Publ. Math. Debrecen
43 / 3–4 (1993), 303–313

On Sasakian anti-holomorphic Cauchy–Riemann submanifolds of locally conformal Kaehler manifolds

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Abstract. We study some properties of the Sasakian anti-holomorphich Cauchy–Riemann submanifolds in a locally conformal Kaehler manifold.

Introduction

The geometry of the Cauchy–Riemann (C.R.) submanifolds of a locally conformal Kaehler (l.c.K.) manifold has been studied in the last ten years, ([B-C], [D_1], [D_2], [D-V], [I-O], [M], [V]).

The concept of normal C.R. submanifold was introduced by A. Bejancu, ([B_1]) in analogy with the theory of the normal almost contact structures, ([B], [H-S]). In [B_1] a theory for the normal C.R. submanifolds in a Kaehler manifold is developed. In particular, a C.R. hypersurface of a Kaehler manifold is a normal contact hypersurface, ([OM]).

Some properties of the normal C.R. submanifolds of l.c.K. manifolds have been studied in a former paper, ([V]).

In this paper, we study the Sasakian anti-holomorphic C.R. submanifolds in a l.c.K. manifold. In the first section, we recall some properties of the l.c.K. manifolds and of the anti-holomorphic C.R. submanifolds that are C.R. submanifolds such that the totally real distribution and the normal bundle have the same dimension. D.E. Blair and B.Y. Chen proved that the totally real distribution of a C.R. submanifold in a l.c.K. manifold is integrable, ([B-C]).

On the contrary, in the section 2, we prove that the holomorphich distribution of a proper contact anti-holomorphic C.R. submanifold in a l.c.K. manifold cannot be integrable. This generalizes the well known result which states that the canonical distribution of a contact structure cannot be integrable, ([B]). In the sections 3 and 4 we consider C.R. submanifolds that are orthogonal to the lee vector field. When the

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curvature of the normal connection is zero, there exists an orthonormal and parallel frame $\{\xi_i\}_{1\leq i\leq q}$ in the normal bundle $(\mathrm{TM})^\perp$, ([C]). Putting $E_i = -J\xi_i$, $i = 1,\dots,q$, then $\{E_i\}_{1\leq i\leq q}$ is an orthonomal frame of the totally real distribution $D^\perp$. The expression of the covariant derivatives ∇E_1, $i = 1,\dots,q$, generalize the formula for the covariant derivative of the Reeb vector field, ([B]).

Finally, we characterize Sasakian anti-holomorphic C.R. submanifolds by means of the covariant derivative of the vector valued 1-form P.

§1. Preliminaries

Let $\left(M^{2n}, g_0, J\right)$ be a Hermitian manifold of complex dimension n, with Kaehler 2-form Ω_0, i.e. $\Omega_0(X,Y) = g_0(X, JY)$, $X, Y \in TM^{2n}$. Then $\left(M^{2n}, g_0, J\right)$ is a *locally conformal Kaehler (l.c.K.) manifold* if there exists a closed 1-form ω_0 on M^{2n} such that

$$d\Omega_0 = \omega_0 \wedge \Omega_0. \tag{1.1}$$

The 1-form ω_0 is called the *Lee form*, then *Lee vector field* is the vector field B_0 such that $g_0\left(B_0, X\right) = \omega_0(X)$, $\quad X \in TM^{2n}$. If $\bar\nabla$ denotes the Riemannian connection of $\left(M^{2n}, g_0\right)$, then one has:

$$\left(\bar\nabla_X J\right) Y = \frac{1}{2}\left\{\theta_0(Y)X - \omega_0(Y)JX - \Omega_0(X,Y)B_0 - g(X,Y)A_0\right\} \tag{1.2}$$

for any $X, Y \in TM^{2n}$, where $\theta_0 = \omega_0 \circ J$ is the *anti-Lee 1-form* and $A_0 = -JB_0$ is the *anti-Lee vector field.*

