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Prime ideals and complex ring homomorphisms on a commutative algebra

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Abstract. We give a characterization of prime ideals $\mathcal{P}$ of a commutative complex algebra $\mathcal{A}$ in order that $\mathcal{P}$ be the kernel of some complex ring homomorphism on $\mathcal{A}$. If, in addition, $\mathcal{A}$ is a uniform algebra on an infinite compact metric space, then we show that there are exactly $2^{\mathfrak{c}}$ complex ring homomorphisms on $\mathcal{A}$, whose kernels are non-maximal prime ideals. Moreover, it turns out that ring homomorphisms on a commutative Banach algebra are deeply connected with the existence of discontinuous homomorphisms.

1. Introduction and the statement of results

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over the complex number field $\mathbb{C}$. We say that a mapping $\rho : \mathcal{A} \to \mathcal{B}$ is a ring homomorphism, provided that

$$\rho(f+g) = \rho(f) + \rho(g)$$

$$\rho(fg) = \rho(f)\,\rho(g)$$

for every $f, g \in \mathcal{A}$. Moreover if ρ is homogeneous, that is $\rho(\lambda f) = \lambda\rho(f)$ for every $f \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, then ρ is an ordinary homomorphism. It is obvious that if $\rho : \mathcal{A} \to \mathcal{B}$ is a ring homomorphism, then $\rho(rf) = r\rho(f)$ for every rational number r and $f \in \mathcal{A}$. If, in addition, ρ is assumed to be continuous, then we see that ρ is real linear, that is, $\rho(tf) = t\rho(f)$ for every real number t and $f \in \mathcal{A}$. So, we consider ring homomorphisms which need not be continuous. The study of ring homomorphisms between two Banach algebras has a long history. In 1944,

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Arnold [1] proved that a ring isomorphism between the two Banach algebras of all bounded linear operators on two infinite dimensional Banach spaces is linear or conjugate linear. Kaplansky [6] generalized this result as follows: If ρ is a ring isomorphism from one semisimple Banach algebra A onto another, then A is the direct sum of closed ideals A_1, A_2 and A_3 such that $\rho|_{A_1}$ is linear, $\rho|_{A_2}$ is conjugate linear and that A_3 is finite dimensional: The finite dimensional part is not trivial in general. In fact, Kestelman [7] proved that there exists a ring homomorphism $\rho : \mathbb{C} \to \mathbb{C}$ such that ρ is neither linear nor conjugate linear. Moreover, Charnow [2, Theorem 3] proved that there exist $2^{\sharp k}$ ring automorphisms for every algebraically closed field k. Here and after, $\sharp S$ denotes the cardinal number of a set S. In particular, there are $2^{\sharp \mathbb{C}}$ ring automorphisms on $\mathbb{C}$. Molnár [10, Theorem 1] essentially gave a representation of a ring homomorphism between two commutative C^*-algebras.

Suppose $\rho : A \to B$ is a ring homomorphism between two commutative Banach algebras A and B with the maximal ideal spaces M_A and M_B, respectively. When studying such mappings, a natural approach would be to consider ring homomorphisms $\varphi \circ \rho : A \to \mathbb{C}$ for every $\varphi \in M_B$, and patch them by a continuous mapping from a suitable subset of M_B into M_A. Indeed, some representations of ring homomorphisms, with additional conditions, are proved in this way (cf. [5, Theorem 2.3], [9, Theorem 2.6], [11, Theorem 5.1, 5.2]). Unfortunately this approach does not work in general because the kernel $\ker(\varphi \circ \rho)$ need not be a maximal ideal of A. On the other hand, Molnár [10, Theorem 2] essentially gave a representation of ring homomorphisms between two commutative C^*-algebras by another approach. Although we are concerned with *ring* homomorphism, the term ideal will mean an *algebra* ideal. Let $C(X)$ denote the commutative Banach algebra of all complex valued continuous functions on a compact Hausdorff space X. Suppose $\tau : \mathbb{C} \to \mathbb{C}$ is a ring homomorphism, and suppose $x_0 \in X$. Šemrl [11, Example 5.4] considered a complex ring homomorphism $\rho : C(X) \to \mathbb{C}$ of the form

