eigenvectors of the operator $\operatorname{ad} X'$ as well. Y_3 spans $[\mathfrak{n}, \mathfrak{n}]$, while Y_3 and Y_4 span the center $\mathfrak{z}$. This means that the only nonzero Lie bracket of the form $[Y_i, Y_j]$, $i < j$ is $[Y_1, Y_2]$ which gives Y_3 if we choose a suitable normalization of the eigenvector Y_3. Let X_1 and X_2 be the linear combinations of the vectors X, X' characterized by the equations $\alpha_i(X_j) = \delta_{ij}$, $1 \leq i, j \leq 2$. Then the commutation table of the Lie algebra $\mathfrak{g}$ with respect to the basis $\{X_1, X_2, Y_1, Y_2, Y_3, Y_4\}$ takes the form

$[\,,\,]$	X_1	X_2	Y_1	Y_2	Y_3	Y_4
X_1	0	0	Y_1	0	Y_3	λY_4
X_2	0	0	0	Y_2	Y_3	μY_4
Y_1	$-Y_1$	0	0	Y_3	0	0
Y_2	0	$-Y_2$	$-Y_3$	0	0	0
Y_3	$-Y_3$	$-Y_3$	0	0	0	0
Y_4	$-\lambda Y_4$	$-\mu Y_4$	0	0	0	0

.

Let us denote the isomorphism class of the Lie algebra given by this table of commutators by $\Phi_{6,2}\{\lambda,\mu\} \cong \Phi_{6,2}\{\mu,\lambda\}$. Computing the Pfaffian of $\Phi_{6,2}\{\lambda,\mu\}$ we can see that $\Phi_{6,2}\{\lambda,\mu\}$ is a Frobenius Lie algebra if and only if $\lambda \neq \mu$.

Case 3.1.2: $\beta = \alpha_1 + \alpha_2$. Choose a basis $\{Y_3, Y_4\}$ in $\mathfrak{z}$ such that Y_3 spans $[\mathfrak{n},\mathfrak{n}]$. With respect to this basis, the representation of $\mathfrak{g}$ on $\mathfrak{z}$ corresponds to the matrix representation

$$X \mapsto \begin{pmatrix} (\alpha_1+\alpha_2)(X) & 0 \\ \theta(X) & (\alpha_1+\alpha_2)(X) \end{pmatrix}, \tag{11}$$

where θ is a linear form on $\mathfrak{g}$, the kernel of which contains $\mathfrak{n}$, consequently, $\theta = \mu_1\alpha_1 + \mu_2\alpha_2$ for some $\mu_1, \mu_2 \in \mathbb{F}$. Since $\theta(X) = 0$ if and only if the operator $\operatorname{ad} X|_{\mathfrak{z}}$ is semisimple, the pair of ratios $\{\{\mu_1 : \mu_2\}, \{\mu_2 : \mu_1\}\}$ is an invariant of the isomorphism class of $\mathfrak{g}$. Choose an element $X \in \mathfrak{g}$ for which $\alpha_1(X) = 1$, $\alpha_2(X) = 2$, and let X', Y_1, Y_2 be the eigenvectors of the restriction of the operator $\operatorname{ad} X$ onto the kernel of α_1. Then Y_1 and Y_2 are eigenvectors of the operator $\operatorname{ad} X'$ as well, and the only nonzero commutator of the form $[Y_i, Y_j]$ is $[Y_1, Y_2]$, which is equal to Y_3 under a suitable normalization of the vectors Y_i. Defining X_1, X_2 just as in the case 3.1.1, we see, that the commutation table of $\mathfrak{g}$ with respect to the basis $\{X_1, X_2, Y_1, Y_2, Y_3, Y_4\}$ is the following:

$[\,,\,]$	X_1	X_2	Y_1	Y_2	Y_3	Y_4
X_1	0	0	Y_1	0	Y_3	$Y_4 + \mu_1 Y_3$
X_2	0	0	0	Y_2	Y_3	$Y_4 + \mu_2 Y_3$
Y_1	$-Y_1$	0	0	Y_3	0	0
Y_2	0	$-Y_2$	$-Y_3$	0	0	0
Y_3	$-Y_3$	$-Y_3$	0	0	0	0
Y_4	$-Y_4 - \mu_1 Y_3$	$-Y_4 - \mu_2 Y_3$	0	0	0	0

.

Let $\Phi_{6,3}\{\mu_1 : \mu_2\}$ be the isomorphism class of the Lie algebra given by this table. This class depends only on the ratio between the numbers μ_1 and μ_2

since replacing the basis vector Y_4 with λY_4 ($0 \neq \lambda \in \mathbb{F}$) the commutation table preserves its form, but μ_1 and μ_2 change for $\lambda\mu_1$ and $\lambda\mu_2$ respectively. Computing the commutation table of $\Phi_{6,3}\{\mu_1 : \mu_2\}$ with respect to the basis (X_2, X_1, Y_2, Y_1, $-Y_3$, $-Y_4$) we see immediately that $\Phi_{6,3}\{\mu_1 : \mu_2\} \cong \Phi_{6,3}\{\mu_2 : \mu_1\}$. $\Phi_{6,3}\{\mu_1 : \mu_2\}$ is a Frobenius Lie algebra if and only if $(\mu_1 : \mu_2) \neq (1 : 1)$.

Case 3.1.3: $\beta = \alpha_1$. This case can be treated in the same manner as the previous one. Beside Lie algebras of the type $\Phi_{6,2}\{1, 0\}$, we obtain a one-parameter family of Frobenius Lie algebras parameterized by the ratio between two numbers, that is, by a point of the projective line. Elements of this family will be denoted by $\Phi_{6,4}(\mu_1 : \mu_2)$. With respect to a suitable basis, they can be given by the commutation table

$[\,,\,]$	X_1	X_2	Y_1	Y_2	Y_3	Y_4
X_1	0	0	$Y_1 + \mu_1 Y_4$	0	Y_3	Y_4
X_2	0	0	$\mu_2 Y_4$	Y_2	Y_3	0
Y_1	$-Y_1 - \mu_1 Y_4$	$-\mu_2 Y_4$	0	Y_3	0	0
Y_2	0	$-Y_2$	$-Y_3$	0	0	0
Y_3	$-Y_3$	$-Y_3$	0	0	0	0
Y_4	$-Y_4$	0	0	0	0	0

.