We use the notation and the properties stated in [V_1], [V_2]. A submanifold M^m of M^{2n} is called a *Cauchy–Riemann (C.R.) submanifold* of M^{2n} if the tangent bundle TM^m is expressed as a direct sum of two distributions D and $D^\perp$, such that D is holomorphic (i.e. $J_x\left(D_x\right) = D_x, x \in M^m$) and $D^\perp$ is totally real (i.e. $J_x(D_x^\perp) \subset (T_xM^m)^\perp, x \in M^m$). Let p be the complex dimension of the holomorphic distribution D and let q be the real dimension of the totally real distribution $D^\perp$. If $q = 0$, M^m is called *holomorphic submanifold*; if $p = 0$, M^m is called *totally real submanifold.* In this paper, we examine the case $p \neq 0$, $q \neq 0$, that is M^m is a *proper C.R. submanifold*, ([B_3]).

Let $\tan_x$ and nor_x be the projections naturally associated with the direct sum decomposition $T_xM^{2n} = T_xM^m \oplus \left(T_xM^m\right)^\perp$, $x \in M^m$. We put $PX = \tan(JX)$, $FX = \mathrm{nor}(JX)$, $t\xi = \tan(J\xi)$ and $f\xi = \mathrm{nor}(J\xi)$ for any $X \in TM^m$, $\xi \in \left(TM^m\right)^\perp$. Then, for any $X \in TM^m$ one has $PX \in D$. Moreover, the following identities hold: $P^2 = -I - tF$, $f^2 = -I - Ft$, $FP = 0$, $fF = 0$, $tf = 0$, $Pt = 0$, $P^3 + P = 0$, $f^3 + f = 0$, ([K-Y]). The

Gauss and Weingarten formulas are still valid, that is:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{1.3}$$

for any $X, Y \in TM^m$, $\xi \in (TM^m)^\perp$, here ∇, h, A_ξ and $\nabla^\perp$ stand, respectively, for *the induced connection, the second foundamental form, the Weingarten operator* (associated with $\xi \in (TM^m)^\perp$) and *the normal connection* in $(TM^m)^\perp$. The forms θ, ω and Ω are naturally induced on the submanifold M^m by θ_0, ω_0 and Ω_0 respectively. One has:

$$\theta = \omega \circ P + \omega_0 \circ F, \quad \Omega(X,Y) = g(X, PY), \quad X, Y, \in TM^m. \tag{1.4}$$

As a consequence of (1.2) and (1.3) one has:

$$\begin{aligned}(\nabla_X P)\,Y = {} & A_{FY}X + th(X,Y) + \\ & + \frac{1}{2}\{\theta(X)Y - \omega(Y)PX - \Omega(X,Y)B - g(X,Y)A\}\end{aligned} \tag{1.5}$$

$$\begin{aligned}(\nabla_X F)\,Y = {} & fh(X,Y) - h(X,PY) - \\ & - \frac{1}{2}\left\{\omega(Y)FX + \Omega(X,Y)B^\perp + g(X,Y)A^\perp\right\}\end{aligned} \tag{1.6}$$

for any X, $Y \in TM^m$, where $A = \tan(A_0)$, $B = \tan(B_0)$, $A^\perp = \mathrm{nor}(A_0)$ and $B^\perp = \mathrm{nor}(B_0)$. We put:

$$S(X,Y) = [P,P](X,Y) - 2t(dF)(X,Y), \quad X, Y \in TM^m. \tag{1.7}$$

Here $[P,P]$ is the Nijenhuis torsion of P and dF is the differential of the vector valued 1-form F, which can be expressed as follows:

$$2(dF)(X,Y) = \nabla^\perp_X(FX) - \nabla^\perp_Y(FX) - F[X,Y], \quad X, Y \in TM^m. \tag{1.8}$$

A C.R. submanifold is called *normal* if $S = 0$, ([B$_1$]). The C.R. submanifold M^m is called *anti-holomorphic* if $J_x\left(D^\perp_x\right) = \left(T_x M^m\right)^\perp$ for any $x \in M^m$.