$$\rho(f) = \tau(f(x_0)) \qquad (f \in C(X)) \tag{$*$}$$

and gave the following example: If $\mathbb{N}$ is the set of all natural numbers and if K is the closure of $\{1/n : n \in \mathbb{N}\}$ with its usual topology, then there is a non-zero ring homomorphism $\phi : C(K) \to \mathbb{C}$ such that ϕ is not of the form $(*)$. In particular, $\ker \phi$ is a non-maximal prime ideal of $C(K)$. The first author [9, Lemma 2.1] gave a characterization of a ring homomorphism ρ between two commutative Banach algebras in order that $\ker \rho$ be a maximal ideal (cf. [5, Lemma 2.2]).

In this note, we are concerned with complex ring homomorphisms ρ on a commutative complex algebra $\mathcal{A}$. If ρ is non-zero, then it is easy to see that

$\ker\rho$ is a prime ideal of $\mathcal{A}$. Recall that if $\mathcal{A}$ is a unital commutative Banach algebra, then there is a one-to-one correspondence between non-zero complex homomorphisms on $\mathcal{A}$ and maximal ideals of $\mathcal{A}$. With this in mind, one might expect that there is also a correspondence between complex ring homomorphisms and prime ideals of a complex commutative algebra $\mathcal{A}$. In this note, we give a characterization of prime ideals that can be represented as the kernels of some complex ring homomorphisms. Before we state our main result, we need some terminology. If $\mathcal{A}$ is unital, then we define $\mathcal{A}_e \overset{\text{def}}{=} \mathcal{A}$; otherwise, $\mathcal{A}_e$ denotes the commutative complex algebra obtained by adjunction of a unit element e to $\mathcal{A}$. As usual, we may identify $f \in \mathcal{A}$ with $(f, 0) \in \mathcal{A}_e$. Now, we are ready to state our main result.

Theorem 1.1. *Suppose $\mathcal{A}$ is a commutative complex algebra and $\mathcal{P}$ is a prime ideal of $\mathcal{A}$. Put $\mathfrak{c} = \sharp\mathbb{C}$. Then each of the following four properties implies the other three:*

(a) *There exists a non-zero ring homomorphism $\rho : \mathcal{A} \to \mathbb{C}$ such that $\ker\rho = \mathcal{P}$.*

(b) *The quotient algebra $\mathcal{A}/\mathcal{P}$ has the cardinal number $\mathfrak{c}$.*

(c) *There exists a prime ideal $\tilde{\mathcal{P}}$ of $\mathcal{A}_e$ such that $\mathcal{P} = \tilde{\mathcal{P}} \cap \mathcal{A}$ and that $\mathcal{A}_e/\tilde{\mathcal{P}}$ has the cardinal number $\mathfrak{c}$.*

(d) *There exists a non-zero ring homomorphism $\tilde{\rho} : \mathcal{A}_e \to \mathbb{C}$ such that $\mathcal{A} \cap \ker\tilde{\rho} = \mathcal{P}$.*

Let K be the closure of $\{1/n : n \in \mathbb{N}\}$ with its usual topology. As stated above, Šemrl gave a complex ring homomorphism on $C(K)$, whose kernel is a non-maximal prime ideal. In the following corollary we see that there exist $2^{\mathfrak{c}}$ such mappings. Moreover, the following is true.

Corollary 1.2. *If A is a uniform algebra on an infinite compact metric space, then there are exactly $2^{\mathfrak{c}}$ complex ring homomorphisms on A, whose kernels are non-maximal prime ideals.*

2. A proof of results

Recall that an ideal $\mathcal{P}$ of a commutative algebra is prime if $\mathcal{P}$ is proper and $fg \notin \mathcal{P}$ whenever $f \notin \mathcal{P}$ and $g \notin \mathcal{P}$.

It is advisable to note that the quotient field of an integral domain R is well defined even if R has no unit: If $a \in R \setminus \{0\}$, then the "fraction" a/a is a unit, and we may identify $b \in R$ with ab/a.

