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Finslerian metric function of totally anisotropic type

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Abstract. The work focuses upon the geometric properties of a Minkowski space endowed with a metric function of the Berwald–Moor type. The zero curvature of the indicatrix is a remarkable property of the approach. We demonstrate how the associated geodesic equations can be solved in a transparent way, thereby obtaining a possibility to introduce unambiguously distance, angle, and scalar product. The invariance group for the metric tensor is found.

1. Introduction and motivation

The pseudo-Euclidean metric function suits the cases when the space-time is uniform in all directions. Alternatively, we may imagine a situation when there exist N geometrically distinguished directions and propose the fundamental metric function

$$F(y) = \sqrt[N]{\prod_{A=1}^{N} |y^A|} \tag{1.1}$$

to measure the length of vectors $y = \{y^A\}$. For historical reasons, the metric function is frequently called after Berwald [1] and Moor [2], belonging to the domain of Finsler Geometry [3], [4]. Such a choice reveals numerous remarkable geometrical properties, many of which have been described in [3]. However, no possibility can be found in the literature to construct appropriate distance and angle. On the other hand, in the previous work [5,6] we have developed a detailed technique to obtain such notions in case of the Finsleroid metric function. It turns that the technique can successfully be applied to the Minkowski space with

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Finslerian metric function of the Berwald–Moor type. Our reasoning is based on the following three definitions.

Definition. Given a centered vector space V_N with a point O being the origin and with the members $y \in V_N$ issued from the point O. Let N directions $\{e_A\},\ A = 1, 2, \ldots, N$, be presupposed in V_N. We may decompose vectors y with respect to such a basis, obtaining the component representation $y = \{y^A\}$. Under these conditions, we define the A_N-*space*:

$$A_N \ := \{V_N, e_A, F(y)\}. \tag{1.2}$$

According to the known methods of Finsler geometry [3], [4], we construct on the basis of the function F the covariant vector $\hat{y} = \{y_A\}$ and the Finslerian metric tensor $\{g_{AB}\}$:

$$y_A \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial F^2}{\partial y^A}, \qquad g_{AB} \stackrel{\text{def}}{=} \frac{\partial y_A}{\partial y^B} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^A \partial y^B}. \tag{1.3}$$

We call the $(N-1)$-dimensional hyperplanes defined by the zeros $\{y^A = 0\}$ of the function F *singular hyperplanes*. They break down the space A_N into a collection of N^2 sectors, including the *up-sector* $A_N^{\{+\}}$.

Definition. The *up-sector* $A_N^{\{+\}} \in A_N$ is defined by the conditions

$$\{y^A\} \in A_N^{\{+\}} : y^A > 0. \tag{1.4}$$

In what follows, we shall deal with that sector (unless explicitly stated otherwise), so that the moduli in the right-hand part of the primary definition (1.1) can be omitted:

$$F(y) = \sqrt[N]{y^1 y^2 \ldots y^N} = \sqrt[N]{\prod_{A=1}^{N} y^A}. \tag{1.5}$$

Applying the rules (1.3) to (1.5) yields the explicit component values

$$y_A = \frac{F^2}{N y^A} \tag{1.6}$$

and

$$g_{AB} = \frac{2 y_A y_B}{F^2} - \frac{F^2}{N y^A y^B} \delta_{AB}; \tag{1.7}$$

the contravariant version of the last tensor is $\{g^{AB}\}$ with

$$g^{AB} = \frac{2 y^A y^B}{F^2} - \frac{N y^A y^B}{F^2} \delta^{AB}, \tag{1.8}$$

so that $g^{AC}g_{BC}=\delta^A_B$; δ stands for the Kronecker symbol. The indefinite nature of the metric tensor

$$\text{signature}\,\{g_{AB}\}=(+-\cdots-), \tag{1.9}$$

as well as the constant determinant

$$\det(g_{AB})=(-1)^{(N+1)}N^{-N}, \tag{1.10}$$

are remarkable properties of the space under study (cf. [3]). Owing to (1.9), the metric tensor may be represented as

$$g_{AB}=h^0_Ah^0_B-h^1_Ah^1_B-\cdots-h^{N-1}_Ah^{N-1}_B \tag{1.11}$$

in terms of the *associated orthonormal frame* $\{h^A_p\}$. The reciprocal representation reads

$$g^{AB}=h^A_0h^B_0-h^A_1h^B_1-\cdots-h^A_{N-1}h^B_{N-1} \tag{1.12}$$

subject to the reciprocity

$$h^p_Ah^A_q=\delta^p_q \tag{1.13}$$

(the indices $p,q,\ldots$ will be specified over the range $0,1,\ldots,N-1$ unless explicitly stated otherwise). By comparing (1.10) with (1.11) we may conclude that

$$\det(h^p_A)=N^{-N/2}. \tag{1.14}$$

Definition. The *indicatrix* $I^{\{+\}}_N\in A^{\{+\}}_N$ is defined as follows:

$$y\in I^{\{+\}}_N:\ \{y\in A^{\{+\}}_N,\,F(y)=1\}. \tag{1.15}$$

Using the *unit vectors*

$$l^A\stackrel{\text{def}}{=}\frac{y^A}{F(y)} \tag{1.16}$$

(so that $F(l)=1$) and choosing a convenient parameterization $\{u^a\}$, $a,b=1,\ldots,N-1$ over the indicatrix to have the representation

$$l^A=t^A(u^a), \tag{1.17}$$

we may construct the *projection factors*

$$t^A_a(u)\stackrel{\text{def}}{=}\frac{\partial t^A}{\partial u^a} \tag{1.18}$$

for the indicatrix to obtain the *induced metric tensor* over the indicatrix

$$i_{ab}(u) \stackrel{\text{def}}{=} -t_a^A t_b^B g_{AB}; \tag{1.19}$$

here the minus in front of the right-hand side reflects the indefinite signature (1.9). As was demonstrated in [3], it is convenient to treat the indicatrix in terms of the coordinate

$$z^0 \stackrel{\text{def}}{=} \ln F. \tag{1.20}$$

Depending on the sign of the coordinate z^0, the space under study is broken into the union

$$A_N^{\{+\}} = A^{\{+\}}_{N\,\{z^0>0\}} \cup A^{\{+\}}_{N\,\{z^0=0\}} \cup A^{\{+\}}_{N\,\{z^0<0\}} \tag{1.21}$$

of the three following regions:

$$A^{\{+\}}_{N\,\{z^0>0\}} := \{y \in A_N^{\{+\}} : F(y) > 1\}, \tag{1.22}$$

$$A^{\{+\}}_{N\,\{z^0=0\}} := \{y \in A_N^{\{+\}} : F(y) = 1\}, \tag{1.23}$$

$$A^{\{+\}}_{N\,\{z^0<0\}} := \{y \in A_N^{\{+\}} : 1 > F(y) > 0\}. \tag{1.24}$$

Notice that the sector (1.23) is the indicatrix:

$$A^{\{+\}}_{N\,\{z^0=0\}} = I_N^{\{+\}}. \tag{1.25}$$

It is the known that if we juxtapose (1.20) to an indicatrix coordinate set $\{u^a\}$ to obtain the coordinates

$$z^p = \{z^0, z^a = u^a\}, \tag{1.26}$$

then the respective transformation of the Finslerian metric tensor would lead to the result

$$g_{AB}(y)\frac{\partial y^A}{\partial z^p}\frac{\partial y^B}{\partial z^q} = \mathrm{e}^{2z^0}\, g^*_{pq} \tag{1.27}$$

which is remarkable in that

$$g^*_{00} = 1, \qquad g^*_{0a} = 0, \qquad g^*_{ab} = -i_{ab}, \tag{1.28}$$

where $\{i_{ab}\}$ is just the indicatrix metric tensor (1.19) (see [3]). Also, in case of the Finslerian metric function (1.5) the tensor $\{i_{ab}\}$ proves to be exactly Euclidean, so that the *conformal representation*

$$g^*_{pq} = \mathrm{e}^{2z^0}\, r_{pq} \tag{1.29}$$

holds with $\{r_{pq}\}$ being the pseudo-Euclidean metric tensor.

Therefore, it is advantageous to introduce the *associated conformally-pseudo-Euclidean space* C_N:

$$C_N \ := \{V_N, z^p \in V_N, g^*_{pq}\} \tag{1.30}$$

to have the *isometry*

$$A^{\{+\}}_N \Longleftrightarrow C_N \tag{1.31}$$

with the decomposition

$$C_N = C^{\{+\}}_N \cup C^{\{0\}} \cup C^{\{-\}}_N, \tag{1.32}$$

where

$$C^{\{+\}}_N := \{z^p \in C^{\{+\}}_N : z^0 > 0\}, \tag{1.33}$$

$$C^{\{0\}}_N := \{z^p \in C^{\{0\}}_N : z^0 = 0\}, \tag{1.34}$$

$$C^{\{-\}}_N := \{z^p \in C^{\{-\}}_N : z^0 < 0\}, \tag{1.35}$$

so that

$$A^{\{+\}}_{N\,\{z^0>0\}} \Longleftrightarrow C^{\{+\}}_N, \quad A^{\{+\}}_{N\,\{z^0=0\}} \Longleftrightarrow C^{\{0\}}_N, \quad A^{\{+\}}_{N\,\{z^0<0\}} \Longleftrightarrow C^{\{-\}}_N. \tag{1.36}$$

Now the question is what is the particular and convenient choice for the set $\{u^a\}$ under which the tensor $\{i_{ab}\}$ is exactly the diagonal unity, that is, when do we get

$$i_{ab} = \delta_{ab}. \tag{1.37}$$

Obviously, in the last case the fundamental length interval ds can be given merely by

$$(ds)^2 = e^{2z^0}\Big((dz^0)^2 - \delta_{ab}dz^a dz^b\Big). \tag{1.38}$$

To suppose true a due and possible answer to the question, it is useful to note that the choice

$$l^A = \exp(C^A_a u^a) \tag{1.39}$$

with any constant C^A_a subjected to the condition

$$\sum_{A=1}^{N} C^A_a = 0 \tag{1.40}$$

would parametrize the indicatrix because of the product structure of the Finslerian metric function (1.5) under study. Also, if we subject the constants to the condition

$$\sum_{A=1}^{N} C_a^A C_b^A = N\delta_{ab}, \tag{1.41}$$

then, because of the particular structure of the right-hand part in the metric tensor (1.7), we just obtain δ_{ab} in the right-hand part of (1.19). When verifying this assertion, it is convenient to note that the projection coefficients (1.18) constructed on the basis of (1.39) bear the structure

$$t_a^A = C_a^A \cdot l^A \tag{1.42}$$

at any value of the index A. Owing to the exponential nature of the right-hand part in the representation (1.39), it is convenient to call the set $\{u^a\}$ the *indicatrix variables.*

It is convenient to supplement the constants by the members

$$C_0^A = 1, \tag{1.43}$$

so that

$$\sum_{A=1}^{N} C_p^A C_q^A = Ne_{pq}, \tag{1.44}$$

where $\{e_{pq}\} = \text{diagonal}(1, -1, \dots, -1)$ is the pseudo-Euclidean metric tensor. This entails

$$\sum_{A=1}^{N} C_A^a = 0. \tag{1.45}$$

The inverse constants C_A^p obeying the relations

$$C_A^p C_q^A = \delta_q^p \tag{1.46}$$

must show the properties

$$C_A^0 = \frac{1}{N} \tag{1.47}$$

and

$$\sum_{A=1}^{N} C_a^A = 0. \tag{1.48}$$

Under these conditions, the representation (1.39) can be inverted to yield

$$u^a = C_A^a \ln l^A = C_A^p \ln y^A \tag{1.49}$$

and

$$z^p = C^p_A \ln y^A, \tag{1.50}$$

which in turn yields for the orthonormal frames

$$h^p_A = F z^p_A = C^p_A \cdot \frac{1}{l^A}, \tag{1.51}$$

where

$$z^p_A = \frac{\partial z^p}{\partial y^A}. \tag{1.52}$$

From (1.51) it follows that

$$g_{AB} = F^2 c_{AB} \tag{1.53}$$

with the tensor

$$c_{AB} = z^p_A z^q_B e_{pq}, \tag{1.54}$$

which demonstrates that the Finslerian metric tensor associated with the metric function (1.1) is conformal to the pseudo-Euclidean metric tensor. The conformal multiplier is the square F^2 of the metric function F.