Let $\{F_1, \dots, F_p, JF_1, \dots, JF_p\}$ be an orthonormal locally defined frame of D; then the normal vector field

$$H_D = \frac{1}{2p}\sum_{i=1}^{p}\left\{h\left(F_i, F_i\right) + h\left(JF_i, JF_i\right)\right\} \tag{1.9}$$

is well defined and is called the *D-mean curvature vector* of M^m. An anti-holomorphic C.R. submanifold is called *contact anti-holomorphic* if $H_D \neq 0$ and

$$(dF)(X,Y) = -\Omega(X,Y)H_D, \quad X, Y \in TM^m. \tag{1.10}$$

A normal contact anti-holomorphic C.R. submanifold is called a *Sasakian antiholomorphic C.R. submanifold*, ([B$_1$]), ([B$_3$]).

§2. Non integrability of the holomorphic distribution on a contact anti-holomorphic C.R. submanifold

D.E. Blair and B.Y. Chen proved that the totally real distribution $D^\perp$ of a C.R. submanifold is integrable, ([B-C]). In this section, we study the integrability of the holomorphic distribution D, proving that D cannot be integrable.

As a consequence of (1.3) one has:

$$\bar{\nabla}_X(JY) = \nabla_X(PY) - A_{FY}X + h(X,PY) + \nabla^\perp_X(FY) \tag{2.1}$$

$$J\left(\bar{\nabla}_X Y\right) = P(\nabla_X Y) + F(\nabla_X Y) + J(h(X,Y)) \tag{2.2}$$

for any $X, Y \in TM^m$. Using (1.2), (2.1) and (2.2), we obtain:

$$\begin{aligned}\nabla^\perp_X(FX) = F(\nabla_X Y) - h(X,PY) - \\ - \frac{1}{2}\{\omega(Y)FX + \Omega(X,Y)B^\perp + g(X,Y)A^\perp\}, \quad X, Y \in TM^m.\end{aligned} \tag{2.3}$$

Moreover, (1.8) and (2.3) imply:

$$\begin{aligned}2(dF)(X,Y) = {} & h(PX,Y) - h(X,PY) + \\ & + \frac{1}{2}\{\omega(X)FY - \omega(Y)FX\} - \Omega(X,Y)B^\perp\end{aligned} \tag{2.4}$$

for any $X, Y \in TM^m$. From (2.4) we obtain:

$$2(dF)(X,Y) = h(PX,Y) - h(X,PY) - \Omega(X,Y)B^\perp, \quad X, Y \in D. \tag{2.5}$$

If D is integrable, then one has:

$$g([X,Y],Z) = 0, \quad X, Y \in D, \ Z \in D^\perp. \tag{2.6}$$

This condition is equivalent to:

$$g_0\left(J\left(\bar{\nabla}_X Y\right) - J\left(\bar{\nabla}_Y X\right), JZ\right) = 0, \quad X, Y \in D, \ Z \in D^\perp. \tag{2.7}$$

Then, (2.7), the Gauss formula and (1.2) imply:

$$g\left(h(X,PY) - h(PX,Y) + \Omega(X,Y)B^\perp, JZ\right) = 0, \qquad X, Y \in D, \ Z \in D^\perp. \tag{2.8}$$

By means of (2.8), (2.5) and (1.10), we obtain:

$$g_0\left(H_D, JZ\right)\Omega(X,Y) = 0, \quad X, Y \in D, \ Z \in D^\perp. \tag{2.9}$$

Since the C.R. submanifold M^m is anti-holomorphic, i.e. $(TM^m)^{\perp} = J\left(D^{\perp}\right)$, there exists $Z_0 \in D^{\perp}$ such that $JZ_0 = H_D$. From (2.9) we obtain, for a given $X \in D$, $X \neq 0$:

$$-\|Z_0\|^2\|X\|^2 = g_0\left(JZ_0, JZ_0\right)\Omega(X, JX) = 0$$

which contradicts the hypothesis $X \neq 0$, $Z_0 \neq 0$. In this way the following result is proved.