Lemma 2.1. *Suppose $\mathcal{A}$ is a commutative complex algebra and $\rho : \mathcal{A} \to \mathbb{C}$ is a non-zero ring homomorphism. Then*

(a) *the kernel* $\ker\rho$ *is a prime ideal of* $\mathcal{A}$*, and*

(b) ρ *is of the form* $\rho = \tau \circ \pi$*, where* τ *is a non-zero field homomorphism on the quotient field* $\mathcal{F}$ *of* $\mathcal{A}/\ker\rho$ *into* $\mathbb{C}$*, and* $\pi : \mathcal{A} \to \mathcal{A}/\ker\rho$ *is the quotient mapping.*

Proof. Choose $a \in \mathcal{A}$ such that $\rho(a) \neq 0$: This is possible since ρ is assumed to be non-zero. (a) Pick $f \in \ker\rho$ and $\lambda \in \mathbb{C}$ arbitrarily. It follows that

$$\rho(\lambda f)\,\rho(a) = \rho(f)\,\rho(\lambda a) = 0,$$

and hence $\lambda f \in \ker\rho$. We thus obtain that $\ker\rho$ is an (algebra) ideal. Now it is obvious that $\ker\rho$ is a prime ideal.

(b) Let $\mathcal{F}$ be the quotient field of $\mathcal{A}/\ker\rho$. $\mathcal{F}$ is well defined since $\mathcal{A}/\ker\rho$ is an integral domain by (a). We define the mapping $\tau : \mathcal{F} \to \mathbb{C}$ by

$$\tau(\pi(f)/\pi(g)) = \frac{\rho(f)}{\rho(g)} \quad (\pi(f)/\pi(g) \in \mathcal{F}).$$

A simple calculation shows that τ is a well defined non-zero field homomorphism. As usual we may identify $\pi(f) \in \mathcal{A}/\ker\rho$ with $\pi(fa)/\pi(a) \in \mathcal{F}$. We get

$$\tau(\pi(f)) = \tau(\pi(fa)/\pi(a)) = \frac{\rho(fa)}{\rho(a)} = \rho(f) \qquad (f \in \mathcal{A}),$$

and hence $\rho = \tau \circ \pi$. □

Lemma 2.2. *Suppose $\mathcal{A}$ is a commutative complex algebra and $\mathcal{P}$ is a prime ideal of $\mathcal{A}$. Then*

(a) $\mathfrak{c} = \sharp\mathbb{C} \leq \sharp(\mathcal{A}/\mathcal{P})$*, and*

(b) *if* $a \in \mathcal{A} \setminus \mathcal{P}$*, the set* $\mathcal{P}_e \stackrel{\text{def}}{=} \{(f, \lambda) \in \mathcal{A}_e : fa + \lambda a \in \mathcal{P}\}$ *is a prime ideal of* $\mathcal{A}_e$ *such that* $\mathcal{P} = \mathcal{P}_e \cap \mathcal{A}$*.*

Proof. Pick $a \in \mathcal{A} \setminus \mathcal{P}$ arbitrarily.

(a) Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{P}$ be the quotient mapping. Since $a \notin \mathcal{P}$ and since $\mathcal{P}$ is an ideal, $\pi(\lambda a) = \pi(\mu a)$ implies $\lambda = \mu$ for $\lambda, \mu \in \mathbb{C}$. This shows that the mapping $\lambda \mapsto \pi(\lambda a)$ is an injection, so that

$$\mathfrak{c} = \sharp\mathbb{C} = \sharp\{\pi(\lambda a) : \lambda \in \mathbb{C}\} \leq \sharp(\mathcal{A}/\mathcal{P}).$$

(b) It is easy to see that $\mathcal{P}_e$ is a proper ideal of $\mathcal{A}_e$ such that $\mathcal{P} = \mathcal{P}_e \cap \mathcal{A}$. To show that $\mathcal{P}_e$ is prime, suppose $(f_1, \lambda_1)(f_2, \lambda_2) \in \mathcal{P}_e$. By definition, this implies that $(f_1 f_2 + \lambda_2 f_1 + \lambda_1 f_2)a + (\lambda_1 \lambda_2)a \in \mathcal{P}$, and so we obtain $(f_1 a + \lambda_1 a)(f_2 a + \lambda_2 a) \in \mathcal{P}$. Since $\mathcal{P}$ is a prime ideal, $(f_1 a + \lambda_1 a)$ or $(f_2 a + \lambda_2 a)$ belongs to $\mathcal{P}$. This implies $(f_1, \lambda_1) \in \mathcal{P}_e$ or $(f_2, \lambda_2) \in \mathcal{P}_e$, and hence $\mathcal{P}_e$ is prime. □