Case 3.1.4. Finally, we show that $\beta = 0$ is not possible. Indeed, $\beta = 0$ would imply that the action of the commutative Lie algebra $\mathfrak{g}/\mathfrak{n}$ on the center $\mathfrak{z}$ of the nilradical would be diagonalizable, since then the weights $\alpha_1 + \alpha_2$ and $\beta = 0$ of this representation would be different. However, in that case, the one-dimensional submodule corresponding to the weight $\beta = 0$ would lie in the center of the Lie algebra $\mathfrak{g}$ contradicting the fact that Frobenius Lie algebras have trivial center.

Case 3.2. When α_1 and α_2 are proportional, $\alpha_1 + \alpha_2$ and β must be linearly independent. In this case we can find a pair of numbers $\{a, 1 - a\}$, such that $\alpha_1 = a(\alpha_1 + \alpha_2)$ and $\alpha_2 = (1 - a)(\alpha_1 + \alpha_2)$, and it is clear that the pair $\{a, 1 - a\}$ is an invariant of the isomorphism class of the Lie algebra $\mathfrak{g}$.

Case 3.2.1: Suppose that a is not in the set $\{0, 1/2, 1\}$. Then there is an element $X \in \mathfrak{g}$ such that the eigenvalues of the restriction of $\operatorname{ad} X$ onto the kernel of $\alpha_1 + \alpha_2$, that is the numbers 0, $\alpha_1(X)$, $\alpha_2(X)$, $\beta(X)$ and $\alpha_1(X) + \alpha_2(X)$ are distinct. Let X', Y_1, Y_2, Y_3, Y_4 be the eigenvectors of $\operatorname{ad} X$ corresponding to these eigenvalues. Then Y_1, Y_2, Y_3, Y_4 are also eigenvectors of the operator $\operatorname{ad} X'$. Since Y_3 spans $[\mathfrak{n}, \mathfrak{n}]$, the only non-zero commutator of the form $[Y_i, Y_j]$, $i < j$ is $[Y_1, Y_2]$, which equals Y_3 if the vectors are normalized properly. Let X_1 and X_2 be those linear combinations of X and X', for which $(\alpha_1 + \alpha_2)(X_1) = \beta(X_2) = 1$

and $(\alpha_1+\alpha_2)(X_2)=\beta(X_1)=0$. Then the commutation table of $\mathfrak{g}$ with respect to the basis X_1, X_2, Y_1, Y_2, Y_3, Y_4 has the following form

$[\,,\,]$	X_1	X_2	Y_1	Y_2	Y_3	Y_4
X_1	0	0	aY_1	$(1-a)Y_2$	Y_3	0
X_2	0	0	0	0	0	Y_4
Y_1	$-aY_1$	0	0	Y_3	0	0
Y_2	$(a-1)Y_2$	0	$-Y_3$	0	0	0
Y_3	$-Y_3$	0	0	0	0	0
Y_4	0	$-Y_4$	0	0	0	0

. (12)

This shows that $\mathfrak{g}\cong\Phi_{4,1}(a)\oplus\mathfrak{aff}(1,\mathbb{F})$.

Case 3.2.2: $a=0$. Choose $X\in\mathfrak{g}$ such that $(\alpha_1+\alpha_2)(X)=1$, $\beta(X)=2$, and let $\ker(\alpha_1+\alpha_2)=\mathfrak{a}_0\oplus\mathfrak{a}_1\oplus\mathfrak{a}_2$ be the decomposition of the kernel of $\alpha_1+\alpha_2$ into generalized eigensubspaces of the operator $\operatorname{ad}X$, i.e.

$$\mathfrak{a}_\alpha=\{Y\in\mathfrak{g}\mid(\operatorname{ad}X-\alpha\,\mathrm{id}_{\mathfrak{g}})^{\dim\mathfrak{g}}Y=0\}.$$

Choose linearly independent vectors $X',Y_1\in\mathfrak{a}_0$; $Y_2,Y_3\in\mathfrak{a}_1$ and $Y_4\in\mathfrak{a}_2$ in such a way that $\beta(X')=1$; $Y_1\in\mathfrak{a}_0\cap\mathfrak{n}$; $Y_3\in[\mathfrak{n},\mathfrak{n}]\subset\mathfrak{a}_1$. Y_3 and Y_4 belong to the center of the nilradical, therefore $[Y_1,Y_2]\neq0$, consequently, Y_3 can be chosen to be $[Y_1,Y_2]$. The first two rows of the commutation table of $\mathfrak{g}$ with respect to the basis X, X', Y_1, Y_2, Y_3, Y_4 has the form

$[\,,\,]$	X	X'	Y_1	Y_2	Y_3	Y_4
X	0	μY_1	0	$Y_2+\rho Y_3$	Y_3	$2Y_4$
X'	$-\mu Y_1$	0	0	$\rho' Y_3$	0	Y_4

,

where $\mu,\rho,\rho'\in\mathbb{F}$.

The Jacobi identity for the triple X, X', Y_2 yields $\mu=0$ immediately. If we set $X_1=X-2X'-(\rho-2\rho')Y_1$ and $X_2=X'-\rho'Y_1$, then the commutation table of $\mathfrak{g}$ with respect to the basis X_1, X_2, Y_1, Y_2, Y_3, Y_4 will get the form (12) with $a=0$, thus $\mathfrak{g}$ is isomorphic to $\Phi_{4,1}(0)\oplus\mathfrak{aff}(1,\mathbb{F})$.

Case 3.2.3: Assume $a=1/2$. In this case, the representation of $\mathfrak{g}$ on the factor space $\mathfrak{n}/\mathfrak{z}$ with respect to the basis Y_1,Y_2 has the form

$$X\mapsto\begin{pmatrix}\alpha_1(X) & \mu_1(\alpha_1+\alpha_2)(X)+\mu_2\beta(X)\\ 0 & \alpha_1(X)\end{pmatrix}.$$

Just as in case 3.1.2, the ratio $(\mu_1 : \mu_2)$ is an invariant of the isomorphism class of $\mathfrak{g}$. Using the same methods as in the previous cases the commutation table of $\mathfrak{g}$ can be brought to the form

$[\,,\,]$	X_1	X_2	Y_1	Y_2	Y_3	Y_4
X_1	0	0	$\frac{1}{2}Y_1 + \mu_1 Y_2$	$\frac{1}{2}Y_2$	Y_3	0
X_2	0	0	$\mu_2 Y_2$	0	0	Y_4
Y_1	$-\frac{1}{2}Y_1 - \mu_1 Y_2$	$-\mu_2 Y_2$	0	Y_3	0	0
Y_2	$-\frac{1}{2}Y_2$	0	$-Y_3$	0	0	0
Y_3	$-Y_3$	0	0	0	0	0
Y_4	0	$-Y_4$	0	0	0	0

.