In Section 2 we deal with the geodesic equations of the space under study. It proves possible to find the adequate explicit solutions for them in both the initial-value and the fixed-edge forms. The associated distance and the scalar product are also found. This opens up the straightforward way to obtain the angle $\eta(a,b)$ between two vectors $a \in V_N$ and $b \in V_N$ by postulating the cosine theorem. The angle is actually defined by the unit vectors $l^A(a) = a^A/F(a)$ and $l^A(b) = b^A/F(b)$:

$$\eta(a,b) = \sqrt{\frac{1}{N}\sum_{A=1}^{N}\left(\ln\frac{l^A(a)}{l^A(b)}\right)^2} \tag{1.55}$$

(see (2.48)) and it is entirely independent of any choice of the constants C^A_a entering the parametric representation (1.39) of the indicatrix. The angle is additive when the vectors point to a fixed geodesic curve. In fact, this angle measures the Euclidean length in the indicatrix.

In Section 3, we expose the transformations that leave invariant the Finslerian metric function as well as the Finslerian metric tensor. In the space under study, the transformations are found to be in general nonlinear. They realize Euclidean rotations and translations in the indicatrix. That is to say, the group of such transformations is a nonlinear image of the Euclidean invariance group.

The translations in the Euclidean indicatrix give rise to scale (product) transformations in the initial space, so that they form a linear (and abelian) subgroup. Detailed calculations are presented in the Appendices A, B, and C.

The paper ends with short Conclusions to emphasize the key aspects of our approach.

2. Geodesics, distance, and angle in $A_N^{\{+\}}$-spaces

Let be given a conformally-pseudo-Euclidean space $C_N^{\{+\}}$ (see (1.33)) with the metric tensor $\{g^*_{pq}\}$ prescribed by the conformal representation (1.29). Calculating the partial derivatives $g^*_{pq,r} = \partial g^*_{pq}/\partial z^r$, we get $g^*_{pq,a} = 0$ and $g^*_{pq,0} = 2g_{pq}$, so that for the components $\Gamma_{pqr} = \frac{1}{2}(g^*_{pq,r} + g^*_{qr,p} - g^*_{pq,r})$ we shall have the values

$$\Gamma_{000} = g^*_{00}, \quad \Gamma_{a00} = \Gamma_{0a0} = 0, \quad \Gamma_{a0b} = -g^*_{ab}, \\ \Gamma_{ab0} = g^*_{ab}, \quad \Gamma_{abc} = 0, \quad \Gamma_{pq0} = g^*_{pq}. \tag{2.1}$$

The associated Christoffel symbols $\Gamma_p{}^r{}_q = g^{*\,rs}\Gamma_{psq}$ are given by the components

$$\Gamma_0{}^0{}_0 = 1, \quad \Gamma_a{}^0{}_0 = \Gamma_0{}^a{}_0 = 0, \quad \Gamma_a{}^b{}_0 = \delta^b_a, \\ \Gamma_a{}^0{}_b = r_{ab}, \quad \Gamma_a{}^b{}_c = 0, \quad \Gamma_p{}^q{}_0 = \delta^q_p. \tag{2.2}$$

Let us consider a curve $C(s)$ parameterized by the length parameter s (cf. (1.38)) and introduce the respective N-dimensional velocity

$$U^p = \frac{dz^p}{ds}, \tag{2.3}$$

so that the velocity is unit:

$$g^*_{pq}(z)U^pU^q = 1. \tag{2.4}$$

The differential equation for the $C(s)$ to be a geodesic curve

$$\frac{dU^p}{ds} + \Gamma_s{}^p{}_r U^s U^r = 0 \tag{2.5}$$

proves to consist of two parts:

$$\frac{dU^0}{ds} = -[(U^0)^2 + \mathbf{U}^2] = -2(U^0)^2 + \mathrm{e}^{-2z^0} \tag{2.6}$$

and

$$\frac{dU^a}{ds} = -2U^aU^0. \tag{2.7}$$

The equation

$$\frac{d^2z^0}{ds^2} + 2\left(\frac{dz^0}{ds}\right)^2 = \mathrm{e}^{-2z^0} \tag{2.8}$$

can readily be integrated, yielding

$$z^0 = \ln(f(s)) \tag{2.9}$$

with

$$f(s) = \sqrt{a^2 + 2bs + s^2}, \tag{2.10}$$

where a and b are integration constants.

Since $z^0 = \ln F$ (see (1.20)), from (2.10) it follows that the Finslerian metric function varies along the geodesics according to the law

$$F(s) = \sqrt{a^2 + 2bs + s^2}. \tag{2.11}$$

Furthermore, using

$$\frac{dz^0}{ds} = \frac{b+s}{(F(s))^2} = U^0 \tag{2.12}$$

in (2.7) enables us to readily find

$$U^a = \frac{\sqrt{b^2 - a^2}\, n^a}{(F(s))^2}, \tag{2.13}$$

where n^a is a set of constants. To fulfil (2.4), the set must be subjected to the unity length condition:

$$\delta_{ab}n^an^b = 1. \tag{2.14}$$

Using $U^a = dz^a/ds$ (see (2.3)) in (2.13) gives us a differential equation to find the functions $z^a(s)$. The equation can readily be integrated to yield

$$z^a(s) = m^a + n^a \frac{1}{2} \ln \frac{s + b - \sqrt{b^2 - a^2}}{s + b + \sqrt{b^2 - a^2}}, \tag{2.15}$$

where m^a are new integration constants; we assume

$$b^2 - a^2 \geq 0, \quad a > 0. \tag{2.16}$$

Equations (2.11)–(2.14) with the condition (2.16) fulfil (2.4).

In this way we obtain explicitly the following formulae:

$$r_1^0 = \ln a, \quad r_2^0 = \ln(F(\Delta s)), \quad \sqrt{b^2 - a^2} = a^2|\mathbf{v}_1|, \quad b = a\sqrt{1 + a^2|\mathbf{v}_1|^2}, \tag{2.17}$$

$$r^0 = \frac{1}{2}\ln(a^2 + 2bs + s^2), \tag{2.18}$$

$$\mathbf{r}(s) = \mathbf{r}_1 + \frac{1}{2}\mathbf{v}_1 \frac{a^2}{\sqrt{b^2 - a^2}} \ln\bigl(X(s)\bigr), \tag{2.19}$$

where $r^0 = z^0$, $\mathbf{r} = \{z_1^a\}$, $\mathbf{v} = \{v_1^a\}$, and

$$X(s) = \frac{s + b - \sqrt{b^2 - a^2}}{s + b + \sqrt{b^2 - a^2}} \, \frac{b + \sqrt{b^2 - a^2}}{b - \sqrt{b^2 - a^2}}. \tag{2.20}$$

The last function can also be represented in the form

$$X(s) = \frac{\Big[a^2 + \bigl(b + \sqrt{b^2 - a^2}\,\bigr)\, s\Big]^2}{a^2 F^2(s)} = \frac{\Big[a^2 + bs + \sqrt{b^2 - a^2}\, s\Big]^2}{a^2 F^2(s)}. \tag{2.21}$$

Thus we have arrived at

Theorem 2.1. *The initial-value solution of the geodesic equations* (2.5) *under study can explicitly be given by equations* (2.17)–(2.20).