Let $\mathcal{K}$ be a transcendental extension field of a commutative field k, and S a subset of $\mathcal{K}$. We recall that S is said to be algebraically independent over k, if the set of all finite products of elements of S is linearly independent over k. A subset T of $\mathcal{K}$ is called a *transcendence base* of $\mathcal{K}$ over k, if T is algebraically independent over k which is maximal with respect to the inclusion ordering. The existence of a transcendence base of $\mathcal{K}$ over k is well known (cf. [8, Theorem 1.1 of Chapter X]). The maximality of T shows that $\mathcal{K}$ is algebraic over $k(T)$, the field generated by T over k.

Lemma 2.3. *Let $\mathbb{Q}$ be the rational number field and k a transcendental extension field of $\mathbb{Q}$ such that $\sharp k = \mathfrak{c}$. If T is a transcendence base of k over $\mathbb{Q}$, then $\sharp T = \mathfrak{c}$.*

Proof. Suppose T is a transcendence base of k over $\mathbb{Q}$. Let $\mathbb{Q}(T)$ be the field generated by T over $\mathbb{Q}$. Since $T \subset \mathbb{Q}(T) \subset k$, we obtain $\sharp T \leq \sharp\{\mathbb{Q}(T)\} \leq \mathfrak{c}$. So, we show that $\mathfrak{c} \leq \sharp T$. Since k is algebraic over $\mathbb{Q}(T)$, each element of k is a zero point of some function in $\wp$, the set of all monic polynomials over $\mathbb{Q}(T)$. Note that for each monic polynomial, its zero points in k is at most finite. Put $\mathfrak{a} = \sharp\,\mathbb{Q}$, then we have

$$\mathfrak{c} = \sharp\, k \leq (\sharp\wp) \times \mathfrak{a} \leq (\sharp\{\mathbb{Q}(T)\} \times \mathfrak{a}) \times \mathfrak{a} = \sharp\{\mathbb{Q}(T)\},$$

and hence $\mathfrak{c} \leq \sharp\{\mathbb{Q}(T)\}$. We thus obtain

$$\mathfrak{c} = \sharp\{\mathbb{Q}(T)\} \leq \mathfrak{a} \times \sharp\, T. \tag{$**$}$$

If T were finite, then we would have $\sharp\{\mathbb{Q}(T)\} = \mathfrak{a}$, in contradiction to $\sharp\{\mathbb{Q}(T)\} = \mathfrak{c}$. It follows that $\sharp\, T \geq \mathfrak{a}$, and so $\mathfrak{a} \times \sharp\, T = \sharp\, T$. By $(**)$ we get $\mathfrak{c} \leq \sharp\, T$, proving $\sharp\, T = \mathfrak{c}$. □

Proof of Theorem 1.1. Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{P}$ be the quotient mapping and fix $a \in \mathcal{A} \setminus \mathcal{P}$.

(a) ⇒ (b) By (a) of Lemma 2.2, we obtain $\mathfrak{c} \leq \sharp(\mathcal{A}/\mathcal{P})$. To prove the opposite inequality, let $\mathcal{F}$ be the quotient field of $\mathcal{A}/\mathcal{P}$. By (b) of Lemma 2.1, we can write $\rho = \tau \circ \pi$, where $\tau : \mathcal{F} \to \mathbb{C}$ is a field homomorphism, and hence injective. It follows that $\sharp\mathcal{F} \leq \sharp\mathbb{C} = \mathfrak{c}$. If we regard $\mathcal{A}/\mathcal{P}$ as a subset of $\mathcal{F}$, it follows that $\sharp(\mathcal{A}/\mathcal{P}) \leq \sharp\mathcal{F} \leq \mathfrak{c}$, proving $\sharp(\mathcal{A}/\mathcal{P}) = \mathfrak{c}$.