Denote by $\Phi_{6,5}(\mu_1 : \mu_2)$ the isomorphism class of this Lie algebra.

$\Phi_{6,5}(\mu_1 : \mu_2)$ cannot be decomposed into a direct sum of smaller dimensional Lie algebras except for $\Phi_{6,5}(1 : 0) \cong \Phi_{4,2} \oplus \mathfrak{aff}(1, \mathbb{F})$ and $\Phi_{6,5}(0 : 0)$ which is isomorphic to $\Phi_{4,1}(1/2) \oplus \mathfrak{aff}(1, \mathbb{F})$.

Case 4: $\mathfrak{n} = \Gamma_{5,6}$. We have the following chain of characteristic ideals in $\mathfrak{n}$

$$\mathfrak{n} \rhd \mathfrak{z}_{\mathfrak{n}}(\mathfrak{n}^3) \rhd \mathfrak{n}^2 = [\mathfrak{n}, \mathfrak{n}] \rhd \mathfrak{n}^3 = [\mathfrak{n}, \mathfrak{n}^2] \rhd \mathfrak{z}_{\mathfrak{n}},$$

where $\mathfrak{z}_{\mathfrak{n}}(\mathfrak{n}^3)$ is the centralizer of $\mathfrak{n}^3$ in $\mathfrak{n}$. If the root induced on the 1-dimensional $\mathfrak{g}$-module $\mathfrak{n}/\mathfrak{z}_{\mathfrak{n}}(\mathfrak{n}^3)$ is λ, then the roots induced on the 1-dimensional factors of the consecutive ideals of this chain are λ, 2λ, 3λ, 4λ, 5λ respectively. $\mathfrak{g}$ is a Frobenius Lie algebra if and only if $\lambda \neq 0$. Choose $X \in \mathfrak{g}$ such that $\lambda(X) = 1$ and let $Y_1, Y_2, Y_3, Y_4, Y_5 \in \mathfrak{n}$ be eigenvectors of the operator $\operatorname{ad} X$, for which $[X, Y_i] = iY_i$. Choosing a suitable normalization of these vectors the commutation table of $\mathfrak{g}$ with respect to the basis X, Y_1, Y_2, Y_3, Y_4, Y_5 will have the form

$[\,,\,]$	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	Y_1	$2Y_2$	$3Y_3$	$4Y_4$	$5Y_5$
Y_1	$-Y_1$	0	Y_3	Y_4	Y_5	0
Y_2	$-2Y_2$	$-Y_3$	0	Y_5	0	0
Y_3	$-3Y_3$	$-Y_4$	$-Y_5$	0	0	0
Y_4	$-4Y_4$	$-Y_5$	0	0	0	0
Y_5	$-5Y_5$	0	0	0	0	0

,

consequently, the case $\mathfrak{n} = \Gamma_{5,6}$ corresponds to a unique isomorphism class of Frobenius Lie algebras which we denote by $\Phi_{6,6}$.

Case 5: $\mathfrak{n} = \Gamma_{5,3}$. The nilradical $\mathfrak{n}$ has the following characteristic ideals

$$\mathfrak{n} \rhd \mathfrak{z}_{\mathfrak{n}}(\mathfrak{n}^2) \rhd \tilde{\mathfrak{z}} \rhd \mathfrak{n}^2 \rhd \mathfrak{z}_{\mathfrak{n}},$$

where $\tilde{\mathfrak{z}} = \{X \in \mathfrak{n} \mid [X, \mathfrak{n}] \subset \mathfrak{z}_{\mathfrak{n}}\}$. Denote by λ the root of $\mathfrak{g}$ induced on the 1-dimensional $\mathfrak{g}$-module $\mathfrak{z}_{\mathfrak{n}}$. Let $a\lambda$ ($a \in \mathbb{F}$) be the root of $\mathfrak{g}$ induced on the factor $\mathfrak{n}/\mathfrak{z}_{\mathfrak{n}}(\mathfrak{n}^2)$. It can be shown that in this case the roots induced on the factors $\mathfrak{z}_{\mathfrak{n}}(\mathfrak{n}^2)/\tilde{\mathfrak{z}}$, $\tilde{\mathfrak{z}}/\mathfrak{n}^2$ and $\mathfrak{n}^2/\mathfrak{z}_{\mathfrak{n}}$ are $(1-2a)\lambda$, $2a\lambda$ and $(1-a)\lambda$ respectively. It is clear that $a \in \mathbb{F}$ is an invariant of the isomorphism class of $\mathfrak{g}$.

Case 5.1: $a \notin \{0, 1, 1/2, 1/3, 1/4\}$. In this case, if $X \in \mathfrak{g}$ is an element with $\lambda(X) = 1$, then the eigenvalues of the operator $\operatorname{ad} X$ are distinct. Let Y_1, Y_2, Y_3, Y_4, Y_5 be the eigenvectors of $\operatorname{ad} X$ corresponding to the eigenvalues a, $2a$, $1-2a$, $1-a$, 1 respectively. Renormalizing these eigenvectors if necessary, the commutator table will be the following:

[,]	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	aY_1	$2aY_2$	$(1-2a)Y_3$	$(1-a)Y_4$	Y_5
Y_1	$-aY_1$	0	0	Y_4	Y_5	0
Y_2	$-2aY_2$	0	0	Y_5	0	0
Y_3	$-(1-2a)Y_3$	$-Y_4$	$-Y_5$	0	0	0
Y_4	$-(1-a)Y_4$	$-Y_5$	0	0	0	0
Y_5	$-Y_5$	0	0	0	0	0

Denote the obtained one-parameter family of Frobenius Lie algebras by $\Phi_{6,7}(a)$ ($a \in \mathbb{F}$).