Also, it is possible to explicate the representation

$$\mathbf{r}(s) = \mathbf{r}_1 + \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \ln\sqrt{X(s)}, \tag{2.22}$$

with

$$|\mathbf{r}_2 - \mathbf{r}_1| = \ln\sqrt{X(\Delta s)}, \tag{2.23}$$

$$b = \frac{|\mathbf{r}_1||\mathbf{r}_2|\cosh|\mathbf{r}_2 - \mathbf{r}_1| - |\mathbf{r}_1|^2}{\Delta s}, \tag{2.24}$$

$$\sqrt{b^2 - a^2} = \frac{|\mathbf{r}_1||\mathbf{r}_2|\sinh|\mathbf{r}_2 - \mathbf{r}_1|}{\Delta s}, \tag{2.25}$$

and

$$(\Delta s)^2 = |\mathbf{r}_1|^2 + |\mathbf{r}_2|^2 - 2|\mathbf{r}_1||\mathbf{r}_2|\cosh|\mathbf{r}_2 - \mathbf{r}_1|. \tag{2.26}$$

Thus, we have obtained

Theorem 2.2. *The fixed-edge solution of the geodesic equations* (2.5) *under study can explicitly be given by equations* (2.22)–(2.26).

Formula (2.26) can also be written as

$$(\Delta s)^2 = |\mathbf{r}_1|^2 + |\mathbf{r}_2|^2 - 2|\mathbf{r}_1||\mathbf{r}_2| \cosh\bigl(\eta(\mathbf{r}_1, \mathbf{r}_2)\bigr) \tag{2.27}$$

with the following $C_N^{\{+\}}$-*angle*:

$$\eta(\mathbf{r}_1, \mathbf{r}_2) = |\mathbf{r}_2 - \mathbf{r}_1|. \tag{2.28}$$

Theorem 2.3. *The* $C_N^{\{+\}}$*-cosine theorem reads as* (2.26) *or* (2.27).

In view of (2.21) and (2.24)–(2.25), we can write

$$X(\Delta s) = \mathrm{e}^{2\eta}. \tag{2.29}$$

By comparing (2.15) and (2.20) with the unit vector representation of the exponential type (1.39), we can readily conclude that the components of the unit vector l^A vary along geodesics in accordance with the law

$$l^A(s) = l^A(0)\bigl(X(s)\bigr)^{\ln_{X(\Delta s)}(l^A(\Delta s)/l^A(0))}, \tag{2.30}$$

where

$$\prod_{A=1}^{N} l^A(0) = 1, \qquad \prod_{A=1}^{N} l^A(\Delta s) = 1, \qquad \prod_{A=1}^{N} l^A(s) = 1. \tag{2.31}$$

This law is applicable at any dimension $N \geq 2$.

For the vector

$$a^A(s) = F(s) l^A(s) \tag{2.32}$$

we obtain from (2.30) a similar behaviour

$$a^A(s) = \frac{F(s)a^A(0)}{F(0)}\bigl(X(s)\bigr)^{\ln_{X(\Delta s)}(a^A(\Delta s)F(0)/a^A(0)F(\Delta s))}, \tag{2.33}$$

where $F(0) = a$ (in view of (2.11)).

Thus we have arrived at

Theorem 2.4. *Let be given two vectors* $\{a^A_{\{1\}}\}$ *and* $\{a^A_{\{2\}}\}$*. Let* C *be a curve going from the end of the first vector to the end of the second vector. Put* $a^A(0) = \{a^A_{\{1\}}\}$ *and* $a^A(\Delta s) = \{a^A_{\{2\}}\}$*. Attribute the length values* $s = 0$ *and* $s = \Delta s$ *to the vectors, where* Δs *is the length of the curve* C*. If* C *is a geodesics, then the vector stretching to the geodesics point with a length value* s *is explicitly given by* (2.33).

The result (2.11) entails the relation

$$a^2F^2(\Delta s) = (a^2 + b\Delta s)^2 - \left(\sqrt{b^2 - a^2}\,\Delta s\right)^2, \tag{2.34}$$

which can be used to introduce the angle η according to

$$a^2 + b\Delta s = aF(\Delta s)\cosh\eta \tag{2.35}$$

and

$$\sqrt{b^2 - a^2}\,\Delta s = aF(\Delta s)\sinh\eta, \tag{2.36}$$

or

$$\frac{a^2 + b\Delta s}{\sqrt{b^2 - a^2}\,\Delta s} = \tanh\eta. \tag{2.37}$$

Applying (2.35) and (2.36) to (2.21), it follows that

$$X(\Delta s) = (\cosh\eta + \sinh\eta)^2 = \mathrm{e}^{2\eta}, \tag{2.38}$$

so that the equality (2.29) has been reproduced. Therefore, we may write the laws (2.30) and (2.33) in the forms

$$l^A(s) = l^A(0)\big(X(s)\big)^{\frac{1}{2\eta}\ln(l^A(\Delta s)/l^A(0))} \tag{2.39}$$

and

$$a^A(s) = \frac{F(s)a^A(0)}{F(0)}\big(X(s)\big)^{\frac{1}{2\eta}\ln(a^A(\Delta s)F(0)/a^A(0)F(\Delta s))}. \tag{2.40}$$

Since

$$\frac{d\left(\frac{s+b-\sqrt{b^2-a^2}}{s+b+\sqrt{b^2-a^2}}\right)}{ds} = \frac{2\sqrt{b^2 - a^2}}{F^2}\,\frac{s + b - \sqrt{b^2 - a^2}}{s + b + \sqrt{b^2 - a^2}}, \tag{2.41}$$

from (2.16) and (2.40) we can conclude that

$$F(s)\frac{da^A}{ds} = \frac{dF(s)}{ds}a^A(s) + 2\sqrt{b^2 - a^2}\,a^A(s)\frac{1}{2\eta}\ln(l^A(\Delta s)/l^A(0)). \tag{2.42}$$