(b) $\Rightarrow$ (c) Let $\mathcal{P}_e$ be the prime ideal of $\mathcal{A}_e$ as in (b) of Lemma 2.2. Let $\tilde{\pi} : \mathcal{A}_e \to \mathcal{A}_e/\mathcal{P}_e$ be the quotient mapping. Identification of f and $(f, 0)$ shows that $\pi(f) = \pi(g)$ if and only if $\tilde{\pi}(f, 0) = \tilde{\pi}(g, 0)$ for $f, g \in \mathcal{A}$, and so $\mathfrak{c} = \sharp(\mathcal{A}/\mathcal{P}) \leq \sharp(\mathcal{A}_e/\mathcal{P}_e)$. To show the opposite inequality, we define the mapping $\psi : \mathcal{A}_e/\mathcal{P}_e \to \mathcal{A}/\mathcal{P}$ by

$$\psi(\tilde{\pi}(f, \lambda)) = \pi(fa + \lambda a) \quad (\tilde{\pi}(f, \lambda) \in \mathcal{A}_e/\mathcal{P}_e).$$

A simple calculation shows that ψ is a well defined injection. Hence $\sharp(\mathcal{A}_e/\mathcal{P}_e) \leq \sharp(\mathcal{A}/\mathcal{P}) = \mathfrak{c}$, proving $\sharp(\mathcal{A}_e/\mathcal{P}_e) = \mathfrak{c}$.

(c) $\Rightarrow$ (d) Let $\tilde{\mathcal{F}}$ be the quotient field of $\mathcal{A}_e/\tilde{\mathcal{P}}$. Then

$$\mathfrak{c} = \sharp(\mathcal{A}_e/\tilde{\mathcal{P}}) \leq \sharp\tilde{\mathcal{F}} \leq \sharp(\mathcal{A}_e/\tilde{\mathcal{P}}) \times \sharp(\mathcal{A}_e/\tilde{\mathcal{P}}) = \mathfrak{c},$$

so that $\tilde{\mathcal{F}}$ also has the cardinal number $\mathfrak{c}$. Note that $\tilde{\mathcal{F}}$ is a transcendental extension of $\mathbb{Q}$ since $\tilde{\mathcal{F}}$ contains a unital algebra $\mathcal{A}_e/\tilde{\mathcal{P}} \supset \mathbb{C}$.

Let T and $\tilde{T}$ be transcendence bases of $\mathbb{C}$ and $\tilde{\mathcal{F}}$ over $\mathbb{Q}$, respectively. By Lemma 2.3, we see that $\sharp T = \mathfrak{c} = \sharp\tilde{T}$. Thus we can find a bijection $\theta : \tilde{T} \to T$. Since T is algebraically independent over $\mathbb{Q}$, the mapping θ is naturally extended to a field homomorphism $\tilde{\theta} : \mathbb{Q}(\tilde{T}) \to \mathbb{Q}(T)$ so that $\tilde{\theta}(r) = r$ for every $r \in \mathbb{Q}$. Since $\tilde{\mathcal{F}}$ is an algebraic extension of $\mathbb{Q}(\tilde{T})$ and since $\mathbb{C}$ is algebraically closed, $\tilde{\theta}$ can be extended to a field homomorphism on $\tilde{\mathcal{F}}$ into $\mathbb{C}$ (cf. [8, Theorem 2.8 of Chapter VII]), which is also denoted by $\tilde{\theta}$. Define $\tilde{\rho} = \tilde{\theta} \circ \tilde{\pi}$, where $\tilde{\pi} : \mathcal{A}_e \to \mathcal{A}_e/\tilde{\mathcal{P}}$ is the quotient mapping. Then $\tilde{\rho} : \mathcal{A}_e \to \mathbb{C}$ is a ring homomorphism whose kernel is equal to $\tilde{\mathcal{P}}$, proving $\mathcal{A} \cap \ker \tilde{\rho} = \mathcal{A} \cap \tilde{\mathcal{P}} = \mathcal{P}$.