Case 5.2: $a = 0$. Let $\mathfrak{n} = \mathfrak{a}_1 + \mathfrak{a}_2$ be the splitting of the nilradical into the sum of generalized eigenspaces of $\operatorname{ad} X$. Choose a basis $Y_1, Y_2 \in \mathfrak{a}_0$; $Y_3, Y_4, Y_5 \in \mathfrak{a}_1$ in such a way that $Y_2 \in \mathfrak{a}_0 \cap \tilde{\mathfrak{z}}$, $Y_4 \in \mathfrak{n}^2$, $Y_5 \in \mathfrak{z}_{\mathfrak{n}}$. Then the commutation table with respect to this basis must have the form

[,]	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	αY_2	0	$Y_3 + \beta Y_4 + \gamma Y_5$	$Y_4 + \delta Y_5$	Y_5
Y_1	*	0	0	$\zeta Y_4 + \eta Y_5$	θY_5	0
Y_2	*	*	0	ιY_5	0	0
Y_3	*	*	*	0	0	0
Y_4	*	*	*	*	0	0
Y_5	*	*	*	*	*	0

Since $\mathfrak{g}$ is Frobenius, ι and θ are different from 0, therefore, we may assume that $\iota = \theta = 1$. Subtracting suitable multiples of Y_2 from X, Y_1 and that of Y_1 from X

we can eliminate the coefficients γ, η and δ. ζ is not zero, since $\dim \mathfrak{n}^2 = 2$, thus we can assume $\zeta = 1$. With these simplifications only two parameters remain, α and β. They are not independent as the Jacobi identity for the triple X, Y_1, Y_3 implies $\alpha + \beta = 0$.

If $\alpha = \beta = 0$, i.e. if the action of $\mathfrak{g}$ on $\mathfrak{n}/\mathfrak{n}^2$ is diagonalizable, then $\mathfrak{g}$ belongs to the class $\Phi_{6,7}(0)$. When $\alpha = -\beta \neq 0$, then α can be made equal to 1 by a renormalization of the basis $Y_1, \dots, Y_5$ and the isomorphism class of $\mathfrak{g}$, which we denote by $\Phi_{6,8}$, is given by the following commutation table:

$[\,,\,]$	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	Y_2	0	$Y_3 - Y_4$	Y_4	Y_5
Y_1	$-Y_2$	0	0	Y_4	Y_5	0
Y_2	0	0	0	Y_5	0	0
Y_3	$Y_4 - Y_3$	$-Y_4$	$-Y_5$	0	0	0
Y_4	$-Y_4$	$-Y_5$	0	0	0	0
Y_5	$-Y_5$	0	0	0	0	0

.

Case 5.3. If $a = 1$, then the restriction of the operator $\operatorname{ad} X$ onto $\mathfrak{n}$ has eigenvalues $1, 2, -1, 0, 1$ and we can choose a basis $Y_1, \dots, Y_5$ in $\mathfrak{n}$ in such a way that Y_2, Y_3 and Y_4 are eigenvectors of the operator $\operatorname{ad} X$ with eigenvalues 2, -1 and 0 respectively, while Y_1 and $Y_5 \in \mathfrak{z}_\mathfrak{n}$ generate the generalized eigensubspace of $\operatorname{ad} X$ corresponding to the eigenvalue 1. If the operator $\operatorname{ad} X$ is diagonalizable, then $\mathfrak{g}$ belongs to the class $\Phi_{6,7}(1)$, and this can always be achieved by adding a multiple of Y_4 to X. Indeed, Y_4 commutes with the vectors Y_2, Y_3, Y_5 and $[Y_1, Y_4] = \alpha Y_5$ with a nonzero α, while the commutator $[X, Y_1]$ has the form $Y_1 + \beta Y_5$, consequently, if X is replaced by $X + \frac{\beta}{\alpha} Y_4$, then β will be cancelled, while the remaining commutation relations do not change.

Case 5.4. An analogous argument shows that if $a = \frac{1}{2}$ and $\lambda(X) = 1$, then the set $X + \mathfrak{n} = \{X' \in \mathfrak{g} \mid \lambda(X') = 1\}$ contains an element X' with diagonalizable $\operatorname{ad} X'$ and using this element one can prove that $\mathfrak{g}$ belongs to the class $\Phi_{6,7}\big(\frac{1}{2}\big)$.

Cases 5.5, 5.6. When a is equal to $\frac{1}{3}$ or $\frac{1}{4}$, then it may happen that the set $\{\operatorname{ad} X \mid X \in \mathfrak{g},\, \lambda(X) = 1\}$ does not contain any diagonalizable element. These cases yield two isomorphism classes $\Phi_{6,9}$ and $\Phi_{6,10}$ of Frobenius Lie algebras which can be defined by the following commutation tables with respect to a suitable basis:

$\Phi_{6,9}$:

$[\,,\,]$	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	$\frac{1}{3}Y_1 + Y_3$	$\frac{2}{3}Y_2 + Y_4$	$\frac{1}{3}Y_3$	$\frac{2}{3}Y_4$	Y_5
Y_1	$-\frac{1}{3}Y_1 - Y_3$	0	0	Y_4	Y_5	0

$\Phi_{6,9}$:

[,]	X	Y_1	Y_2	Y_3	Y_4	Y_5
Y_2	$-\frac{2}{3}Y_2 - Y_4$	0	0	Y_5	0	0
Y_3	$-\frac{1}{3}Y_3$	$-Y_4$	$-Y_5$	0	0	0
Y_4	$-\frac{2}{3}Y_4$	$-Y_5$	0	0	0	0
Y_5	$-Y_5$	0	0	0	0	0

,

$\Phi_{6,10}$:

[,]	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	$\frac{1}{4}Y_1$	$\frac{1}{2}Y_2$	$\frac{1}{2}Y_3 + Y_2$	$\frac{3}{4}Y_4$	Y_5
Y_1	$-\frac{1}{4}Y_1$	0	0	Y_4	Y_5	0
Y_2	$-\frac{1}{2}Y_2$	0	0	Y_5	0	0
Y_3	$-\frac{1}{2}Y_3 - Y_2$	$-Y_4$	$-Y_5$	0	0	0
Y_4	$-\frac{3}{4}Y_4$	$-Y_5$	0	0	0	0
Y_5	$-Y_5$	0	0	0	0	0

.

Case 6: $\mathfrak{n} = \Gamma_{5,1}$. The structure of the nilradical $\mathfrak{n}$ can be described in the following way. $\mathfrak{n}$ has a 1-dimensional center $\mathfrak{z}$ and if we fix a basis vector Y_5 of $\mathfrak{z}$, then there is a skew symmetric bilinear form $\langle\, , \rangle$ on $\mathfrak{n}$ such that the Lie bracket on $\mathfrak{n}$ has the form

$$[Y, Y'] = \langle Y, Y' \rangle Y_5.$$

The kernel of the form $\langle\, , \rangle$ is the center of $\mathfrak{n}$. Denote by λ the root of $\mathfrak{g}$ induced on the $\mathfrak{g}$-module $\mathfrak{z}$.