Using here

$$g_{AB} = \frac{2a_Aa_B}{F^2} - \frac{F^2}{Na^Aa^B}\delta_{AB}, \qquad a_A = \frac{F^2}{Na^A} \tag{2.43}$$

(see (1.6) and (1.7)), and noting that

$$\prod_{A=1}^{N}\ln(l^A(\Delta s)/l^A(0)) = 0, \tag{2.44}$$

we find the equality

$$g_{AB}(a^C)\frac{da^A}{ds}\frac{da^B}{ds} = \left(\frac{dF}{ds}\right)^2 - (b^2 - a^2)\frac{1}{F^2}\frac{1}{N\eta^2}\sum_{A=1}^{N}\left(\ln(l^A(\Delta s)/l^A(0))\right)^2$$

$$= 1 + \frac{b^2 - a^2}{F^2} - (b^2 - a^2)\frac{1}{F^2}\frac{1}{N\eta^2}\sum_{A=1}^{N}\left(\ln(l^A(\Delta s)/l^A(0))\right)^2. \quad (2.45)$$

The left-hand side here must be 1. Therefore, the angle η can be given by

$$\eta = \sqrt{\frac{1}{N}\sum_{A=1}^{N}\left(\ln\frac{l^A(\Delta s)}{l^A(0)}\right)^2}, \quad (2.46)$$

or equivalently,

$$\eta = \sqrt{\frac{1}{N}\sum_{A=1}^{N}\left(\ln\frac{a^A(\Delta s)F(0)}{a^A(0)F(\Delta s)}\right)^2}. \quad (2.47)$$

If we merely consider two vectors $\{a^A\}$ and $\{b^A\}$, then (2.47) assigns to them the respective angle

$$\eta(a,b) = \sqrt{\frac{1}{N}\sum_{A=1}^{N}\left(\ln\frac{a^A/F(a)}{b^A/F(b)}\right)^2} \equiv \frac{F(b)}{F(a})\sqrt{\frac{1}{N}\sum_{A=1}^{N}\left(\ln\frac{a^A}{b^A}\right)^2}. \quad (2.48)$$

Thus the following assertions are valid.

Theorem 2.5. *The angle between two vectors* $\{a^A\}$ *and* $\{b^A\}$ *is given by* (2.48) *The angle is symmetric:*

$$\eta(a,b) = \eta(b,a). \quad (2.49)$$

Also, the angle is additive:

$$\eta(a,b) + \eta(b,c) = \eta(a,c), \quad (2.50)$$

when the vectors $\{a^A, b^A, c^A\}$ *point to a fixed geodesic curve.*

Rewriting (2.34) as

$$F^2(\Delta s) = (\Delta s)^2 - a^2 + 2(a^2 + b\Delta s) \quad (2.51)$$

and using (2.35), we get the *Finslerian* $A_N^{\{+\}}$*-Cosine Theorem:*

$$(\Delta s)^2 = (F(a))^2 + (F(b))^2 - 2F(a)F(b)\cosh\bigl(\eta(a,b)\bigr). \tag{2.52}$$

Therefore, the *Finslerian* $A_N^{\{+\}}$*-Distance* $|b \ominus a|$ between the endpoints of two given vectors is

$$|b \ominus a|^2 = (F(a))^2 + (F(b))^2 - 2F(a)F(b)\cosh\Bigl(\eta(a,b)\Bigr). \tag{2.53}$$

The *Finslerian* $A_N^{\{+\}}$*-Scalar Product*

$$(ab) = F(a)F(b)\cosh\Bigl(\eta(a,b)\Bigr) \tag{2.54}$$

is obtained.

We may use in the above expression (2.46) the indicatrix representation (1.39) and apply (1.41). On so doing, we obtain

$$\eta = \sqrt{(\Delta u^1)^2 + (\Delta u^2)^2 + \cdots + (\Delta u^{N-1})^2}. \tag{2.55}$$

Since at the same time the variables $\{u^a\}$ are some Euclidean coordinates on the indicatrix (see (1.37)), we may state the following

Theorem 2.6. *The Finslerian angle* η *is tantamount to the indicatrix Euclidean distance.*

It may also be said that, in entire analogy to Euclidean geometry proper, the *Finslerian angle* η *found measures the geodesic lengths on the indicatrix.* However, in Euclidean geometry the arcs are pieces of circles (the Euclidean indicatrix is a unit sphere), while in our present case they are pieces of straightlines (since the indicatrix is a Euclidean plane). It is useful to compare (2.55) with the representation (2.28) of the angle η.

Note. The two-dimensional case $N = 2$ is also comprised by the above formulae. Namely, the $\{N = 2\}$-dimensional precursor to the angle (2.48) reads

$$\eta_{\{N=2\}}(a,b) = \sqrt{\frac{1}{2}\sum_{A=1}^{2}\left(\ln\frac{a^A F(b)}{b^A F(a)}\right)^2} = \sqrt{\frac{1}{2}\left(\left(\ln\frac{a^1 F(b)}{b^1 F(a)}\right)^2 + \left(\ln\frac{a^2 F(b)}{b^2 F(a)}\right)^2\right)}$$

(with $F(a) = \sqrt{a^1 a^2}$ and $F(b) = \sqrt{b^1 b^2}$), or

$$\eta_{\{N=2\}}(a,b) = \ln\frac{a^1 F(b)}{b^1 F(a)}.$$

Therefore,

$$\cosh(\eta_{\{N=2\}}(a,b)) = \frac{1}{2}\left(\frac{a^1F(b)}{b^1F(a)} + \frac{b^1F(a)}{a^1F(b)}\right) = \frac{a^1b^2 + b^1a^2}{F(a)F(b)}. \tag{2.56}$$

On taking

$$m^1 = \frac{a^1 + a^2}{2}, \quad m^2 = \frac{a^1 - a^2}{2}, \qquad n^1 = \frac{b^1 + b^2}{2}, \quad n^2 = \frac{b^1 - b^2}{2},$$

our explication (2.56) just reduces to the ordinary pseudo-Euclidean rule

$$\cosh(\eta_{\{N=2\}}(a,b)) = \frac{m^1n^1 - m^2n^2}{\sqrt{(m^1)^2 - (m^2)^2}\,\sqrt{(n^1)^2 - (n^2)^2}}.$$

Let us illustrate the material of the present section.