Theorem 2.1. *If M^m is a proper contact anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}, then the distribution D of M^m is not integrable.*

Corollary 2.1. *The contact distribution of a contact metric hypersurface of a l.c.K. manifold is not integrable.*

Remark. The result of the corollary 2.1 can be also derived from a remark due to D.E. Blair, ([B]), p. 36).

We recall that a proper C.R. submanifold is called *mixed totally geodesic* if $h(X, Y) = 0$ for any $X \in D$, $Y \in D^{\perp}$.

Proposition 2.1. *Let M^m be a contact anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}. If M^m is orthogonal to the Lee vector field B_0, then M^m is mixed totally geodesic.*

Indeed, one has: $\Omega(X, Y) = g(X, JY) = 0$, for any $X \in D$, $Y \in D^{\perp}$. Since M^m is a contact anti-holomorphic C.R. submanifold, (1.10) implies: $(dF)(X, Y) = 0$, for any $X \in D$, $Y \in D^{\perp}$. Since M^m is orthogonal to the Lee vector field B_0, it follows that $\omega_0(X) = 0$, for any $X \in TM^m$. Moreover, (2.4) gives: $h(PX, Y) = 0$, for any $X \in D$, $Y \in D^{\perp}$, and so $h(X, Y) = 0$, for any $X \in D$, $Y \in D^{\perp}$; since $\mathrm{Im}P = D$.

§3. Sasakian anti-holomorphic C.R. submanifolds with flat normal connection

The curvature tensor $R^{\perp}$ of the normal connection $\nabla^{\perp}$ of a submanifold M^m of M^{2n} is defined by

$$R^{\perp}(X, Y)\xi = \nabla^{\perp}_X\left(\nabla^{\perp}_Y\xi\right) - \nabla^{\perp}_Y\left(\nabla^{\perp}_X\xi\right) - \nabla^{\perp}_{[X,Y]}\xi, \qquad X, Y \in TM^m,\ \xi \in (TM^m)^{\perp}. \tag{3.1}$$

The normal connection $\nabla^{\perp}$ is *flat* if $R^{\perp} = 0$. The following theorem due to B.Y. Chen is well known, ([C], p. 99, Proposition 1.1).

Theorem 3.1. *Let M^m be a submanifold of a Riemannian manifold M^r. Then, the normal connection $\nabla^{\perp}$ of M^m in M^r is flat if and only if* *there exist locally $r-m$ mutually orthogonal unit normal vector fields ξ_i, $i=1,\dots,r-m$, such that each of the ξ_i is parallel in the normal bundle.*

Let M^m be an anti-holomorphic C.R. submanifold of the l.c.K. M^{2n}. A local orthonormal frame $\{\xi_1,\dots,\xi_q\}$ of the normal bundle which satisfies the properties of the theorem 3.1 is called an orthonormal *ξ-frame.*

We put $E_i=-J\xi_i$, $i=1,\dots,q$. Then $\{E_1,\dots,E_q\}$ is a local orthonormal frame of the totally real distribution $D^\perp$.

Proposition 3.1. *Let M^m be a Sasakian anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}. If M^m is orthogonal to the Lee vector field B_0 and the normal connection $\nabla^\perp$ is flat then one has:*

$$\nabla_X E_i = PA_iX-\frac{1}{2}\theta\left(E_i\right)PX \tag{3.2}$$

$$P\circ A_i=A_i\circ P \tag{3.3}$$

for any $X\in TM^m$ and for any orthonormal ξ-frame, where $A_1=A_{\xi_i}$, $i=1,\dots,q$.

The proposition 2.1 implies that M^m is mixed totally geodesic. The formula (3.2) is a consequence of the corollary 2.1 in [V]; moreover (3.3) follows from the proposition 2.1 of [V].