Lemma 4.1. *There is an element $X \in \mathfrak{g}$ such that $\mathfrak{n}$ can be decomposed into the direct sum of* $\operatorname{ad} X$*-invariant subspaces V and $\mathfrak{z}$.*

Proof. Fix a decomposition $\mathfrak{n} = V \oplus \mathfrak{z}$ and an element $\tilde{X} \in \mathfrak{g}$ having the property $\lambda(\tilde{X}) = 1$. The action of the operator $\operatorname{ad} \tilde{X}$ on V is described by the formula

$$\operatorname{ad} \tilde{X}(v) = v' \oplus \phi(v) Y_5,$$

where $v, v' \in V$ and $\phi \in V^*$. Since $\langle\, , \rangle$ is nondegenerate on V, there is an element $v_0 \in V$ for which $\langle v_0, v \rangle = \phi(v)$ for all $v \in V$. Setting $X = \tilde{X} - v_0$ we have $\lambda(X) = 1$ and $\operatorname{ad} X(V) \subset V$. □

Let X and V be chosen as in Lemma 4.1. Denote by V_ξ the generalized eigensubspace of the operator $\operatorname{ad} X|_V$ corresponding to the number $\xi \in \mathbb{F}$. Then V_ξ and V_η are orthogonal with respect to $\langle\, , \rangle$ if $\xi + \eta \neq 1$, while $\langle\, , \rangle$ is a nondegenerate pairing on the pairs V_ξ, $V_{1-\xi}$. Thus, we have the following possibilities:

Case 6.1. $V = V_\xi \oplus V_\eta \oplus V_{1-\xi} \oplus V_{1-\eta}$, where $\xi \neq \eta$; $\xi, \eta \neq \frac{1}{2}$; $\xi \neq 1 - \eta$;

Case 6.2. $V = V_\xi \oplus V_{1-\xi} \oplus V_{\frac{1}{2}}$, where $\xi \neq \frac{1}{2}$ and $\dim V_{\frac{1}{2}} = 2$;
Case 6.3. $V = V_\xi \oplus V_{1-\xi}$, where $\xi \neq \frac{1}{2}$ and $\dim V_\xi = \dim V_{1-\xi} = 2$;
Case 6.4. $V = V_{\frac{1}{2}}$.

In those cases, when $\operatorname{ad} X$ is diagonalizable, e.g. in case 6.1, $\mathfrak{g}$ belongs to the 2-parameter family of Frobenius Lie algebras $\Phi_{6,11}\{\xi, \eta\}$ defined by the commutation table

$[\,,\,]$	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	ξY_1	ηY_2	$(1-\eta)Y_3$	$(1-\xi)Y_4$	Y_5
Y_1	$-\xi Y_1$	0	0	0	Y_5	0
Y_2	$-\eta Y_2$	0	0	Y_5	0	0
Y_3	$-(1-\eta)Y_3$	0	$-Y_5$	0	0	0
Y_4	$-(1-\xi)Y_4$	$-Y_5$	0	0	0	0
Y_5	$-Y_5$	0	0	0	0	0

.

Lie algebras belonging to case 6.2 with non-diagonalizable $\operatorname{ad} X$ form a 1-parameter family $\Phi_{6,12}(\xi)$ of Frobenius Lie algebras given by the commutation table

$[\,,\,]$	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	ξY_1	$\frac{1}{2}Y_2 + Y_3$	$\frac{1}{2}Y_3$	$(1-\xi)Y_4$	Y_5
Y_1	$-\xi Y_1$	0	0	0	Y_5	0
Y_2	$-\frac{1}{2}Y_2 - Y_3$	0	0	Y_5	0	0
Y_3	$-\frac{1}{2}Y_3$	0	$-Y_5$	0	0	0
Y_4	$-(1-\xi)Y_4$	$-Y_5$	0	0	0	0
Y_5	$-Y_5$	0	0	0	0	0

.

Consider case 6.3 assuming that $\operatorname{ad} X$ is not diagonalizable say on V_ξ. Then there is a basis $\{Y_1, Y_2\}$ in V_ξ such that $[X, Y_2] = \xi Y_2$, $[X, Y_1] = \xi Y_1 + Y_2$. Let $Y_4 \in V_{1-\xi}$ be an eigenvector of $\operatorname{ad} X$. Then Y_2 and Y_4 commute. Indeed,

$$\begin{aligned}[Y_1, Y_4] &= [X, [Y_1, Y_4]] = [[X, Y_1], Y_4] + [Y_1, [X, Y_4]] \\ &= [\xi Y_1 + Y_2, Y_4] + [Y_1, (1-\xi)Y_4] = [Y_1, Y_4] + [Y_2, Y_4].\end{aligned}$$

Thus, $V_{1-\xi}$ contains a basis $\{Y_3, Y_4\}$ such that $[X, Y_4] = (1-\xi)Y_4$, $[Y_1, Y_4] = [Y_2, Y_3] = Y_5$, $[Y_1, Y_3] = [Y_2, Y_4] = 0$, and then $[X, Y_3] = (1-\xi)Y_3 + \alpha Y_4$. The following computation shows that α must be -1.

$$\begin{aligned}0 = [Y_1, Y_3] &= [X, [Y_1, Y_3]] = [[X, Y_1], Y_3] + [Y_1, [X, Y_3]] \\ &= [Y_1, (1-\xi)Y_3 + \alpha Y_4] = (1+\alpha)Y_5.\end{aligned}$$

Thus, case 6.3 with nondiagonalizable $\operatorname{ad} X$ yields a one-parameter family $\Phi_{6,13}(\xi)$ of Frobenius Lie algebras.

In case 6.4, choose a basis E_1, E_2, E_3, E_4 in $V = V_{\frac{1}{2}}$ in which the operator $\operatorname{ad} X|_V$ has Jordan normal form. Corresponding to the possible sizes of the Jordan cells, we have five subcases (4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$).