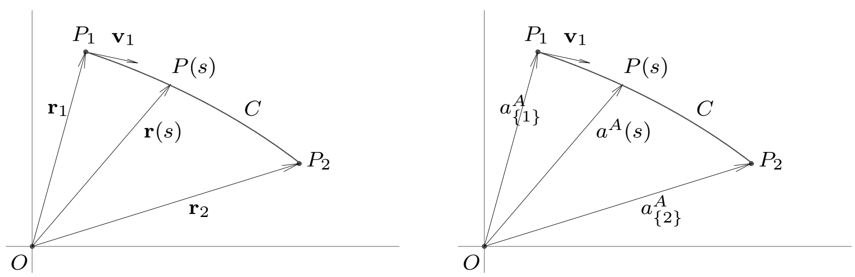

Figure 1. C is a geodesic curve. The length of the curve from p. P_1 to p. P_2 is equal to Δs, and from p. P_1 to p. $P(s)$ it is equal to s. The curve is the solution of the geodesic equation (2.5) (consisted of the parts (2.6) as (2.7)) in the space $A_N^{\{+\}}$. The value of the angle η between the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is presented by the explicit formula (2.28) and between the vectors $a^A_{\{1\}}$ and $a^A_{\{2\}}$ is given by the explicit formula (2.48).

3. Invariance in $A_N^{\{+\}}$-space

Let us consider a non-singular, and in general non-linear, transformation

$$y^A = F^A(\widetilde{y}^B) \tag{3.1}$$

under which the Finslerian metric function remains invariant, that is,

$$F(y) = F(\widetilde{y}). \tag{3.2}$$

Let us construct from the coefficients F^A the derivatives

$$F^A_B \stackrel{\text{def}}{=} \frac{\partial F^A}{\partial \widetilde{y}^B} \tag{3.3}$$

and

$$F^A_{BC} \stackrel{\text{def}}{=} \frac{\partial F^A_B}{\partial \widetilde{y}^C}. \tag{3.4}$$

For our purposes it is worth assuming that the functions F^A are C^3-smooth and positively homogeneous of degree 1 with respect to $\widetilde{y}$, so that

$$F^A(k\widetilde{y}) = kF^A(\widetilde{y}), \qquad k > 0, \tag{3.5}$$

(for any admissible set of arguments). The last condition guarantees the preservation of the homogeneity property for the Finslerian metric function F under the transformations (3.1) and allows rewriting them in the form

$$y^A = F^A_B(\widetilde{y})\widetilde{y}^B \tag{3.6}$$

(as this immediately follows from the Euler theorem for homogeneous functions). Generally speaking, the second derivatives do not vanish identically:

$$F^A_{BC} \neq 0. \tag{3.7}$$

Differentiating (3.2) with respect to $\widetilde{y}^C$ leads to the new identity

$$\widetilde{y}_C = y_B F^B_C, \tag{3.8}$$

which in turn can be differentiated with respect to $\widetilde{y}^D$, yielding

$$g_{CD}(\widetilde{y}) = F^A_C(\widetilde{y})F^B_D(\widetilde{y})g_{AB}\left(F(\widetilde{y})\right) + y_B F^B_{CD} \tag{3.9}$$

(the definition (3.3) has been used).

If the transformation (3.1) fulfils also the condition

$$y_B F^B_{CD} = 0, \tag{3.10}$$

then we call it *metric*, keeping in mind that in such a case the transformation (3.9) leaves also invariant the Finslerian metric tensor

$$g_{CD}(\widetilde{y}) = F_C^A(\widetilde{y})F_D^B(\widetilde{y})g_{AB}\left(F(\widetilde{y})\right). \tag{3.11}$$

Owing to (1.6), the metricity condition (3.10) can be written as

$$F_B F_{CD}^B \equiv 0 \tag{3.12}$$

with the functions

$$F_B = 1/F^B. \tag{3.13}$$

Obviously, the metric transformations form a group.

Definition. Under the above conditions, the set of transformations (3.1) is called the *group of Finslerian metric transformations.*

In case of the particular Finslerian metric function (1.5), an attentive consideration of the role of the indicatrix variables $\{u^a\}$ (see (1.39)) leads to the following conclusions.

Theorem 3.1. *The Euclidean rotations of the indicatrix variables* $\{u^a\}$ *give rise to the nonlinear transformations of the vectors* $\{y^A\}$, *which leave the Finslerian metric function* (1.5) *invariant and simultaneously realize the invariance transformation* (3.11) *for the associated Finslerian metric tensor.*

The explicit form of the required coefficients F^A will be evaluated below in Appendix A, taking as an example the dimension $N = 4$. Namely, under the rotation conditions (A.8)–(A.12) the nonlinear transformations considered prove to be given explicitly by means of the formulae (A.3)–(A.7). They involve three angles of rotations. For the transformations obtained the validity of the metricity condition (3.10) can be verified straightforwardly by applying the required Maple 10-tools (see Appendix B below). The formulae do essentially get simplified in case of one-angle-rotations (see Appendix C below). Additionally, the translations in the indicatrix:

$$\widetilde{u}^a = u^a + n^a \tag{3.14}$$

induce obviously the unimodular dilatations

$$\widetilde{y}^A = y^A \cdot k^A, \qquad \prod_{A=1}^{N} k^A = 1, \tag{3.15}$$

because of the exponential nature of the indicatrix representation (1.39) of unit vectors.

Appendix A: Coefficients for three-angle rotations in the four-dimnensional case

Let us take the dimension $N=4$ and start with an arbitrary linear nonsingular transformation of the indicatrix variables $\{u^a\}$ entering (1.39). Specifying them for definiteness to fulfil $\ln l^1=\alpha+\beta+\gamma$, $\ln l^2=-\alpha+\beta-\gamma$, $\ln l^3=\alpha-\beta-\gamma$, $\ln l^4=-\alpha-\beta+\gamma$ with $\{\alpha,\beta,\gamma\}=\{u^1,u^2,u^3\}$, we have

$$\alpha=l_1\widetilde{\alpha}+l_2\widetilde{\beta}+l_3\widetilde{\gamma},\quad \beta=m_1\widetilde{\alpha}+m_2\widetilde{\beta}+m_3\widetilde{\gamma},\quad \gamma=n_1\widetilde{\alpha}+n_2\widetilde{\beta}+n_3\widetilde{\gamma}, \tag{A.1}$$