We recall that in a Sasakian manifold with the contact structure (ϕ,ξ,η,g) this formula holds:

$$\nabla_X\xi=-\phi X \tag{3.4}$$

for any vector field X tangent to the manifold, ([B], p. 74). We want to generalize (3.4) for the Sasakian anti-holomorphic C.R. submanifolds of a l.c.K. manifold.

Theorem 3.2. *Let M^m be a normal proper anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}. If M^m is orthogonal to the Lee vector field B_0 and the normal connection $\nabla^\perp$ is flat, then the following statements are equivalent:*

a) M^m is a Sasakian submanifold,

b) for any $x\in M^m$ there exist a neighborhood U of x and an orthonormal ξ-frame on U such that

$$\nabla_X E_i=\left(g_0\left(H_D,\xi_i\right)-\theta\left(E_i\right)\right)PX,\quad X\in TM^m,\quad i=1,\dots,q. \tag{3.5}$$

Assume that M^m is a Sasakian submanifold. From the proposition 2.1 one has:

$$\begin{aligned} &g\left(h(PX,Y),\xi_i\right)=g\left(A_iPX,Y\right)=\\ &\qquad=g\left(PA_iX,Y\right)=-g\left(A_iX,PY\right)=-g_0\left(h(X,PY),\xi_i\right)\end{aligned}$$

for any $X, Y \in TM^m$, $i = 1, \dots, q$. This implies:

$$h(PX, Y) = -h(X, PY), \quad X, Y \in TM^m. \tag{3.6}$$

Moreover, (2.4) and (3.6) imply:

$$2(dF)(X, Y) = 2h(PX, Y) - \Omega(X, Y)B_0, \quad X, Y \in TM^m. \tag{3.7}$$

From the proposition 3.1, (1.10) and (3.7) it follows that:

$$\begin{aligned}\sum_{i=1}^{q} g(\nabla_X E_i, Y)\xi_i =& \sum_{i=1}^{q} g\left(PA_i X, Y\right)\xi_i - \frac{1}{2}\sum_{i=1}^{q} g\left(\theta\left(E_i\right)PX, Y\right)\xi_i = \\ &= \sum_{i=1}^{q} g\left(A_i PX, Y\right)\xi_i + \frac{1}{2}\sum_{i=1}^{q}\theta\left(E_i\right)\Omega(X, Y)\xi_i = \\ &= \sum_{i=1}^{q} g_0(h(PX, Y), \xi_i)\xi_i + \frac{1}{2}\Omega(X, Y)\sum_{i=1}^{q} g_0\left(B_0, \xi_i\right)\xi_i = \\ &= h(X, PY) + \frac{1}{2}\Omega(X, Y)B_0 = (dF)(X, Y) + \Omega(X, Y)B_0 = \\ &= -\Omega(X, Y)H_D + \Omega(X, Y)B_0\end{aligned}$$

for any $X, Y \in TM^m$. Therefore, one has:

$$\sum_{i=1}^{q} g\left(\nabla_X E_i, Y\right)\xi_i = g(PX, Y)\sum_{i=1}^{q} g_0\left(H_D - B_0, \xi_i\right)\xi_i \tag{3.8}$$

for any $X, Y \in TM^m$, and this condition is equivalent to (3.5). Now, we consider a neighborhood U of a given $x \in M^m$ and an orthonormal ξ-frame $\{\xi_1, \dots, \xi_q\}$ on U which satisfies (3.5). With the same technique used before one has:

$$\begin{aligned}(dF)(X, Y) = \sum_{i=1}^{q} g\left(\nabla_X E_i, Y\right)\xi_i = \Omega(X, Y)B_0 = \\ = \sum_{i=1}^{q} g_0\left(H_D, \xi_i\right)g(PX, Y)\xi_i - \sum_{i=1}^{q}\theta\left(E_i\right)g(PX, Y)\xi_i - \Omega(X, Y)B_0 = \\ = -\Omega(X, Y)H_D\end{aligned}$$

for any $X, Y \in TM^m$.