(d) $\Rightarrow$ (a) Put $\rho = \tilde{\rho}|_{\mathcal{A}}$. Then $\rho : \mathcal{A} \to \mathbb{C}$ is a non-zero ring homomorphism such that $\ker \rho = \mathcal{P}$. □

Proof of Corollary 1.2. Suppose A is a uniform algebra on an infinite compact metric space X. Let Δ be the set of all non-maximal prime ideals of A. It is well known [4, Corollary 1] that there exist exactly $2^{\mathfrak{c}}$ non-maximal prime ideals of A, and hence $\sharp\Delta = 2^{\mathfrak{c}}$. Since X is separable, $\sharp A = \mathfrak{c}$: For if X_0 is a countable dense subset of X, then the restriction map $f \mapsto f|_{X_0}$ $(f \in A)$ is injective since each element of A is continuous, and hence $\mathfrak{c} \leq \sharp A \leq \mathfrak{a} \times \mathfrak{c} = \mathfrak{c}$. So, there exist exactly $2^{\mathfrak{c}}$ functions on A into $\mathbb{C}$, which need not be continuous nor ring homomorphic. This implies that there are at most $2^{\mathfrak{c}}$ complex ring homomorphisms on A.

Conversely pick $P \in \varDelta$ arbitrarily. By (a) of Lemma 2.2, we have

$$\mathfrak{c} \le \sharp(A/P) \le \sharp A = \mathfrak{c},$$

and hence $\sharp(A/P) = \mathfrak{c}$. So, by Theorem 1.1, to each $P \in \varDelta$ there corresponds a complex ring homomorphism $\rho_P : A \to \mathbb{C}$ such that $\ker \rho_P = P$. Suppose $\rho_{P_1} = \rho_{P_2}$ for $P_1, P_2 \in \varDelta$. It follows from $\ker \rho_{P_j} = P_j$ for $j = 1, 2$ that $P_1 = P_2$, and hence the mapping $P \mapsto \rho_P$ is an injection. We conclude that

$$2^{\mathfrak{c}} = \sharp\varDelta \le \sharp\{\rho_P : P \in \varDelta\},$$

and the proof is complete. □

Remark. Let A be a commutative Banach algebra. It is well-known (cf. [3, Theorem 5.7.32]) that under the continuum hypothesis there is a discontinuous homomorphism on A into some Banach algebra whenever there is a non-maximal prime ideal P of A with $\sharp(A/P) = \mathfrak{c}$. It follows from Theorem 1.1 that under the continuum hypothesis a discontinuous homomorphisms on A exists whenever there is a non-zero ring homomorphism $\rho : A \to \mathbb{C}$ such that $\ker \rho$ is non-maximal.

Example 2.1. Let $\bar{\mathbb{D}}$ denote the closure of the open unit disk $\mathbb{D}$ in $\mathbb{C}$. The disk algebra $A(\bar{\mathbb{D}})$ is a typical example of uniform algebras. Hatori, Ishii with the first and second author ([5, Corollary 5.3]) proved that if $\rho : A(\bar{\mathbb{D}}) \to A(\bar{\mathbb{D}})$ is a ring homomorphism whose range contains a non-constant function, then ρ is linear or conjugate linear.

Here, let us consider complex ring homomorphisms on $A(\bar{\mathbb{D}})$, that is, the range contains only constant functions. It is well known that the set of all non-zero complex homomorphisms on $A(\bar{\mathbb{D}})$ can be identified with $\bar{\mathbb{D}}$. So, there are $\mathfrak{c}$ complex homomorphisms on $A(\bar{\mathbb{D}})$. On the other hand, by Corollary 1.2, we see that there are $2^{\mathfrak{c}}$ ring homomorphisms whose kernels are non-maximal prime ideals.

Finally, we give a pathological feature of complex ring homomorphisms (cf. [5, Corollary 5.2]).

Example 2.2. If $H(\Omega)$ is the algebra of all analytic functions on a region $\Omega \subset \mathbb{C}$, then, as we shall show, $H(\Omega)$ is a subring of $\mathbb{C}$. In particular, it will follow that every subalgebra $\mathcal{A}$ of $H(\Omega)$, which is with or without unit, is a subring of $\mathbb{C}$. In fact, the ideal (0) containing only zero is a prime ideal of $H(\Omega)$. Moreover $\sharp\, H(\Omega) = \mathfrak{c}$ since $\mathfrak{c} = \sharp\, \mathbb{C} \le \sharp\, H(\Omega) \le \sharp\, C(\Omega) = \mathfrak{c}$, and so by Theorem 1.1 there exists a non-zero complex ring homomorphism ρ on $H(\Omega)$ such that $\ker \rho = (0)$. Therefore, ρ is an injective complex ring homomorphism on $H(\Omega)$.

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