Case 6.4.1. If the Jordan normal form of $\operatorname{ad} X$ is one single Jordan cell, then $[X, E_i] = \frac{1}{2}E_i + E_{i+1}$ for $i = 1, 2, 3$ and $[X, E_4] = \frac{1}{2}E_4$. Since $[E_i, E_j]$ is a multiple of Y_5, we have $[X, [E_i, E_j]] = [E_i, E_j]$. Thus, Jacobi identities for the triples $\{X, E_i, E_j\}$ reduce to

$$[E_1, E_3] = [E_2, E_3] + [E_1, E_4] = [E_2, E_4] = [E_3, E_4] = 0.$$

Set $[E_1, E_2] = \alpha Y_5$ and $[E_2, E_3] = \beta Y_5$. It is clear that $\beta \neq 0$, and we can define a new basis $Y_1, \ldots, Y_4$ in V by

$$Y_1 = \frac{1}{\sqrt{\beta}}\left(E_1 + \sqrt{-\alpha/\beta}E_2\right); \quad Y_2 = \frac{1}{\sqrt{\beta}}\left(E_2 + \sqrt{-\alpha/\beta}E_3\right);$$
$$Y_3 = \frac{1}{\sqrt{\beta}}\left(E_3 + \sqrt{-\alpha/\beta}E_4\right); \quad Y_4 = -\frac{1}{\sqrt{\beta}}E_4.$$

The commutation table of the basis vectors $X, Y_1, \ldots, Y_5$ is equal to

$[\,,\,]$	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	$\frac{1}{2}Y_1 + Y_2$	$\frac{1}{2}Y_2 + Y_3$	$\frac{1}{2}Y_3 - Y_4$	$\frac{1}{2}Y_4$	Y_5
Y_1	$-\frac{1}{2}Y_1 - Y_2$	0	0	0	Y_5	0
Y_2	$-\frac{1}{2}Y_2 - Y_3$	0	0	Y_5	0	0
Y_3	$-\frac{1}{2}Y_3 + Y_4$	0	$-Y_5$	0	0	0
Y_4	$-\frac{1}{2}Y_4$	$-Y_5$	0	0	0	0
Y_5	$-Y_5$	0	0	0	0	0

.

The isomorphism class of this Lie algebra is denoted by $\Phi_{6,14}$.

Case 6.4.2. If the Jordan normal form of $\operatorname{ad} X|_V$ has Jordan blocks of sizes 3×3 and 1×1, then $[X, E_i] = \frac{1}{2}E_i + E_{i+1}$ for $i = 1, 2$, and $[X, E_i] = \frac{1}{2}E_i$ for $i = 3, 4$. Applying the Jacobi identity for the triples $\{X, E_i, E_j\}$ as in the previous subcase we obtain $[E_1, E_3] = [E_2, E_3] = [E_2, E_4] = [E_3, E_4] = 0$, however, this is impossible, since maximal isotropic subspaces of V with respect to $\langle\,,\,\rangle$ have dimension two, while E_2, E_3, E_4 span an isotropic subspace.

Case 6.4.3. If the Jordan normal form of $\operatorname{ad} X|_V$ has two 2×2 Jordan blocks, then $[X, E_i] = \frac{1}{2}E_i + E_{i+1}$ for $i = 1, 3$, and $[X, E_i] = \frac{1}{2}E_i$ for $i = 2, 4$ and the Jacobi identity for the triples $\{X, E_i, E_j\}$ yields $[E_2, E_3] + [E_1, E_4] =$

$[E_2, E_4] = 0$. If the commutator $[[X, Y], Y]$ is identically equal to 0 for $X \in \mathfrak{g}$ and $Y \in \mathfrak{n}$, then, similarly to case 6.3, we can show that the Lie algebra belongs to the class $\Phi_{6,13}\left(\frac{1}{2}\right)$. On the other hand, if $[[X, Y], Y]$ is not identically 0 for $(X, Y) \in \mathfrak{g} \times \mathfrak{n}$, then the vectors $E_1, \ldots, E_4$ can be chosen also in such a way that $[E_1, E_2] = Y_5 = [E_3, E_4]$ and $[E_1, E_3] = [E_1, E_4] = [E_2, E_3] = [E_2, E_4] = 0$. Thus, we obtain a unique isomorphism class $\Phi_{6,15}$ whose commutation table with respect to the basis $\{X, Y_1 = E_1, Y_2 = E_3, Y_3 = E_4, Y_4 = E_2, Y_5\}$ is the following

$[\,,\,]$	X	Y_1	Y_2	Y_3	Y_4	Y_5
X	0	$\frac{1}{2}Y_1 + Y_4$	$\frac{1}{2}Y_2 + Y_3$	$\frac{1}{2}Y_3$	$\frac{1}{2}Y_4$	Y_5
Y_1	$-\frac{1}{2}Y_1 - Y_4$	0	0	0	Y_5	0
Y_2	$-\frac{1}{2}Y_2 - Y_3$	0	0	Y_5	0	0
Y_3	$-\frac{1}{2}Y_3$	0	$-Y_5$	0	0	0
Y_4	$-\frac{1}{2}Y_4$	$-Y_5$	0	0	0	0
Y_5	$-Y_5$	0	0	0	0	0

Case 6.5.4. If $[X, E_1] = \frac{1}{2}E_1 + E_2$ and $[X, E_i] = \frac{1}{2}E_i$ for $i = 2, 3, 4$, then $[E_2, E_3] = [E_2, E_4] = 0$ follows from the Jacobi identity for $\{X, E_i, E_j\}$. Consequently, $[E_1, E_2] \neq 0$, thus, subtracting suitable multiples of E_2 from E_3 and E_4 we may assume that $[E_1, E_3] = [E_1, E_4] = 0$. Then repeating the ideas used in case 6.3 we can show that $\mathfrak{g}$ belongs to $\Phi_{6,12}\left(\frac{1}{2}\right)$.

Case 6.5.5. Finally, if $\operatorname{ad} X|_V$ is diagonalizable, then $\mathfrak{g}$ is in $\Phi_{6,11}\left\{\frac{1}{2}, \frac{1}{2}\right\}$.

This completes the classification of solvable 6-dimensional Frobenius Lie algebras.

Assume that $\mathfrak{g}$ is a non-solvable Frobenius Lie algebra with Levi–Malcev decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$, where $\mathfrak{r}$ is the radical of $\mathfrak{g}$, $\mathfrak{s}$ is the semisimple part. Since $\mathfrak{s} \neq 0$ is a semisimple Lie algebra of dimension less than 6, $\mathfrak{s} \cong \mathfrak{sl}(2, \mathbb{F})$. It is known that finite dimensional $\mathfrak{s}$-modules are completely reducible, and that up to isomorphism, there is a unique irreducible $\mathfrak{sl}(2, \mathbb{F})$-module V^n of dimension n for any $n \in \mathbb{N}$. Consider the $\mathfrak{s}$-module $\mathfrak{r}$.

The representation of $\mathfrak{s}$ on $\mathfrak{r}$ cannot be trivial, because in that case we would have $\mathfrak{g} \cong \mathfrak{r} \times \mathfrak{s}$ and $\operatorname{ind} \mathfrak{g} = \operatorname{ind} \mathfrak{r} + \operatorname{ind} \mathfrak{s} > 0$.