Remark. It is easy to prove that the formula (3.5) is equivalent to:

$$A_i(PX) = g_0\left(H_D - \frac{1}{2}B_0, \xi_i\right)PX \tag{3.9}$$

for any $X \in TM^m$ and for any orthonormal ξ-frame.

§4. The covariant derivative of the vector valued 1-form P

The following result can be easily obtained by mean of a straightforward calculation.

Lemma 4.1. *Let M^m be a C.R. submanifold of the 1.c.K. manifold M^{2n}. Then, one has:*

(4.1)
$$\begin{aligned}
2g(\nabla_X P)Y, Z) = \\
&= 3(d\Omega)(X, PY, PZ) - 3(d\Omega)(X, Y, Z) + g([P, P](Y, Z), PX) + \\
&\quad + 2g_0(dF)(PY, Z), FX) + 2g_0((dF)(PY, X), FZ) - \\
&\quad - 2g_0(dF)(PZ, X)FY) - 2g_0((dF)(PZ, Y), FX)
\end{aligned}$$

for any $X, Y, Z \in TM^m$.

Remark. If the manifold M^{2n} is Kaehler, then $d\Omega = 0$ and (4.1) gives a formula due to A. Bejancu, ([B_3], p. 51, Proposition 3.1).

Proposition 4.1. *Let M^m be a Sasakian anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}. If M^m is orthogonal to the Lee vector field B_0 and the normal connection $\nabla^\perp$ is flat, then one has:*

$$(4.2) \qquad (\nabla_X P)\, Y = g(PX, PY) JH_D + g_0\,(FY, H_D)\, \iota X, \quad X, Y \in TM^m$$

where $\iota : TM^m \to D$ denotes the natural projection operator associated with the holomorphic distribution D.

Since M^m is orthogonal to the Lee vector field B_0, Ω is closed. Moreover, by a direct calculation one has:

$$(4.3) \qquad g([P, P](Y, Z), PX) = 0, \quad X, Y, Z \in TM^m.$$

Then, the lemma 4.1 and (4.3) imply:

$$\begin{aligned}
g\,((\nabla_X P)\, Y, Z) &= g_0((dF)(PY, Z), FX) + g_0((dF)(PY, X), FZ) - \\
&\quad - g_0((dF)(PZ, X), FY) - g_0((dF)(PZ, Y), FX) = \\
&= - \Omega(PY, Z) g_0\,(H_D, FX) - \Omega(PY, X) g_0\,(H_D, FZ) + \\
&\quad + \Omega(PZ, X) g_0\,(H_D, FY) + \Omega(PZ, Y) g_0\,(H_D, FX) = \\
&= - g(PX, PY) g_0\,(H_D, JZ) + g(PZ, PX) g_0\,(H_D, FY) = \\
&= g(PX, PY) g_0\,(JH_D, Z) + g(Z, \iota X) g_0\,(H_D, FY) = \\
&= g(PX, PY) JH_D + g_0\,(FY, H_D)\, \iota X
\end{aligned}$$

for any $X, Y, Z \in TM^m$.

Theorem 4.1. *Let M^m be a Sasakian anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}. If M^m is orthogonal to the Lee vector field B_0 and the normal connection $\nabla^{\perp}$ is flat, then one has:*

$$(\nabla_X \Omega)(Y,Z) = g_0(H_D, FZ)\, g(PX, PY) - g_0(H_D, FY)\, g(PX, PZ), \quad X, Y, Z \in TM^m.$$

As a consequence of the proposition 4.1 one has:

$$\begin{aligned}(\nabla_X \Omega)(Y,Z) &= g((\nabla_X P)Z, Y) = \\ &= g_0(H_D, FZ)\, g(\iota X, Y) + g(PX, PZ) g_0(JH_D, Y) = \\ &= g_0(H_D, FZ)\, g(J\iota X, JY) - g(PX, PZ) g_0(H_D, JY) = \\ &= g_0(H_D, FZ)\, g(PX, PY) - g(PX, PZ) g_0(H_D, FY)\end{aligned}$$

for any $X, Y, Z \in TM^m$.