where

$$\{m_1,m_2,m_3,n_1,n_2,n_3,l_1,l_2,l_3\} \tag{A.2}$$

is a set of constants. This entails

$$\ln y^1=(l_1\widetilde{\alpha}+l_2\widetilde{\beta}+l_3\widetilde{\gamma})+(m_1\widetilde{\alpha}+m_2\widetilde{\beta}+m_3\widetilde{\gamma})+(n_1\widetilde{\alpha}+n_2\widetilde{\beta}+n_3\widetilde{\gamma}),$$

$$\ln y^2=-(l_1\widetilde{\alpha}+l_2\widetilde{\beta}+l_3\widetilde{\gamma})+(m_1\widetilde{\alpha}+m_2\widetilde{\beta}+m_3\widetilde{\gamma})-(n_1\widetilde{\alpha}+n_2\widetilde{\beta}+n_3\widetilde{\gamma}),$$

$$\ln y^3=(l_1\widetilde{\alpha}+l_2\widetilde{\beta}+l_3\widetilde{\gamma})-(m_1\widetilde{\alpha}+m_2\widetilde{\beta}+m_3\widetilde{\gamma})-(n_1\widetilde{\alpha}+n_2\widetilde{\beta}+n_3\widetilde{\gamma}),$$

$$\ln y^4=-(l_1\widetilde{\alpha}+l_2\widetilde{\beta}+l_3\widetilde{\gamma})-(m_1\widetilde{\alpha}+m_2\widetilde{\beta}+m_3\widetilde{\gamma})+(n_1\widetilde{\alpha}+n_2\widetilde{\beta}+n_3\widetilde{\gamma}),$$

or

$$\ln y^1=(l_1+m_1+n_1)\widetilde{\alpha}+(l_2+m_2+n_2)\widetilde{\beta}+(l_3+m_3+n_3)\widetilde{\gamma},$$

$$\ln y^2=(-l_1+m_1-n_1)\widetilde{\alpha}+(-l_2+m_2-n_2)\widetilde{\beta}+(-l_3+m_3-n_3)\widetilde{\gamma},$$

$$\ln y^3=(l_1-m_1-n_1)\widetilde{\alpha}+(l_2-m_2-n_2)\widetilde{\beta}+(l_3-m_3-n_3)\widetilde{\gamma},$$

$$\ln y^4=(-l_1-m_1+n_1)\widetilde{\alpha}+(-l_2-m_2+n_2)\widetilde{\beta}+(-l_3-m_3+n_3)\widetilde{\gamma},$$

from which it follows that the coefficients of the sought transformation

$$y^A=F^A(\widetilde{y}^B) \tag{A.3}$$

can explicitly be given by the list

$$F^1=(\widetilde{y}^1)^{f^{11}}(\widetilde{y}^2)^{f^{12}}(\widetilde{y}^3)^{f^{13}}(\widetilde{y}^4)^{f^{14}}, \tag{A.4}$$

$$F^2=(\widetilde{y}^1)^{f^{21}}(\widetilde{y}^2)^{f^{22}}(\widetilde{y}^3)^{f^{23}}(\widetilde{y}^4)^{f^{24}}, \tag{A.5}$$

$$F^3=(\widetilde{y}^1)^{f^{31}}(\widetilde{y}^2)^{f^{32}}(\widetilde{y}^3)^{f^{33}}(\widetilde{y}^4)^{f^{34}}, \tag{A.6}$$

$$F^4 = (\widetilde{y}^1)^{f^{41}}(\widetilde{y}^2)^{f^{42}}(\widetilde{y}^3)^{f^{43}}(\widetilde{y}^4)^{f^{44}}, \tag{A.7}$$

with the exponents f^{AB} being linear combinations of the constants (A.2). The explicit representation (obtainable after lengthy straightforward calculations) for the coefficients f^{AB} is clear from the right-hand sides of the functions $F1$, $F2$, F, 3, $F4$ presented below in the Appendix B.

Finally, we are to subject the set (A.2) to the condition that the transformation (A.1) realizes a Euclidean rotation of the set $\{\alpha, \beta, \gamma\}$. To this end it is convenient to accept the (Euler) three angle choice:

$$c_1 = \cos\theta, \qquad c_2 = \cos\psi, \qquad c_3 = \cos\phi, \tag{A.8}$$

$$s_1 = \sin\theta, \qquad s_2 = \sin\psi, \qquad s_3 = \sin\phi \tag{A.9}$$

to have

$$l_1 = c_2c_3 - c_1s_2s_3, \quad m_1 = s_2c_3 + c_1c_2s_3, \quad n_1 = s_1s_3, \tag{A.10}$$

$$l_2 = -c_2s_3 - c_1s_2c_3, \quad m_2 = -s_2s_3 + c_1c_2c_3, \quad n_2 = s_1c_3, \tag{A.11}$$

$$l_3 = s_1s_2, \quad m_3 = -s_1c_2, \quad n_3 = c_1. \tag{A.12}$$

Appendix B: Three-angle.mws (by Maple 10)

The program presented below (created by means of Maple 10) does evaluate the metricity condition (3.12) which fulfilment means that the transformation leaves the Finslerian metric tensor invariant (that is, the equality (3.11) holds). The variables $\widetilde{y}^1$, $\widetilde{y}^2$, $\widetilde{y}^3$, $\widetilde{y}^4$ will be denoted by e^1, e^2, e^3, e^4.

```
> c1:=cos(theta);c2:=cos(psi);c3:=cos(phi);
> s1:=sin(theta);s2:=sin(psi);s3:=sin(phi);
> l1:=c2*c3-c1*s2*s3;l2:=-c2*s3-c1*s2*c3;l3:=s1*s2;
> m1:=s2*c3+c1*c2*s3;m2:=-s2*s3+c1*c2*c3;m3:=-s1*c2;
> n1:=s1*s3;n2:=s1*c3;n3:=c1;
```

$$c1 := \cos(\theta); \quad c2 := \cos(\psi); \quad c3 := \cos(\phi);$$

$$s1 := \sin(\theta); \quad s2 := \sin(\psi); \quad s3 := \sin(\phi);$$

$$l1 := c2\ c3 - c1\ s2\ s3; \quad l2 := -c2\ s3 - c1\ s2\ c3; \quad l3 := s1\ s2;$$

$$m1 := s2\ c3 + c1\ c2\ s3; \quad m2 := -s2\ s3 + c1\ c2\ c3; \quad m3 := -s1\ c2;$$