On the other hand, $\mathfrak{r}$ must contain a 1-dimensional $\mathfrak{s}$-submodule, otherwise we would have

$$[\mathfrak{g}, \mathfrak{g}] \supset [\mathfrak{s}, \mathfrak{s} \oplus V^3] = [\mathfrak{s}, \mathfrak{s}] + [\mathfrak{s}, V^3] = \mathfrak{s} + V^3 = \mathfrak{g},$$

since $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ and $\mathfrak{s}V^n = V^n$ for $n > 1$, and this would contradict Corollary 2.1.

We conclude that $\mathfrak{r} \cong V^1 \oplus V^2$ as an $\mathfrak{s}$-module. Then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{s} + V^2$, $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}] = V^2$. Since $[\mathfrak{r}, \mathfrak{r}]$ is an $\mathfrak{s}$-submodule in V^2 and $[\mathfrak{r}, \mathfrak{r}] \neq 0$ (otherwise $\mathfrak{g}$

were not Frobenius), therefore, $[\mathfrak{r},\mathfrak{r}] = V^2$. $[V^2, V^2]$ is also an $\mathfrak{s}$-submodule of V^2 and $[V^2, V^2] \neq V^2$, consequently V^2 is commutative.

V^1 acts on V^2 and its action commutes with that of $\mathfrak{s}$, thus, by Schur's lemma, V^1 acts by scalar multiplications.

This information determines the Lie algebra structure of $\mathfrak{g}$ uniquely up to isomorphism, so there is only one non-solvable Frobenius Lie algebra of dimension 6, $\mathfrak{aff}(2,\mathbb{F})$, the Lie algebra of the group of affine transformations of the plane.

5. Summary

In the tables below we describe Lie algebras by listing the nonzero elements above the diagonal of their commutation tables with respect to an ordered basis. In the four-dimensional case, basis vectors are denoted by X_1, X_2, X_3, X_4. In the 6-dimensional case, we use bases of the form $X_1, \dots, X_m$, $Y_1, \dots, Y_{6-m}$ and when $m = 1$, X_1 is denoted by X.

We proved the following two classification theorems.

Theorem 5.1. *Isomorphism classes of Frobenius Lie algebras of dimension* 4 *over a field of characteristic* $\neq 2$ *are listed in the following table.*

Φ'	$[X_1,X_4] = [X_2,X_3] = -X_1$, $[X_2,X_4] = -X_2/2$, $[X_3,X_4] = -X_3/2$.
$\Phi''(\Delta)$, $\Delta \in \mathbb{F}$	$[X_1,X_4] = [X_2,X_3] = -X_1$, $[X_2,X_4] = -X_3$, $[X_3,X_4] = -X_3 + \Delta X_2$.
$\Phi'''(\varepsilon)$, $0 \neq \varepsilon \in \mathbb{F}$	$[X_1,X_3] = [X_2,X_4] = -X_1$, $[X_1,X_4] = \varepsilon X_2$, $[X_2,X_3] = -X_2$.

$\Phi'''(\varepsilon_1) \cong \Phi'''(\varepsilon_2)$ *if and only if the quotient* $\varepsilon_1/\varepsilon_2$ *is the square of an element of* $\mathbb{F}$, *any other pairs of Lie algebras of the table are non-isomorphic. The only decomposable* 4*-dimensional Frobenius Lie algebra* $\mathfrak{aff}(1,\mathbb{F}) \oplus \mathfrak{aff}(1,\mathbb{F})$ *is isomorphic to* $\Phi'''(-1)$.

Theorem 5.2. *Isomorphism classes of non-decomposable Frobenius Lie algebras of dimension* 6 *over an algebraically closed field* $\mathbb{F}$ *of characteristic* 0 *are listed in the following table.*

$\Phi_{6,1}$	$[X_1,Y_1] = Y_1$, $[X_1,Y_3] = Y_3$, $[X_1,Y_4] = 2Y_4$, $[X_2,Y_2] = Y_2$, $[X_2,Y_3] = Y_3$, $[X_2,Y_4] = Y_4$, $[Y_1,Y_2] = Y_3$, $[Y_1,Y_3] = Y_4$.