Proposition 4.2. *Let M^m be a normal anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}. Moreover, M^m is orthogonal to the Lee vector field B_0 and the normal connection $\nabla^{\perp}$ is flat. If one has:*

$$(\nabla_X P)\, Y = g(PX, PY) JH_D + g_0(FY, H_D)\, \iota X - \theta(Y) X \tag{4.4}$$

for any $X, Y \in TM^m$, then M^m is a Sasakian submanifold.

Let $\{\xi_1, \ldots, \xi_q\}$ be an orthonormal ξ-frame and $X \in TM^m$. Applying (4.4), we obtain:

$$\begin{aligned}\nabla_X E_i &= -P^2(\nabla_X E_i) = P((\nabla_X P) E_i) = \\ &= g_0(FE_i, H_D) P\iota X - \theta(E_i) PX = g_0(H_D, \xi_i) PX - \theta(E_i) PX.\end{aligned}$$

The statement follows applying the theorem 3.2.

Corollary 4.1. *Let M^m be a normal anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}. Moreover, M^m is orthogonal to the Lee vector field B_0 and the normal connection $\nabla^{\perp}$ is flat. If one has:*

$$(\nabla_X P)\, Y = g(PX, PY) JH_D + g_0(FY, H_D)\, \iota X - \theta(Y) X \tag{4.4}$$

for any $X, Y \in TM^m$, then $\theta = 0$.

The statement is a consequence of the propositions 4.1 and 4.2.

Proposition 4.3. *Let M^m be a normal anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}. Moreover, M^m is orthogonal to the Lee vector field B_0 and the normal connection $\nabla^{\perp}$ is flat. If one has:*

$$\begin{aligned}&(\nabla_X \Omega)(Y,Z) = \\ &= g_0(H_D, FZ)\, g(PX, PY) - g_0(H_D, FY)\, g(PX, PZ) - \theta(Z) g(X, Y)\end{aligned}$$

for any $X, Y, Z \in TM^m$, then M^m is a Sasakian submanifold.

Infact, one has:

$$\begin{aligned} &g\left(\left(\nabla_X P\right) Z, Y\right) = \left(\nabla_X \Omega\right)(Y, Z) = \\ &= g_0\left(H_D, FZ\right) g(PX, PY) - g_0\left(H_D, FY\right) g(PX, PZ) - \theta(Z) g(X, Y) = \\ &= g_0\left(H_D, FZ\right) g(\iota X, Y) - g_0\left(H_D, JY\right) g(PX, PZ) - \theta(Z) g(X, Y) = \\ &= g_0\left(H_D, FZ\right) g(\iota X, Y) + g_0\left(JH_D, Y\right) g(PX, PZ) - \theta(Z) g(X, Y) = \\ &= g\left(g(PX, PZ) JH_D + g_0\left(H_D, FZ\right) \iota X - \theta(Z) X, Y\right) \end{aligned}$$

for any $X, Y, Z \in TM^m$. The statement is a consequence of the proposition 4.2.

Corollary 4.2. *Let M^m be a normal anti-holomorphic C.R. submanifold of the l.c.K. manifold M^{2n}. Moreover, M^m is orthogonal to the Lee vector field B_0 and the normal connection $\nabla^\perp$ is flat. If one has:*

$$\begin{aligned} &\left(\nabla_X \Omega\right)(Y, Z) = \\ &= g_0\left(H_D, FZ\right) g(PX, PY) - g_0\left(H_D, FY\right) g(PX, PZ) - \theta(Z) g(X, Y) \end{aligned}$$

for any $X, Y, Z \in TM^m$, then $\theta = 0$.

The statement is a consequence of the proposition 4.3 and of the theorem 4.1.

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(Received March 30, 1992)