$$n1 := s1\ s3; \quad n2 := s1\ c3; \quad n3 := c1$$

```
> F1:=(e1)^((l1+m1+n1+l2+m2+n2+l3+m3+n3+1)/4)*
      (e2)^((-l1-m1-n1+l2+m2+n2-l3-m3-n3+1)/4)*
      (e3)^((l1+m1+n1-l2-m2-n2-l3-m3-n3+1)/4)*
      (e4)^((-l1-m1-n1-l2-m2-n2+l3+m3+n3+1)/4):

> F2:=(e1)^((-l1+m1-n1-l2+m2-n2-l3+m3-n3+1)/4)*
      (e2)^((l1-m1+n1-l2+m2-n2+l3-m3+n3+1)/4)*
      (e3)^((-l1+m1-n1+l2-m2+n2+l3-m3+n3+1)/4)*
      (e4)^((l1-m1+n1+l2-m2+n2-l3+m3-n3+1)/4):

> F3:=(e1)^((l1-m1-n1+l2-m2-n2+l3-m3-n3+1)/4)*
      (e2)^((-l1+m1+n1+l2-m2-n2-l3+m3+n3+1)/4)*
      (e3)^((l1-m1-n1-l2+m2+n2-l3+m3+n3+1)/4)*
      (e4)^((-l1+m1+n1-l2+m2+n2+l3-m3-n3+1)/4):

> F4:=(e1)^((-l1-m1+n1-l2-m2+n2-l3-m3+n3+1)/4)*
      (e2)^((l1+m1-n1-l2-m2+n2+l3+m3-n3+1)/4)*
      (e3)^((-l1-m1+n1+l2+m2-n2+l3+m3-n3+1)/4)*
      (e4)^((l1+m1-n1+l2+m2-n2-l3-m3+n3+1)/4):

> a:=array(1..4,1..4):
> for i from 1 to 4
  do
     for j from 1 to 4
     do
        a[i,j]:=diff(F||i,e||j);
     end do:
  end do:

> b:=array(1..4,1..4):
> for i from 1 to 4
  do
     for j from 1 to 4
     do
        b[i,j]:=simplify(add(1/F||k*diff(a[k,i],e||j),k=1..4),symbolic);
     end do:
  end do:

> print(b);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The result that all the entries of the matrix are zeros means that the metricity condition (3.12) holds true (in the dimension $N = 4$).

Appendix C: One-angle rotation

Let us take the particular case

$$\alpha = \widetilde{\alpha}\cos\eta + \widetilde{\beta}\sin\eta, \qquad \beta = -\widetilde{\alpha}\sin\eta + \widetilde{\beta}\cos\eta, \qquad \gamma = \widetilde{\gamma} \tag{C.1}$$

which represents the rotation by one angle, η, in the γ-plane. We get

$$\ln y^1 = \widetilde{\alpha}\cos\eta + \widetilde{\beta}\sin\eta - \widetilde{\alpha}\sin\eta + \widetilde{\beta}\cos\eta + \widetilde{\gamma},$$

$$\ln y^2 = -\widetilde{\alpha}\cos\eta - \widetilde{\beta}\sin\eta - \widetilde{\alpha}\sin\eta + \widetilde{\beta}\cos\eta - \widetilde{\gamma},$$

$$\ln y^3 = \widetilde{\alpha}\cos\eta + \widetilde{\beta}\sin\eta + \widetilde{\alpha}\sin\eta - \widetilde{\beta}\cos\eta - \widetilde{\gamma},$$

$$\ln y^4 = -\widetilde{\alpha}\cos\eta - \widetilde{\beta}\sin\eta + \widetilde{\alpha}\sin\eta - \widetilde{\beta}\cos\eta + \widetilde{\gamma}.$$

The respective generalized rotation coefficients are given by the list:

$$F^1 = (\widetilde{y}^1)^{(2\cos\eta+1)/4}(\widetilde{y}^2)^{(2\sin\eta-1)/4}(\widetilde{y}^3)^{(-2\sin\eta-1)/4}(\widetilde{y}^4)^{(-2\cos\eta+1)/4}, \tag{C.2}$$

$$F^2 = (\widetilde{y}^1)^{(-2\sin\eta-1)/4}(\widetilde{y}^2)^{(2\cos\eta+1)/4}(\widetilde{y}^3)^{(-2\cos\eta+1)/4}(\widetilde{y}^4)^{(2\sin\eta-1)/4}, \tag{C.3}$$

$$F^3 = (\widetilde{y}^1)^{(2\sin\eta-1)/4}(\widetilde{y}^2)^{(-2\cos\eta+1)/4}(\widetilde{y}^3)^{(2\cos\eta+1)/4}(\widetilde{y}^4)^{(-2\sin\eta-1)/4}, \tag{C.4}$$

$$F^4 = (\widetilde{y}^1)^{(-2\cos\eta+1)/4}(\widetilde{y}^2)^{(-2\sin\eta-1)/4}(\widetilde{y}^3)^{(2\sin\eta-1)/4}(\widetilde{y}^4)^{(2\cos\eta+1)/4}. \tag{C.5}$$

Conclusions

The $(N = 2)$-dimensional precursor of the space (1.2) is the ordinary hyperbolic space $A_2 := \{V_2, e_1, e_2, F^{\{\text{two-dimensional}\}}(y)\}$ with $F^{\{\text{two-dimensional}\}} = \sqrt{|y^1y^2|} \equiv \sqrt{|t^2 - x^2|}$, where $t = (y^1 + y^2)/2$ and $x = (y^1 - y^2)/2$. The methods exposed in the previous sections enable one to increase the dimension N and to arrive at the anisotropic A_N-space that differs drastically from the isotropic conventional pseudo-Euclidean space. Our methods of analysis were adequately founded upon use of the indicatrix geometry and indicatrix coordinates. The conformal nature of the associated Finslerian metric tensor (exhibited by (1.53)) has played a crucial role in Section 2 in our getting explicit solutions to the $A_N^{\{+\}}$-geodesic equations. They are the solutions that entailed the distance, angle, and scalar product for the $A_N^{\{+\}}$-space. Study of the invariance properties of the $A_N^{\{+\}}$-space faced us to conclude in Section 3 that the associated group of invariance is a nonlinear representation of the Euclidean group of rotations and translations given rise to by the induced Euclidean structure of the generalized $A_N^{\{+\}}$-hyperboloid (which is the indicatrix of the space under study).

Generally, search for novel physical and other applied aspects produced tentatively by the anisotropic structures seems to be an urgent task for modern science (see in particular [7]–[9]). It can be hoped that the evidence of the basic notions revealed in the present paper should make the space with the metric function considered applicable in various mathematical and physical scenarios.

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