$\Phi_{6,2}\{\xi,\eta\}$ $\xi\neq\eta$	$[X_1,Y_1]=Y_1$, $[X_1,Y_3]=Y_3$, $[X_1,Y_4]=\xi Y_4$, $[X_2,Y_2]=Y_2$, $[X_2,Y_3]=Y_3$, $[X_2,Y_4]=\eta Y_4$, $[Y_1,Y_2]=Y_3$.
$\Phi_{6,3}\{\xi:\eta\}$ $(\xi:\eta)\neq(1:1)$	$[X_1,Y_1]=Y_1$, $[X_1,Y_3]=Y_3$, $[X_1,Y_4]=Y_4+\xi Y_3$, $[X_2,Y_2]=Y_2$, $[X_2,Y_3]=Y_3$, $[X_2,Y_4]=Y_4+\eta Y_3$, $[Y_1,Y_2]=Y_3$.
$\Phi_{6,4}(\xi:\eta)$ $(\xi:\eta)\neq(0:0)$	$[X_1,Y_1]=Y_1+\xi Y_4$, $[X_1,Y_3]=Y_3$, $[X_1,Y_4]=Y_4$, $[X_2,Y_1]=\eta Y_4$, $[X_2,Y_2]=Y_2$, $[X_2,Y_3]=Y_3$, $[Y_1,Y_2]=Y_3$.
$\Phi_{6,5}(\xi:\eta)$ $\eta\neq 0$	$[X_1,Y_1]=Y_1/2+\xi Y_2$, $[X_1,Y_2]=Y_2/2$, $[X_1,Y_3]=Y_3$, $[X_2,Y_1]=\eta Y_2$, $[X_2,Y_4]=Y_4$, $[Y_1,Y_2]=Y_3$.
$\Phi_{6,6}$	$[X,Y_1]=Y_1$, $[X,Y_2]=2Y_2$, $[X,Y_3]=3Y_3$, $[X,Y_4]=4Y_4$, $[X,Y_5]=5Y_5$, $[Y_1,Y_2]=Y_3$, $[Y_1,Y_3]=Y_4$, $[Y_1,Y_4]=Y_5$, $[Y_2,Y_3]=Y_5$.
$\Phi_{6,7}(\xi)$	$[X,Y_1]=\xi Y_1$, $[X,Y_2]=2\xi Y_2$, $[X,Y_3]=(1-2\xi)Y_3$, $[X,Y_4]=(1-\xi)Y_4$, $[X,Y_5]=Y_5$, $[Y_1,Y_3]=Y_4$, $[Y_1,Y_4]=[Y_2,Y_3]=Y_5$.
$\Phi_{6,8}$	$[X,Y_1]=Y_2$, $[X,Y_3]=Y_3-Y_4$, $[X,Y_4]=Y_4$, $[X,Y_5]=Y_5$, $[Y_1,Y_3]=Y_4$, $[Y_1,Y_4]=[Y_2,Y_3]=Y_5$.
$\Phi_{6,9}$	$[X,Y_1]=Y_1/3+Y_3$, $[X,Y_2]=2Y_2/3+Y_4$, $[X,Y_3]=Y_3/3$, $[X,Y_4]=2Y_4/3$, $[X,Y_5]=Y_5$, $[Y_1,Y_3]=Y_4$, $[Y_1,Y_4]=[Y_2,Y_3]=Y_5$.
$\Phi_{6,10}$	$[X,Y_1]=Y_1/4$, $[X,Y_2]=Y_2/2$, $[X,Y_3]=Y_3/2+Y_2$, $[X,Y_4]=3Y_4/4$, $[X,Y_5]=Y_5$, $[Y_1,Y_3]=Y_4$, $[Y_1,Y_4]=[Y_2,Y_3]=Y_5$.
$\Phi_{6,11}\{\xi,\eta\}$	$[X,Y_1]=\xi Y_1$, $[X,Y_2]=\eta Y_2$, $[X,Y_3]=(1-\eta)Y_3$, $[X,Y_4]=(1-\xi)Y_4$, $[X,Y_5]=[Y_1,Y_4]=[Y_2,Y_3]=Y_5$.
$\Phi_{6,12}(\xi)$ $\cong\Phi_{6,12}(1-\xi)$	$[X,Y_1]=\xi Y_1$, $[X,Y_2]=Y_2/2+Y_3$, $[X,Y_3]=Y_3/2$, $[X,Y_4]=(1-\xi)Y_4$, $[X,Y_5]=[Y_1,Y_4]=[Y_2,Y_3]=Y_5$.
$\Phi_{6,13}(\xi)$ $\cong\Phi_{6,13}(1-\xi)$	$[X,Y_1]=\xi Y_1+Y_2$, $[X,Y_2]=\xi Y_2$, $[X,Y_3]=(1-\xi)Y_3-Y_4$, $[X,Y_4]=(1-\xi)Y_4$, $[X,Y_5]=[Y_1,Y_4]=[Y_2,Y_3]=Y_5$.
$\Phi_{6,14}$	$[X,Y_1]=Y_1/2+Y_2$, $[X,Y_2]=Y_2/2+Y_3$, $[X,Y_3]=Y_3/2-Y_4$,$[X,Y_4]=Y_4/2$, $[X,Y_5]=[Y_1,Y_4]=[Y_2,Y_3]=Y_5$.
$\Phi_{6,15}$	$[X,Y_1]=Y_1/2+Y_4$, $[X,Y_2]=Y_2/2+Y_3$, $[X,Y_3]=Y_3/2$, $[X,Y_4]=Y_4/2$, $[X,Y_5]=[Y_1,Y_4]=[Y_2,Y_3]=Y_5$.
$\mathfrak{aff}(2,\mathbb{F})$	

When the parameters $\xi, \eta \in \mathbb{F}$ are separated by a colon, the isomorphism class does not change if both parameters are multiplied by a non-zero number. We put curly brackets around the parameters when the isomorphism class does not depend on the order of them. $\Phi_{6,11}\{\xi, \eta\}$ depends only on the set $\{\xi, 1-\xi, \eta, 1-\eta\}$. Except for these isomorphisms, the isomorphism classes of the table are pairwise distinct.

Remark 5.3. Irreducible almost algebraic Frobenius Lie algebras were denoted by $L_{6,1}$, $L_{6,2}(a_1, a_2)$, $L_{6,3}(a)$, $L_{6,4}$ and $L_{6,5}$ in the list of A. G. Elashvili in [7]. These Lie algebras are isomorphic to the following Lie algebras of our list: $L_{6,1} \cong \mathfrak{aff}(2, \mathbb{F})$, $L_{6,2}(a_1, a_2) \cong \Phi_{6,11}(a_1, a_2)$, $L_{6,3}(a) \cong \Phi_{6,7}(a)$, $L_{6,4} \cong \Phi_{6,6}$ and $L_{6,5} \cong \Phi_{6,1}$. (The commutation table of $L_{6,3}(a)$ contains some misprints in [7].)

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References

[1] N. Anh, Sur une conjecture de Wolf et Moore caractérisant les groupe de Lie ayant une série discrete, *C.R. Acad. Sci. Paris Sér. A* **287** (1978), 919–921.

[2] R. Basili, Resolutions of singularities of varieties of Lie algebras of dimension 3 and 4, *Journal of Lie Theory* **12**, no. 2 (2002), 397–407.

[3] B. Csikós, Frobenius Lie algebras and their representations, CSc thesis, *Moscow State University*, 1988 (in *Russian*).

[4] J. Dixmier, Sur les representations unitaires des groupes de Lie nilpotents, III, *Canad. J. Math.* **10** (1958), 321–348.

[5] V. G. Drinfel'd, Hamiltonian Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equation, *Sov. Math. Dokl.* **27** (1983), 68–71.

[6] A. G. Elashvili, Frobenius Lie algebras, I, *Funkts. Anal. Prilozh.* **16**, no. 4 (1982), 94–95 (in *Russian*).

[7] A. G. Elashvili, Frobenius Lie algebras, II, *Tr. Tbilis. Mat. Inst. Razmadze* **32** (1985), 228–231 (in *Russian*).

[8] V. V. Morozov, Classification of nilpotent Lie algebras of order 6, *Izv. Vyssh. Uchebn. Zaved., Mat.* **4**, no. 5 (1958), 161–171.

[9] A. I. Ooms, On Lie algebras having a primitive universal enveloping algebra, *J. Algebra* **32** (1974), 488–500.

[10] A. I. Ooms, On Lie algebras with primitive envelopes, supplements, *Proc. Am. Math. Soc.* **58** (1976), 67–72.

[11] A. I. Ooms, On Frobenius Lie algebras, *Communications in Algebra* **8** (1980), 13–52.

[12] D. I. Panyusev, The index of a Lie algebra, the centralizer of a nilpotent element, and the normaliser of the centralizer, *arXiv:math.AG/0107031 v1* (July 2001).

[13] M. Raïs, La représentation coadjointe du groupe affine, *Ann. Inst. Fourier* **28**, no. 1 (1978), 207–237.

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